Robust periodic economic MPC for linear systems

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ABSTRACT

Economic Model Predictive Control differs from conventional tracking model predictive control by directly addressing a plant’s economic cost as the stage cost, consequently leading to better economic performance. This paper extends current economic model predictive control theory to linear time-invariant systems with periodic disturbances and cost functions, under mild assumptions. To ensure an increased region of attraction and to continuously guarantee feasibility of the controller despite changing economic conditions, a periodic terminal condition is used in place of terminal constraints. The approach draws on constraint tightening techniques in order to guarantee robust satisfaction of constraints as well as convergence of the controller. A Lyapunov based approach is used to show stability of the proposed controller and characterise a region about the optimal trajectory to which the system converges.

1. Introduction

Model Predictive Control (MPC) is often employed as part of a hierarchical control architecture in order to minimise a plant’s economic cost. In such schemes, a high level, static Real Time Optimiser (RTO), selects steady state set-points which minimise an economic cost (Qin & Badgwell, 2003). Typically, a Tracking MPC (TMPC) problem is formulated using an error system from the set-points, with convergence of the closed-loop system guaranteeing set-point tracking (Mayne & Rawlings, 2009). The TMPC is designed to reject disturbances and regulate the system to the given set-points, however it does not generally take into account the real economic cost of the state trajectories. Therefore, if the set-points change frequently with respect to the plant dynamics, this approach may result in poor economic performance. An alternative is to combine the RTO and TMPC into a single controller, known as Economic Model Predictive Control (EMPC) (Rawlings, Angeli, & Bates, 2012). Though improved transient performance drives the development of EMPC, asymptotic performance has also been shown to be at least as good as the best steady state operating condition as achievable by a TMPC (Angeli, Amrit, & Rawlings, 2012).

In order to demonstrate closed loop stability of the optimal steady state, early EMPC formulations used terminal point constraints, combined with assumptions of strong duality with the steady state problem (Diehl, Amrit, & Rawlings, 2011). A Lyapunov function can be constructed, based on the strong duality assumption (Diehl et al., 2011), which has been extended to allow periodic cost functions and systems (Angeli et al., 2012; Zanon, Gros, & Diehl, 2013). The strong duality assumption can also be relaxed to a dissipativity assumption (Amrit, Rawlings, & Angeli, 2011; Angeli et al., 2012; Grüne, 2013), while the robustness of this assumption has also been investigated (Müller & Allgöwer, 2012). Alternatively, a strong convexity assumption can be used, as demonstrated in Huang, Biegler, and Harinath (2012) for infinite horizon problems. Each of these assumptions restricts the cost functions and systems for which stability guarantees can be provided, however regularisation of the cost function can ensure these assumptions are fulfilled (Maree & Imsland, 2014).

After removing terminal point constraints, it can be shown that as horizon lengths approach infinity, closed loop performance converges towards optimal performance (Grüne, 2013; Huang et al., 2012). In practice however, infinite horizons have to be approximated by sufficiently long finite horizons, which may still result in computational intractability for many applications. Another approach is to replace the terminal point constraint with a terminal set and corresponding terminal weight (Amrit et al., 2011). The use of a terminal set rather than terminal point constraint increases the region of attraction, however these sets can be complex to calculate.
for non-linear systems and costs, and may still be restrictive for short horizons. Critically for EMPC, when using terminal point or set constraints, infeasibility of the optimisation can arise due to changes in economic parameters. These changes in economic parameters may come about due to changes in market conditions, plant production rates or material costs and result in a changed optimal operating point.

Recent extensions to EMPC (Fagiano & Teel, 2013; Ferramosca, Limón, & Camacho, 2014; Müller, Angeli, & Allgöwer, 2013) have taken advantage of techniques used in TMPC (Fagiano & Teel, 2013; Ferramosca, Limón, Alvarado, Alamo, & Camacho, 2009; Limon, Alvarado, Alamo, & Camacho, 2008), whereby the terminal state may be any steady state and an offset function is used to penalise the distance to the optimal steady state. This approach gives feasibility guarantees for changing economic parameters, and an enlarged region of attraction as compared to terminal point constraints. The application of these generalised terminal constraints to EMPC requires some modification of the stage cost (Ferramosca et al., 2014) or an additional constraint on the terminal condition (Fagiano & Teel, 2013). The modified stage cost used to ensure convergence in Ferramosca et al. (2014) may result in the stage cost being optimised not resembling the true economic cost, while in Fagiano and Teel (2013) the additional constraint effectively constrains the terminal condition to enforce convergence of the system. While time-varying terminal weights can improve performance (Müller et al., 2013), this additional constraint may be overly restrictive.

Early development in EMPC theory assumed perfect prediction models to ensure continual feasibility. In practice however, this assumption is invalid due to external disturbances or modelling error. Tube MPC approaches have been demonstrated in EMPC, however since the planned inputs are not directly implemented, it is important to integrate the cost spatially over the resulting tube as demonstrated in Bayer, Müller, and Allgöwer (2014). Stochastic EMPC approaches are demonstrated in Howgaard, Larsen, and Jorgensen (2011) to handle approximate constraints, however the economic cost and system are limited to the linear case and the formulation results in a second order cone programming problem. Both of these approaches, as with other robustness approaches such as Min–Max MPC have increased computational complexity over standard MPC (Mayne, Rawlings, Rao, & Scokaert, 2000). Constraint tightening for MPC is a robustness technique which achieves constraint robustness with little to no increased computational complexity. Introduced in Gossner, Kouvaritakis, and Rossiter (1997) and generalised in Chisci, Rossiter, and Zappa (2001), Kuwarta, Richards, and How (2007) and Richards (2005) for conventional MPC, constraint tightening can be combined with EMPC, resulting in a robust EMPC formulation (Broomhead, Manzie, Shekhar, Brear, & Hield, 2014). Constraint tightening EMPC has been shown to subsume tube MPC approaches with constant feedback gains, due to increased regions of attraction (Trodden, 2009). Additionally, as opposed to tube EMPC, constraint tightening EMPC formulations directly determine the control inputs to be applied to the plant at the next time-step, which gives opportunity for increased economic performance.

