Model reduction and MIMO model predictive control of gas turbine systems

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Abstract

This paper presents the development and experimental implementation of an online, fully nonlinear model predictive controller (NMPC) for a gas turbine. The reduced-order, internal model used in the controller was developed from an original high-order, physics-based model using rigorous time scale separation arguments that may be extended to any gas turbine system. A control problem for a prototype gas turbine system is then formulated as a MIMO optimisation problem that may be addressed through NMPC. The controller objective is to regulate compressor exit pressure while tracking compressor bleed demand. The constraint set includes the range of the control actuators, the valid range of the internal model, and safety limits such as the compressor surge margin and maximum turbine inlet temperature. This formulation also includes derivation of the conditions sufficient to prove nominal asymptotic stability of the closed-loop system.

Controller performance is then evaluated through representative tracking and regulation experiments. This highlights the advantages of the proposed controller’s constraint handling and disturbance rejection. More broadly, these experiments demonstrate that a nonlinear model predictive controller can be successfully deployed on a gas turbine, in real time, without the need for any linearisation.

1. Introduction

Gas turbine systems typically feature high power density, low vibration and high reliability. However, the transient response of gas turbines is a common issue, as are practical limitations on sensor placement, system nonlinearity and operating constraints, the latter of which include those related to component temperatures, exhaust emissions and compressor surge and/or stall. These limitations potentially make robust application of classical control theory difficult.

An example of a gas turbine system that exhibits all of these positive features and limitations is presented in this work. A prototype gas turbine air compressor (GTAC) (Wiese, Blom, Brear, Manzie, & Kitchener, 2013a) based on a single stage microturbine is shown schematically in Fig. 1, where the available measurements are denoted by ‘○’ labels, and the two actuators are the fuel valve position governing $m_f$, and the position of the bleed valve, BV. Delivery air is bled from the cycle between the compressor and combustion chamber, and so the GTAC can be considered equivalent to a conventional gas turbine that delivers shaft work to a second air compressor of the same pressure ratio. The objective of the GTAC prototype is therefore conventional – to deliver compressed air at a regulated pressure whilst tracking its time-varying demand.

In order to meet this objective, a control system must be designed which also adheres to the constraints necessary to ensure the safe operation of a gas turbine, including temperature and compressor surge limits. While significant research effort has been directed towards alternative control strategies that solve one or more of the problems arising from nonlinearity and sensor availability e.g. Kapasouris, Athans, and Austin Spang III (1985), Garg (1993), and Tavakoli, Griffin, and Fleming (2005), none are able to address them all. Most commonly, constraint handling is added after the controller has been designed. This may lead to conservative controller design, and potentially degrades the stability and/or performance of the designed controller.