Systems with periodic disturbances and costs commonly arise in practical applications (Broomhead et al., 2013; Huang et al., 2012), while time-invariant applications can be seen as a special case of these, where the period equals one. The main contribution of this paper is a novel robust EMPC formulation for systems with periodic disturbances and costs, which may also be impacted by unknown disturbances. The formulation uses a generalised terminal constraint as in Fagiano and Teel (2013), Ferramosca et al. (2014) and Müller et al. (2013), allowing feasibility guarantees despite changing economic parameters, further generalised to the periodic case. The stage cost does not need to be modified as in Ferramosca et al. (2014), and additional contractive constraints are not required as in Fagiano and Teel (2013) and Müller et al. (2013). Instead, additional constraint tightening towards the end of the horizon provides a margin of control authority, which is used to ensure convergence towards the optimal terminal condition. The approach is combined with standard constraint tightening techniques to achieve robust constraint satisfaction. Stability of the controller and convergence to a region around the optimal trajectory, which can be made arbitrarily small in the undisturbed case, is demonstrated using Lyapunov based techniques. A brief simulation example is used to highlight the efficacy of the approach.

1.1. Notation

The \( \sim \) operator is used to denote the Pontryagin difference, where \( A \sim X := \{a \mid a + b \in A \forall b \in X\} \), implying

\[
c \in A \sim B \Rightarrow c + b \in A \quad \forall b \in B.
\]  

(1)

The set of integers between \( a \) and \( b \) inclusive is denoted by \( [a, b] \). The Euclidean norm of \( x \) is denoted \( |x| \) and \( |x|_1 \) denotes the \( L_1 \) norm of \( x \). Real system states, inputs and outputs are defined at time \( k \) as \( x(k), u(k) \) and \( y(k) \) respectively. At time-step \( k \), the prediction of the variable \( x \) at position \( j \) along the horizon is denoted \( x_j^k \). Stacked vectors are in bold, i.e. the stacked vector of \( x_j^k \), \( j \in [0, T] \) is denoted by \( \mathbf{x}^T \). The notation \( [j] = j \mod P \) where \( P \) is the relevant period. The set of all class \( \mathcal{K} \) functions is denoted \( \mathcal{K} \), and the set of all class \( \mathcal{K}_\infty \) functions is denoted \( \mathcal{K}_\infty \). \( B_d(x) \) represents an \( n \)-ball of radius \( d \) about \( x \). The least non-zero singular value of \( A \) is denoted \( \sigma(A) \), while the largest singular value of \( A \) is \( \sigma(A) \). The Kronecker product is denoted \( \otimes \), while the operation \( \bigoplus_{i=1}^{m} x_i \) represents the Minkowski sum of sets \( x_i \) to \( x_m \).

2. Controller formulation

Consider the discrete linear time-invariant system

\[
x(k + 1) = Ax(k) + Bu(k) + B_d[k] + w(k),
\]  

(2)

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control input, \( d \in \mathbb{R}^n \) is a known periodic disturbance with period \( P \) and \( w \in \mathcal{W} \subseteq \mathbb{R}^n \) is an unknown disturbance or error.

The system outputs, which are constructed to represent system constraints, can be defined as

\[
y(k) := Cx(k) + Du(k) + D_d[k].
\]  

(3a)

\[
y(k) \in \mathcal{Y}.
\]  

(3b)

Remark 1. The outputs \( y(k) \) and corresponding constraint set \( \mathcal{Y} \) are used to capture input, state and output constraints. Input slew-rate constraints can also be considered with this formulation by appropriate augmentation of the state vector and state equations.

Assumption 2. The sets \( \mathcal{W} \) and \( \mathcal{Y} \) are compact and convex. Furthermore, \( \mathcal{W} \) contains the origin, and is contained within an \( n \)-ball of radius \( w_r \).

The following assumption, made implicitly in previous works (Zanon et al., 2013), is required to construct the MPC.

Assumption 3. The period of the cost and disturbance, \( P \), is known.

The following assumption is used to ensure that any disturbances can be eliminated in a finite number of steps and allows a feasible candidate trajectory to be constructed.

Assumption 4. There exists a set of nilpotent controller gains, \( K_j \in \mathbb{R}_{N_{nil}-1} \), which drive the system \( x(j + 1) = (A + BK_j)x(j) \), \( j \in [0, N_{nil}-1] \) to the origin in \( N_{nil} \) steps.
The aim is to generate a robust feedback control law to minimise the economic costs $f^k(x(k), u(k), p(k))$ over one period, where the economic parameter $p$ may vary over time. For a given $p$, the optimal periodic trajectory of the undisturbed system is

$$l(x^*, u^*, y^*) := \arg \min \sum_{j=0}^{P-1} f(x_j^*, u_j^*, p)$$

s.t. $x_{j+1}^* = Ax_j^* + Bu_j^* + Bd_j, \quad x_0^* = x^0$,

where $y_j^* = Cx_j^* + Du_j^* + Dd_j, \quad y_j^* \in \mathbb{Y}_j, \quad \forall j \in \mathbb{I}_0, p-1$,

which is assumed unique and its dependence on $p$ is removed for brevity.

**Remark 5.** The formulation of the MPC controller does not require $x^*$ to be found explicitly, hence (4) does not necessarily have to be solved to construct the MPC.

Convergence of the controller is guaranteed, in part, by ensuring that the periodic terminal condition can be moved strictly towards the target terminal trajectory at each time step. To ensure such a move is feasible, constraint tightening is employed not only for robustness, but also for stability. To utilise constraint tightening in this way, tubes containing the changes in periodic input and state trajectories must be found. Denote $X_c \subset \mathbb{R}^n$ and $U_c \subset \mathbb{R}^m$ as the state and input sets which contain these tubes. Given that at any phase in the trajectory, the periodic terminal condition should be moved by some minimum distance, $b$, these sets can be described as

$$(I - \sigma X)^t \subseteq \mathbb{H} U_c, \quad X_0 \subseteq X_c$$

which represents the product set $X_0 \times X_1 \times X_2 \times \ldots \times (P \text{ terms})$, with an equivalent definition for $U_c$, while $\sigma$ and $\mathbb{H}$ are defined as

$$\sigma := \begin{bmatrix} 0 & A \end{bmatrix}, \quad \mathbb{H} := \begin{bmatrix} I_{p-1} \otimes A & 0 \end{bmatrix}.$$}

where the $P$-periodic system lifted into a higher dimensional time-invariant system.

**Assumption 6.** Let the diameter of $X_c$ and $U_c$ be denoted by $b_u$, $b_u$ respectively. Then $b_u + b_u \leq \bar{\sigma}(b)$ where $\bar{\sigma} \in \mathcal{K}$.

In the following, a simple method is described to find approximations $\mathcal{X}_c$ and $\mathcal{U}_c$ which satisfy Assumption 6 and (5), assuming that $(I - \sigma X)$ is invertible. First, some control authority is reserved, defined as

$$\mathcal{U}_c := \mathcal{B}_b(0),$$

where $b_b > 0$ is chosen by the practitioner. The corresponding state trajectories are guaranteed to lie within the set $X_c$, described by

$$X_c := \bigoplus_{i=1}^{\mathcal{M}_t} \mathcal{U}_c$$

where $\mathcal{U}_c \in \mathbb{R}^{n \times m}$ are defined as

$$\mathcal{U}_c := \begin{bmatrix} \mathcal{U}_{c1} & \cdots & \mathcal{U}_{cp} \\ \vdots & \ddots & \vdots \\ \mathcal{U}_{cp1} & \cdots & \mathcal{U}_{cpP} \end{bmatrix} := (I - \sigma X)^{-1} \mathbb{H}.$$

The reserved control authority is then guaranteed to allow the terminal trajectory to be moved by

$$b = \min_{i \in \mathcal{I}_p} \sigma(\mathcal{U}_c_i)b_u.$$ Alternatively, if there exists $n$ non-zero singular values for each $\mathcal{M}_t$, $i \in \mathcal{I}_p$, then the less conservative bound $b = \sum_{i=1}^{\mathcal{M}_t} \sigma(\mathcal{M}_t_i)b_u$ can be used. For the purposes of constraint tightening, it also known that $X_c \subseteq \mathcal{B}_b(0)$ where $\phi = \sum_{i=1}^{\mathcal{M}_t} \sigma(\mathcal{M}_t_i)$.

**Remark 7.** Since the set $\mathcal{U}_c$ is constant over the period, only a single row of the matrix in (8) is required. In fact, since (8) is Toeplitz, any row could be used. If $P$ distinct amounts of control authority were reserved, i.e. $P$ distinct $\mathcal{U}_c$, then there would be $P$ distinct corresponding $X_c$, and the whole matrix (8) would be required.

**Remark 8.** The method of finding $X_c$ and $\mathcal{U}_c$ presented here may be conservative. However, finding $X_c$ and $\mathcal{U}_c$ which satisfy Assumption 6 and (5) with more complex geometries may be computationally challenging.

The constraint tightening procedure can be described as

$$y_0 = y, \quad \text{min}_{\mathcal{M}_c} \sum_{j=0}^{P-1} f(\tilde{x}_j, \tilde{u}_j, p) \text{ s.t. } \forall j \in \mathcal{I}_0, p-1$$

alternatively, if there exists $N$ non-zero singular values for each $\mathcal{M}_t$, $i \in \mathcal{I}_p$, then the less conservative bound $b = \sum_{i=1}^{\mathcal{M}_t} \sigma(\mathcal{M}_t_i)b_u$ can be used. For the purposes of constraint tightening, it also known that $X_c \subseteq \mathcal{B}_b(0)$ where $\phi = \sum_{i=1}^{\mathcal{M}_t} \sigma(\mathcal{M}_t_i)$.

**Remark 9.** Since the set $\mathcal{U}_c$ is constant over the period, only a single row of the matrix in (8) is required. In fact, since (8) is Toeplitz, any row could be used. If $P$ distinct amounts of control authority were reserved, i.e. $P$ distinct $\mathcal{U}_c$, then there would be $P$ distinct corresponding $X_c$, and the whole matrix (8) would be required.

**Remark 10.** Following from Remark 5 and as in Müller et al. (2013), the offset functions may take the form $V_j(x, u) = \beta f(x, u, p) \forall i \in \mathcal{I}_0, p-1$, if $\beta \in \mathbb{R}$ is found such that the conditions (12) are satisfied.

Use of constraint tightening requires the following assumption to ensure a feasible solution exists.

**Assumption 11.** The constraint set $\mathcal{Y}_c$ is sufficiently large with respect to the disturbance set $\mathcal{W}$, tightening sets $\mathcal{X}_c$, $\mathcal{U}_c$ and gains $K$, such that a solution to (11) exists.
Remark 12. Since \( \mathcal{Y} \) is convex, \( 0 \in \mathcal{W} \) and \( 0 \in \mathcal{U} \), then \( \mathcal{Y}_j \forall j \in \mathbb{I}_{0,N-1} \) are also convex.