Model predictive control (MPC) is an excellent candidate for gas turbine control, due to its inherent capacity for control of constrained nonlinear systems. MPC has been successfully applied in a variety of industrial settings, e.g. Tanaskovic, Minnetian, Fagiano, and Morari (2014), and particularly for energy conversion applications e.g. Di Cairano, Yanakiev, Bemporad, Kolmanovsky, and Hrovat (2012), Venkat, Hiskins, Rawlings, and Wright (2008), Vahidi, Stefanopoulou, and Peng (2006), Wahlström and Eriksson (2013b).
Several authors have proposed MPC approaches for gas turbines, but due to the computational complexity, each has involved some degree of linearisation. The two most common approaches are to either use a linear representation of the plant dynamics, with nonlinear functions (Brunell, Bitmead, & Connolly, 2002; Kim, Powell, & Edgar, 2013), or maps (Diwanji, Godbole, & Waghode, 2006; Jurado, 2006; Jurado & Carpio, 2006), to calculate the engine outputs, or to implement successive linearisation. The accuracy of the first approach is subject to the linearity of the engine, and the proximity to the setpoint about which the dynamics were linearised. In the case of Brunell et al. (2002), the simplified real-time model introduced transient errors of up to 22% when compared with their component level model. Successive linearisation potentially improves accuracy by generating a new linearised model at each sampling instant (Brunell, Viassolo, & Prasanth, 2004; Mu, Rees, & Liu, 2005; van Essen & de Lange, 2001; Vroemen, van Essen, van Steenhoven, & Kok, 1999). This approach was demonstrated on a laboratory gas turbine in van Essen and de Lange (2001) for slow variations in commanded shaft speed, (i.e. sinusoidal transients with periods 150 s). The successive linearisation method has also been combined with multiplexed MPC for a jet engine in Richter, Singaraju, and Litt (2008), where the computation time of an MPC using a higher-order model was reduced by only considering a subset of the available actuators at each sampling instant. This improvement in computation time was achieved at a small cost in thrust and temperature tracking, and fuel consumption when compared with an equivalent controller without multiplexing. While the successive linearisation approach may improve on a uniformly linearised model, constraint adherence and closed-loop performance is dependent on the non-linearity of the plant, and the model update rate. Furthermore, while Richter et al. (2008) include some preliminary linear stability results, it appears that the gas turbine MPC studies to date involved transient errors of up to 22% when compared with their component level model. Successive linearisation potentially improves accuracy by generating a new linearised model at each sampling instant (Brunell, Viassolo, & Prasanth, 2004; Mu, Rees, & Liu, 2005; van Essen & de Lange, 2001; Vroemen, van Essen, van Steenhoven, & Kok, 1999). This approach was demonstrated on a laboratory gas turbine in van Essen and de Lange (2001) for slow variations in commanded shaft speed, (i.e. sinusoidal transients with periods 150 s). The successive linearisation method has also been combined with multiplexed MPC for a jet engine in Richter, Singaraju, and Litt (2008), where the computation time of an MPC using a higher-order model was reduced by only considering a subset of the available actuators at each sampling instant. This improvement in computation time was achieved at a small cost in thrust and temperature tracking, and fuel consumption when compared with an equivalent controller without multiplexing. While the successive linearisation approach may improve on a uniformly linearised model, constraint adherence and closed-loop performance is dependent on the non-linearity of the plant, and the model update rate. Furthermore, while Richter et al. (2008) include some preliminary linear stability results, it appears that the gas turbine MPC studies to date have not formally considered the stability of the controllers involved. Implementing a fully nonlinear MPC may be able to preserve the feasibility of the controller, and facilitate a formal stability analysis.
model. Component-level dynamic models of gas turbines are commonly constructed using an inter-component volume approach e.g. Camporeale, Fortunato, and Mastrovito (2006), or a finite difference/volume approximation of the conservation equations e.g. Badmus, Eveker, and Nett (1995) and Wiese, Blom, Brear, Manzie, and Kitchener (2013b). The combination of physical insight and validity over wide operating ranges makes these 1-D nonlinear models appealing for use in gas turbine control design and testing. However, these approaches typically lead to high-order models that also require small time steps to obtain a numeric solution. To overcome this, earlier control work utilised linearised models obtained by Taylor series expansion of component level models about discrete operating points. Model order was then reduced using singular value decomposition (Kapasouris et al., 1985), balanced realisation (Garg, 1993), or more commonly, modal decomposition (DeHoff & Hall, 1976; Kapasouris et al., 1985; Pfeil, 1984). In the latter case, the bandwidth of the identified modes are compared with actuator bandwidths, and only those states whose corresponding modes are within the actuator bandwidth are retained. Much of the model reduction work in the open literature concerns larger gas turbine engines, and Davison and Birk (2006) note that the conclusions drawn from these large engine studies may not extend to microturbines with fast transients, the latter being of particular concern for constraint handling if the model is part of an MPC. These findings are reflected in the control-oriented models for smaller scale turbines, e.g. Traverso, Calzolari, and Massardo (2005) and Ailer, Sánta, Szederkényi, and Hangos (2001), which retain some gas dynamic phenomena. Consequently, any reduction of a microturbine model will need to both assess the relative importance of the various dynamics, and to quantify the likely error resulting from model reduction.

This paper begins with an experimentally validated, physics-based model of the GTAC (Wiese et al., 2013b), then systematically applies nonlinear model reduction techniques. Performing this reduction requires non-dimensional forms of the constitutive equations. While the model framework (Badmus et al., 1995) that the GTAC model (Wiese et al., 2013b) is based on features of a non-dimensional form of the gas path equations, the shaft and thermal storage equations from Badmus et al. (1995) are unsuitable for such an analysis. This is due to their dependence on user-defined reference parameters. Subsequently, the first section of this paper

Fig. 2. GTAC transient simulation layout.
derives suitable non-dimensional forms of the equations describing shaft and heat transfer dynamics. The resulting reduced order model is validated in simulation against the higher-order model, and the expected transient error is quantified. Even though this model reduction is demonstrated for the GTAC prototype, the framework is applicable to gas turbines generally. Having established the reduced order model, a nonlinear optimal control problem is formulated for gas turbine control. The corresponding conditions for the nominal stability of such a controller are then derived. To account for plant-model mismatch, the NMPC is augmented with integral action to achieve steady-state disturbance rejection. Finally, the augmented controller is experimentally demonstrated on the GTAC prototype, and the closed-loop performance of the prototype is assessed.

2. Reduced-order gas turbine model

A 32-state transient model of the GTAC was developed in Wiese et al. (2013b) and was shown to demonstrate good agreement with experimentally observed dynamics. This model is representative of a class of microturbine systems. However, the complexity of the model renders it unsuitable for use in real-time control algorithms. To reduce the computational expense of a model based controller, the order of the transient model is systematically reduced, based on qualitative comparison of time-scales and singular perturbation theory (Khalil, 2002).