The economic cost function for the controller is

\[
V^{[k]}(x^k, u^k, p) := \sum_{j=0}^{N-p-1} \ell^{[k+j]}(x^j_k, u^j_k, p) + \sum_{j=N-p}^{N-1} \ell^{[k+j]}(x^j_k, u^j_k) \tag{13}
\]

and at time \( k \) and state \( x(k) \) the MPC problem is

\[
\mathcal{P}(x(k), p) : V^{[k]}_0(x(k), p) := \min_u V^{[k]}(x^k, u^k, p)
\]

\[
\text{s.t.} \ x_0^k = x(k), \quad x_{N-1}^k = x_{N-P}^k,
\]

\[
y^k_j = Cx^k_j + Du^k_j + B_d d_{[k+j]}, \quad j \in \mathbb{I}_j,
\]

\[
y^k_j \in \mathcal{Y}_j, \quad \forall j \in \mathbb{I}_{0,N-1},
\]

where the MPC control law defined as \( \kappa(x(k)) = u_0^k \).

3. Stability analysis

In Section 3.1, recursive feasibility of the controller is demonstrated by proposing a candidate solution, shown to be a feasible successor to any feasible optimisation. In Section 3.2, a rotated cost function and auxiliary optimisation problem is introduced, which gives the same solution as the original problem, and for which the value function represents a candidate Lyapunov function. Using this Lyapunov function, it is shown that each phase of the periodic trajectory is input-to-state stable, which is used in turn to show ultimate boundedness of the closed loop system about the optimal trajectory. Before conducting the stability analysis, additional assumptions on the nature of the system and cost function are stated. As in [40], the following assumption ensures a suitable Lyapunov function can be found. By lifting the periodic system into a higher dimensional time-invariant system, this assumption can also be shown to satisfy the strong duality assumption in [41].

Assumption 13 (Strong Duality). There exists a set of multipliers \( \lambda_j \) such that \( (x^*, u^*) \) uniquely solves \( \min_{x^0, u^0} \sum_{i=0}^{N-1} \ell(x^i, u^i, p) \) s.t. \( Cx^i + Du^i + B_d d_{i} \in \mathcal{Y} \), where \( \bar{\mathcal{Y}} \) denotes the rotated stage cost

\[
\bar{\ell}(x, u, p) = \bar{\ell}(x, u, p) - \ell(x^i_k, u^i_k, p) + \lambda^1_j(x - x^i_k)
\]

\[
-\ell_{[i+1]}(A(x - x^i_k) + B(u - u^i_k) + B_d d_i).
\]

Moreover, there exists \( \alpha \in \mathcal{K} \) such that \( \bar{\ell}(x, u, p) \geq \alpha(\mathcal{Y} - x^i_k) \), \( \forall i \in \mathbb{I}_{0,P-1}, \forall (x, u) \) satisfying \( Cx + Du + B_d d_i \in \mathcal{Y} \).

Remark 14. Linear systems with strictly convex costs and constraints satisfy Assumption 13 under the mild assumption that they also satisfy the Slater condition ([42]).

Assumption 15. The rotated cost \( \bar{\ell}(x, u, p) \) is Lipschitz continuous in \( \mathcal{Y} \) with constant \( M \) such that \( \bar{\ell}(x, u, p) - \bar{\ell}(x_1, u_1, p) \leq M(|x_1 - x_2| + |u_1 - u_2|) \).

3.1. Recursive feasibility

Theorem 16 (Feasibility), Consider a system described by (2), subject to constraints (3) and let Assumptions 2–11 hold. Under the control law given by \( \mathcal{P}(x, p) \) and given that \( x(k) \in \mathcal{X}_N \), where \( \mathcal{X}_N \) is the feasible region of \( \mathcal{P}(x, p) \), then \( x(j) \in \mathcal{X}_N, \forall j \geq k \).

Proof. At time \( k \), consider any \( x(k) \in \mathcal{X}_N \). Define a feasible solution to \( \mathcal{P}(x(k), p) \) as \( u^k \), with corresponding states \( x^k \) and outputs \( y^k \). Consider the candidate solution depicted in Fig. 1. The candidate solution is comprised of the shifted solution at time \( k \), with perturbations added for disturbance rejection and perturbations added to move the shift of the periodic terminal condition closer to the target terminal condition by the distance \( b \). This move can be seen as a shift of the origin and a corresponding disturbance to be rejected in the opposing direction, beginning from position \( N = N - N_{mil} - P \). Describe the candidate solution as

\[
\begin{align*}
u^{k+1}_j & := \begin{cases} u^k_j + K_L l w_k & j \in \mathbb{I}_{0,N-1} \quad (16) \\
u^k_{N-P} + \Delta u^{h+1}_{[0-N-1]} & j = N - 1, \end{cases} \\
y^{k+1}_j & := \begin{cases} y^k_j + (C + DK) I u w_k & j \in \mathbb{I}_{0,N-1} \quad (17) \\
y^k_{N-P} + \Delta y^{h+1}_{[0-N-1]} & j = N - 1, \end{cases}
\end{align*}
\]

where

\[
\begin{align*}
\Delta u^h & := h^{h+1}_0 (u^h_{[N-P+1]} - u^h_{[N-P+1]}), \quad \forall i \in \mathbb{I}_{0,P-1} \quad (19) \\
\Delta y^h & := h^{h+1}_0 (x^h_{[N-P+1]} - x^h_{[N-P+1]}), \quad \forall i \in \mathbb{I}_{0,P-1} \quad (20) \\
\Delta y^h & := h^{h+1}_0 (x^h_{[N-P+1]} - x^h_{[N-P+1]}), \quad \forall i \in \mathbb{I}_{0,P-1} \quad (21) \\
\Delta y^h & := \min(h^h_0 (x^h_{[N-P+1]} - x^h_{[N-P+1]}), (22)
\end{align*}
\]

Recursive feasibility is now demonstrated by showing the candidate solution at time \( k + 1 \) satisfies all constraints. The real state at \( k + 1 \) is given by the system dynamics (2). Substitution into the internal model (14b) gives \( x^h_{[0]} = x^h_0 + u^h_0 \), and since \( L_0 = I \), the initial condition constraint (14a) is satisfied. Substitution of the initial condition and candidate solution (18) into the internal model (14b) and output equation (14c) gives (17) and (18), satisfying constraints (14b)–(14c) by construction. Substitution of the candidate state trajectory into the periodicity constraint of (14a) gives \( x^h_{[N-P+1]} + \Delta x^h_{[0-N-1]} = A x^h_{[0-N-1]} + B u^h_{[0-N-1]} + B_d d_{[0]} \).

Due to periodicity of the terminal condition at time \( k \), \( x^h_{[N-P]} = x^h_{[N]} \), and \( \Delta x^h_{[0-N-1]} + \Delta u^h_{[0-N-1]} \) through superposition, hence the equality constraint (14a) is satisfied. Due to construction, \( \Delta x^h_{[0-N-1]} \) in \( \mathcal{X}_N \), \( \forall i \in \mathbb{I}_{0,P-1} \) and \( \Delta u^h_{[0-N-1]} \in \mathcal{U} \), \( \forall i \in \mathbb{I}_{0,P-1} \). As a result, substitution of the candidate outputs (18) and the constraint tightening policy (10) into the Pontryagin difference property (1) reveals that if \( y^h \in \mathcal{Y} \), then \( y^h \in \mathcal{Y}_N \) satisfying the output inequality constraint (14d) up to \( k + N - P \). Feasibility at time \( k \) ensures that \( y^h \in \mathcal{Y}_N \) is satisfied, \( \forall i \in \mathbb{I}_{0,P-1} \). By definition, then due to convexity of \( y^h_{[0]} \), \( y^h_0 \) is strictly positive, ensuring the candidate solution satisfies the output inequality constraints. Since the candidate solution is a feasible solution for \( \mathcal{P}(x, p) \), a feasible solution exists for any \( x \in \mathcal{X}_N \).

Remark 17. In contrast to the control formulations in [42] and [40], no constraints depend on \( x^* \), which is dependent on \( p \). This allows arbitrary variation of the economic
Theorem 18. Consider a system described by \( x(k+1) = f(x(k), d, k) \), where \( d > 0 \). Assume there exists a continuous function \( V : \mathcal{X}_N \to \mathbb{R} \) as well as \( a, b \in \mathcal{K} \) and \( a_0, \omega_2 \in \mathcal{K} \) such that
\[
\alpha((x(k) - x^*)) \leq V(x(k)) \leq b(|x(k) - x^*)| + \omega_2(d), \quad x(k) \in \mathcal{X}_N,
\]
(23)
\[
V(x(k+1)) - V(x(k)) \leq -\alpha((x(k) - x^*)) + \omega_2(d), \quad x(k) \in \mathcal{X}_N,
\]
(24)
and \( \mathcal{B}_{a-1}(0)(x^*) \subseteq \mathcal{X}_N \) where \( \eta > \omega_1(d) + \omega_2(d) + \beta(\alpha^{-1}(\omega_2(d))). \)
Then there exists \( k \) such that for all \( k > k^* \), \( x^k \in \mathcal{B}_{a-1}(0)(x^*) \) thereafter.

**Proof.** Define \( \mu = \alpha^{-1}(\omega_2(d)) \), then from (24), \( |x^k - x^*| > \mu \) implies \( V(x^{k+1}) - V(x^k) < 0 \). Ultimate boundedness of \( x^k \) in \( \mathcal{B}_{a-1}(0)(x^*) \) is demonstrated in two stages. The first system is shown to enter the set \( \mathcal{B}_{a-1}(0)(x^*) \) and then the state is shown to be bounded within \( \mathcal{B}_{a-1}(0)(x^*) \).

Admission into set \( \mathcal{B}_{a-1}(0)(x^*) \) is shown by contradiction. Suppose \( |x^k - x^*| > \theta^{-1}(\eta) \), \( k > 0 \). It follows from (24) that \( \theta = \min_{\eta, \theta} \theta^{-1}(\eta) > 0 \). Hence from (23) that \( |x^k - x^*| > \mu \). If \( |x^k - x^*| > \mu \), \( k > j \), then it follows from (23) that \( |x^k - x^*| \leq |x^j - x^*| + \alpha^{-1}(\mu + \omega_2(d)) \leq \alpha^{-1}(\eta) \). If there exists \( k > j \) such that \( |x^k - x^*| > \mu \), then since \( |x^k - x^*| \leq \mu \), it follows that \( k \leq k^* \) such that \( |x^{k^*} - x^*| \leq \mu \) and \( |x^k - x^*| > \mu \), \( \forall k \leq k^* \). Hence it follows from (23) and (24) that
\[
\alpha((x^k - x^*)) \leq V(x^k) \leq V(x^*) \leq V(x^{k^*}) + \omega_2(d) \leq \eta
\]
which implies that \( x^k \in \mathcal{B}_{a-1}(0)(x^*) \). Next consider \( \mu < |x^k - x^*| \leq \theta^{-1}(\eta) \). For every \( k > j \) such that \( \mu < |x^k - x^*| \), \( k \leq j \) it follows from (23) and (24) that
\[
\alpha((x^k - x^*)) \leq V(x^k) \leq V(x^*) \leq \eta
\]
which implies that \( |x^k - x^*| \leq \alpha^{-1}(\eta) \), \( \forall k \in j \). If there exists \( k \) such that \( |x^k - x^*| \leq \mu \), then it follows from the first case that
\[
|x^k - x^*| \leq \alpha^{-1}(\eta), \quad \forall k \leq k^*.
\]
Therefore it has been shown that, \( \mathcal{B}_{a-1}(0)(x^*) \) is an invariant set for the system.

Satisfaction of the conditions for Theorem 18 imply that the closed loop system will asymptotically converge to a ball about the state \( x^* \). If \( d \) can be made arbitrarily small, the system will converge arbitrarily close to \( x^* \).

The auxiliary optimisation problem is now introduced, demonstrated in Lemma 19 to give the same result as the optimisation problem \( \mathcal{P}(x, p) \). Define the rotated offset functions as
\[
\tilde{V}_f^k(x, u) := V_f^k(x, u) - V_f^k(x^*_0, \bar{u}_0).
\]
(25)
Using the rotated stage cost (15) and rotated offset functions (25), define the rotated cost function as
\[
\tilde{V}^k(x^k, u^k, p) = \sum_{j=0}^{N-1} \tilde{V}^{k+j}_f(x^k, u^k, p) + \sum_{j=N-1}^{N-1} \tilde{V}^{k+j}_f(x^k, u^k, p) + \frac{\lambda}{2} \left( |x^k - x^*|_p^2 - x^*_{(k+N-1)} - x^*_{(N-k+1)} \right),
\]
(26)
and auxiliary optimisation problem respectively as
\[
\tilde{\mathcal{P}}(x(k), p) := \tilde{V}_f^0(x(k), p) = \min_{u^k} \tilde{V}^k(x^k, u^k, p)
\]
s.t. (14).

**Lemma 19.** The optimisation problem \( \tilde{\mathcal{P}}(x, p) \) gives the same optimal control sequence as \( \mathcal{P}(x, p) \).

The closeness between the optimal periodic solution \( x^* \) and the tightened solution \( \tilde{x}^* \) is a function of constraint tightening. Lemma 20 shows that the closeness of the trajectories can be upper bounded by the amount of constraint tightening, represented by \( b \) and \( w_j \). This result is used to establish periodic descent conditions for the Lyapunov function in Lemma 22. These two results are used with Theorem 18 to demonstrate stability of the overall closed loop system in Theorem 23.

**Lemma 20.** There exists \( v \in \mathcal{K} \) such that \( \sum_{i=0}^{N-1} |x^*_i - x^*_i| \leq v(b + w_j) \).

A number of definitions are now stated, used in the following stability proof. Define an upper bound on the economic cost for any periodic condition, \( \chi := \max_{0 \leq i \leq p} \tilde{P}(x_0, u_0, p) \) s.t. (11b)–(11c), \( k \in \mathbb{N} \), on the cost associated with moving the periodic terminal condition or disturbance rejection, 
\[
\sigma := \sum_{j=0}^{N-1} (|L_j| + |K_j|),
\]
(27)
and define the largest multiplier \( \lambda_m := \max_{0 \leq i \leq p-1} |L_i| \). Define the following constant
\[
\gamma_0 := \frac{\chi}{b} + \lambda_m.
\]
(28)

**Assumption 21.** The weighting of the offset function is such that (12) holds with \( \alpha \geq \gamma_0 \).

**Lemma 22.** Consider a system described by (2), with constraints (3) and let Assumptions 3–21 hold. At time \( k \), given \( x(k) \in \mathcal{X}_N \) there exists \( \alpha \in \mathcal{K}_\infty \), and \( \omega_2 \in \mathcal{K} \) such that under the control law given by \( \mathcal{P}(x, p) \),
\[
\tilde{V}_f^k(x(k) + P, p) - \tilde{V}_f^k(x(k), p) \leq -\alpha((x(k) - x^*_k)) + \omega_2(b + w_j).
\]
(29)

The main result of the paper can now be stated. Theorem 23 uses the result of Theorem 18 to show that each of the \( P \) phases of the
closed loop system, corresponding to the P phases of the cost function or disturbance signal are each input-to-state stable. Following this, stability of the overall closed loop system can be concluded.

**Theorem 23 (Asymptotic Stability).** Consider a system described by (2), subject to constraints (3). Let Assumptions 3–21 hold. There exists \( \eta \in \mathcal{K} \) such that under the control law given by \( \mathcal{T}(x, p) \) and given \( x \in \mathcal{X}_N \), the closed loop system is stable about the tube given by \( \mathcal{X}_N^{B} = \{ x : |x - x^*_i| \leq \eta(b + w_i), \forall i \} \)

**Proof.** Given any state \( x(k) \in \mathcal{X}_N \) at time \( k \), let \( (\hat{x}^k, \hat{u}^k) \) be the optimal state and input trajectories and cost corresponding to \( \mathcal{T}(x(k), p) \). Due to Assumption 13 and (12) we have, \( \mathcal{V}_0(x(k), p) \) \( \geq \alpha(|x(k) - x^*_i|) + \lambda^T_{i}|x^*_i - x_{i+N} + p_{i+N} + y| |x^*_i - x_{i+N} + p_{i+N} + y| \), then since \( y > |\lambda_m| \), \( \alpha(|x(k) - x^*_i|) \leq \mathcal{V}_0^0(x(k), p), \forall k \). (30)

At time \( k \), define the following feasible solution to \( \mathcal{T}(x(k), p), \hat{x}^k = [\hat{x}^*_i, \ldots, \hat{x}^*_{i+N}] \) and \( \hat{u}^k = [\hat{u}^*_i, \ldots, \hat{u}^*_{i+N}] \). From optimality \( \mathcal{V}_0^0(x(k), p) \leq \mathcal{V}_0^s(\hat{x}^k, \hat{u}^k, p) \) and using arguments similar to Lemma 20 with Assumption 15 it can be shown that \( \mathcal{V}_0^s(\hat{x}^k, \hat{u}^k, p) \leq \omega(b + w_i) \), \( \forall i \in \mathcal{K} \). Then due to the controllability Assumption 4 there exist \( \beta \in \mathcal{K}_\infty \) such that \( \mathcal{V}_0^s(x(k), p) \leq \beta(|x(k) - x^*_i|) + \omega(b + w_i) \). Using the triangle inequality, and Lemma 20 can then be shown that \( \mathcal{V}_0^0(x(k), p) \leq \beta(|x(k) - x^*_i|) + \omega(b + w_i), \forall k \). (31)

Consider P systems, each representing one phase of the system (2) under the control law given by \( \mathcal{T}(x, p) \) over \( P \) successive steps. For each \( i \in \mathcal{I}_{0, P-1} \), these systems can be denoted as \( x((k + 1)P + i) = f^i(x(kP + i), b + w_i, kP + i) \), (32)

where the time dependence is due to the time-varying disturbance signal. Each of these \( P \) systems satisfies the conditions of Theorem 18 using (30), (31), Lemma 22 and the result of Theorem 16. Since \( \omega_i, \alpha, \beta \in \mathcal{K} \) there exists \( \eta \in \mathcal{K} \) such that \( \alpha(|x^*(i)| + \omega_i^2) + \beta(\alpha^{-1}(\omega_i)) \). Each phase of the closed loop system will converge to \( x^*(i) \), \( \forall i \in \mathcal{I}_{0, P-1} \) as \( k \to \infty \) and hence the closed loop system converges to the tube \( \mathcal{X}_N^{B} = \{ x : |x - x^*_i| \leq \eta(b + w_i), \forall i \} \), \( \forall k \). The result extends to \( \mathcal{T}(x, p) \) due to Lemma 19.

**Remark 24.** This result is similar to the periodic Lyapunov stability result in Böhm, Lührn, and Allgöwer (2010), however without requiring its Assumption 8, which bounds one phase’s convergence to the previous such that when one phase has converged, all phases are converged. As a result, Theorem 23 shows that each phase of the system is independently stable, however cannot guarantee that one phase’s entrance into its robust invariant set implies the others.

**Remark 25.** The constraint tightening sets \( \mathcal{X}_i \) and \( \mathcal{U}_i \) can be made arbitrarily small, with \( \eta_0 \) becoming large. In the case where \( \mathcal{W} = \{0\} \), this allows the system to converge arbitrarily close to \( x^* \).

**Remark 26.** The developed controller and associated proof can be applied to the steady state problem as a special case where \( P = 1 \).

4. Numerical example

In this example, a point mass model in a single axis of motion is considered, as shown in Fig. 2. Using unity values for mass \( m \), spring constant \( k \) and damping coefficient \( c \), an exact discretisation with a time-step of 0.1 with first order holds for the input and disturbance and including an additive disturbance yields the system description (2) where

\[
A := \begin{bmatrix} 0.9952 & 0.0950 \\ -0.0950 & 0.9092 \end{bmatrix}, \quad B := \begin{bmatrix} 0.0048 \\ 0.0950 \end{bmatrix}, \quad B_d := B,
\]

and \( x = [z \dot{z}]^T \). The known disturbance may represent, for example, the impact of oceanic waves on a system as in Broomhead et al. (2013). Described by \( \dot{z} = 5 \cos(2 \pi t) \), the disturbance has period \( P = 10 \), satisfying Assumption 3. The unknown disturbance represents measurement and modelling error, and is randomly drawn from \( \mathcal{W} \) with uniform distribution where \( \mathcal{W} = \{ w | -0.001 \leq w \leq [0.001 \ 0.011] \} \). The constraints are given by \( -8 \leq u \leq 8 \) and \( [-1.5 \ -0.75]^T \leq x \leq [1.5 \ 0.75]^T \), satisfying Assumption 2, and the nilpotent control law is calculated using the LQR method described in Richards (2005) using \( N_{0,2} = 2 \) and identity weighting matrices, satisfying Assumption 4 within numerical tolerances. Satisfaction of Assumption 11 is verified by the existence of any feasible solution in the example. The control authority set \( \mathcal{U} \) is chosen to have a radius of \( b_u = 0.05 \). For the purposes of constraint tightening, the set \( \mathcal{X}_i \) is conservatively approximated by \( \mathcal{X}_i = \mathcal{B}_{b_0}(0) \) where \( \phi = \sum_{i=1}^{n} \sigma_i(\mathcal{X}_i) \), resulting in \( \mathcal{X}_i = \mathcal{B}_{b_0}(0) \) in this example. Choosing a smaller \( b_u \) will result in a smaller discrepancy between \( x^* \) and \( \hat{x}^* \), as seen in Fig. 3, however this value is kept to highlight its impact.

The economic cost is \( f(x, u) = 100(\dot{z} - \sin(2\pi t))^2 + (sz^2)^2 \), \( i \in \mathcal{I}_{0, P-1} \), chosen for its non-steady state optimal periodic solution and which satisfies Assumptions 13–15. The prediction horizon length is chosen as \( N = 15 \) and the offset function is chosen as \( \mathcal{V}_i(x, u) = \gamma|x - x^*_i| \), \( \gamma = 46 \cdot 10^{-4} \), sufficiently large to satisfy the Assumption 21.

Since all assumptions are satisfied, application of Theorem 16 guarantees the existence of successive solutions at all times, while application of Theorem 23 guarantees closed-loop stability of the system. In Fig. 3 it can be seen that the closed loop trajectories converge close to the optimal solution as expected, and will be bound within a tube surrounding this trajectory. Fig. 4 shows the economic cost over one period for the trajectories where it can be seen that the closed loop trajectories perform better than if the candidate trajectory was used recursively (see \( x^* \)) and instead approach the optimal periodic cost of \( x^* \). Fig. 5 shows the Lyapunov function over time, which is periodic in nature, however decreases periodically until close to zero.
Fig. 4. Economic cost over one period for $x^*$ (dashed), $\tilde{x}^*$ (dot-dashed), and closed-loop trajectories (solid black) for three initial conditions.

Fig. 5. Lyapunov function value for closed-loop trajectories of three initial conditions. Note the periodicity of the function.

5. Conclusions and outlook

In this work, a robust constraint tightening EMPC algorithm has been proposed for systems with unknown and periodic disturbances and periodic economic cost functions. The controller has been shown to be recursively feasible despite unknown disturbances and potential changes in economic parameters. The formulation removes the need for a terminal point constraint, replacing it with a terminal periodic condition, without the need to modify the stage cost or impose constraining constraints. Convergence of the closed loop system to a tube about the optimal closed loop trajectory is guaranteed, which can be made arbitrarily small for the undisturbed case.

Future work may include extensions for non-linear prediction models, to relax the strong duality assumption to a dissipative assumption, to generalise the terminal weight to be a time-varying terminal weight as in Müller et al. (2013) and to utilise the analysis as in Grüne (2013) and Müller and Allgöwer (2012) to further characterise closed loop performance.

Appendix

Proof of Lemma 19

Since optimisation problems $\tilde{J}_N(x, p)$ and $J_N(x, p)$ utilise the same constraints, it is sufficient to show that the optimal solution to both cost functions is the same. Expanding (26) and substituting in (14b) yields

$$\tilde{J}^{[k]}(x^k, u^k, p) = \sum_{j=0}^{N-p} [1^{[k+j]}(x^k_{[k+j]}, u^k_{[k+j]}, p) - \sum_{j=0}^{N-p} \bar{V}_j(x^k_{[k+j]}, \tilde{u}^k_{[k+j]}) + \sum_{j=0}^{N-p} \bar{V}_j(x^k_{[k+j]} - x^*_{[k+j]}, \tilde{u}^k_{[k+j]})$$

showing that the cost functions differ only by a constant term. As a result, the optimisation problems $\tilde{J}(x, p)$ and $J(x, p)$ give the same optimal solution.

Proof of Lemma 20

Let $\Phi = \sum_{i=0}^{P-1} [x_i^2 - x_i^*]$. Under Assumption 6, when $b + w_r = 0$, $y_{N-1} = y$. In this case, $x^*$ is a uniquely minimising solution to (11) and hence $\Phi = 0$. Further, $\Phi$ is continuous about $b + w_r = 0$ due to convexity of constraints and Assumption 15. Following from this, there exists a $v \in K$ such that $\Phi \leq v(b + w_r)$.

Proof of Lemma 22

Let $(x^k, u^k)$ represent any feasible solution to $\tilde{J}(x, p)$ at time $k$ and successive trajectories $(x^{k+1}, u^{k+1}), \forall j \in I_{1:N-1}$ be determined by the candidate trajectory (16)–(18). Comparing successive costs, noting that $|x_{k+1}^* - x_{k+1}^*| - |x_{k+1}^* - x_{k+1}| = h_{k+1}|x_{k+1} - x_{k+1}^*|$, $\forall j \in I_{N-1}$ and utilising (12) results in

$$\tilde{J}^{[k+1]}(x^{k+1}, u^{k+1}, p) - \tilde{J}^{[k]}(x^k, u^k, p) \leq -\tilde{L}^{[k]}(x^k, u^k, p) + \tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p) + \sigma w_r$$

$$\leq -\tilde{L}^{[k]}(x^k, u^k, p) + \tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p) + \sigma w_r$$

where $\sigma$ is from (27). From Assumption 6, it can be shown that $|x_{k+1}^*| \leq \sigma(b), \forall j \in I_{N-1}$. Then there exists $\Phi \in K$ such that

$$\tilde{J}^{[k+1]}(x^{k+1}, u^{k+1}, p) - \tilde{J}^{[k]}(x^k, u^k, p) \leq -\tilde{L}^{[k]}(x^k, u^k, p) + \tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p)$$

$$\leq -\tilde{L}^{[k]}(x^k, u^k, p) + \tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p) + \sigma w_r$$

$$\leq -\tilde{L}^{[k]}(x^k, u^k, p) + \tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p)$$

Due to the conditional statement (22), two cases must be considered. First consider $|x_{k+1}^*| = b$, then $h_{k+1} \sum_{j=0}^{N-p} [x_{k+j} - x_{k+j}^*]^2 \geq h_{k+1}$ and

$$\tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p) \leq \tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p) \leq \chi.$$ 

In the second case, consider $|x_{k+1}^*| < b$, then the terminal condition is at $x^* = x^*$ and hence

$$\tilde{L}^{[k+1]}(x^{k+1}, u^{k+1}, p) \leq M |x_{k+N-1} - x_{k+N-1}^*|.$$ 

In either case, using Assumption 13 and that $\gamma \geq \frac{\lambda}{b}$, (34) becomes

$$\tilde{J}^{[k+1]}(x^{k+1}, u^{k+1}, p) - \tilde{J}^{[k]}(x^k, u^k, p) \leq -\sigma |x_{k} - x_{k}^*| + M |x_{k+N-1} - x_{k+N-1}^*|.$$
\[ -\lambda_0 h^{k+1} + \sum_{j=0}^{1} \left| x_j - \bar{x}_{[k+j]} \right| + \psi(b + w_i) \\
= \sum_{j=0}^{1} \left( |x_j - \bar{x}_{[k+j]}| + \psi(b + w_i) \right) \]

Summing (35) over \( P \) successive steps and noting \( \sum_{i=0}^{P-1} h^{k+1+i} \),
\[ \bar{v}_{i(k+p)}(x^{k+p}, u^{k+p}, p) - \bar{v}_{i(k)}(x^k, u^k, p) \leq -\alpha(x_k^i - x_{[k]}^i) + M \sum_{i=0}^{P-1} |x_i - x_{[k]}^i| + P\psi(b + w_i) \]
\[ + \sum_{i=0}^{P-1} h^{k+1+i} (x_{k+1+i} - x_{[k+1+i]}^i) \]

The Cauchy–Schwarz inequality and Lemma 20 then yield
\[ \bar{v}_{i(k+p)}(x^{k+p}, u^{k+p}, p) - \bar{v}_{i(k)}(x^k, u^k, p) \leq -\alpha(x_k^i - x_{[k]}^i) + \psi(b + w_i) \]

where \( \psi(b + w_i) \).

References


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