2.1. Initial component level model

The component level transient model developed in Wiese et al. (2013b) is presented schematically in Fig. 2. These equations comprising the transient model are reproduced in abbreviated form in Appendix A. However, for notational convenience, the model is stated here in vector form as

\[
E \frac{dz}{dt} = F(z, T_m, \omega, u, u_{md}).
\]  

(1)

\[
J \frac{d\omega}{dt} = f_1(z, T_m, \omega, u, u_{md}).
\]

(2)

\[
C_w \frac{dT_w}{dt} = f_2(z, T_m, \omega, u, u_{md}).
\]

(3)

where \( z \in \mathbb{R}^{38}, T_m \in \mathbb{R}^3 \), and \( \omega \in \mathbb{R} \) denote the gas path states \((M, \bar{p}, \dot{s})\), wall temperatures \((T_w)\), and the angular speed of the shaft respectively, while \( u \in \mathbb{R}^2 \) and \( u_{md} \in \mathbb{R}^2 \) denote the actuator positions and measured disturbances respectively. This model includes the following assumption regarding actuator dynamics:

Assumption 1. The closed loop response of the fuel valve and bleed valve positions are sufficiently fast to be treated as instantaneous.

Remark 1. This simplifying assumption is made based on the fast, closed-loop performance of the two actuators, and the fact that both actuators use only a small fraction of their total range of movement.

2.2. Non-dimensional inertia modelling

Before any time-scale separation between these dynamics can be identified, the inertia terms must each be expressed in a physically sensible, non-dimensional form. As \( (1) \) is an extension of a non-dimensional form of the conservation equations derived in Badmus et al. (1999), the gas path inertia parameters \( c_k \) (collated in the vector \( E \)) are already non-dimensional terms, \( c_k = \frac{L_k}{t_R k_{k-1}} \)

\[ (4) \]

roughly approximating the ratio of the residence time of a pressure disturbance travelling at sonic velocity through the \( k \)th component of the gas path, to a reference time scale \( t \).

To non-dimensionalise the shaft inertia \( J \), it is proposed to divide (2) by a representative compressor torque \( \tau \). The torque of a centrifugal compressor \( \tau_c \) is expressed as

\[
\tau_c = m_r \bar{r}_{tip} \sigma_{tip} \omega.
\]

(5)

where \( m_r, r_{tip} \) and \( \sigma_{tip} \) are the compressor mass flow rate, radius to the exducer tip and slip ratio (Saravanamuttoo, Rogers, & Cohen, 2001) respectively. Then, a reference speed \( \omega_0 \) is introduced

\[
\hat{\omega} = \frac{\omega}{\omega_0}
\]

and used to define a reference torque

\[
\hat{\tau} = m_r \bar{r}_{tip} \sigma_{tip} \hat{\omega}.
\]

(7)
The shaft equation (2) can be divided by (7) to obtain

\[
\frac{1}{tm_r \bar{r}_{tip} \sigma_{tip}} \frac{d\hat{\omega}}{dt} = \frac{1}{\tau_c} \hat{f}_1,
\]

(8)

where the non-dimensional inertia term

\[
\hat{J} = \frac{1}{tm_r \bar{r}_{tip} \sigma_{tip}}.
\]

(9)
is suitable for comparison with the other inertia terms.

A similar approach can be adopted to non-dimensionalise the thermal storage dynamics. As there are multiple mechanisms of heat transfer, there are also several feasible choices for a representative heat transfer. Based on steady-state modelling of the GTAC (Wiese et al., 2013a), it appears that convection from the hot gases to the wall is the dominant mechanism of heat transfer. Subsequently, the most appropriate choice of representative heat transfer, \( \tilde{Q} \), is

\[
\tilde{Q} = h_{int} A_{int} \tilde{T},
\]

(10)

where the internal convective heat transfer parameters \( h_{int} \) and \( A_{int} \) were estimated as part of the system identification presented in Wiese et al. (2013a). Dividing (3) by (10) and introducing a non-dimensional wall temperature,

\[
\tilde{T}_w = \ln \left( \frac{T_w}{T} \right),
\]

(11)

lead to a non-dimensional thermal inertia

\[
\tilde{C}_w = \frac{C_w}{h_{int} A_{int}}.
\]

(12)

The representative, non-dimensional inertia terms for the expected range of system states are collated in Table 1 for each physical component of the simulation model in Fig. 2. As the reference time \( T \) is yet to be determined, the values in Table 1 are expressed as multiples of \( T \).

From Table 1, it can be seen that the ratios \( \max(\hat{J} | \hat{C}_w) \) and \( \max(E | \hat{C}_w) \) are \( O(10^{-3}) \) and \( O(10^{-4}) \), suggesting the existence of sufficient time scale separation that singular perturbation theory may be applied. This would result in a reduced order model where \( T_m \) are the only states retained. While this would significantly reduce the number of states, and eliminate the need for a stiff ODE
was informed by experimental data in (10). Previous analysis by Davison and Birk (2006) suggests that the potential level of model reduction for large scale systems is similar to that found above for the GTAC prototype.

With the exception of the representative heat transfer term $\dot{Q}$ (10), the non-dimensional inertia modelling presented in this section may be applied to gas turbines of varying scale. In this application, the choice of $\dot{Q}$ was informed by experimental data specific to the prototype, and a similar assessment would be required for different gas turbines. Similarly, the resulting reduced order model is specific to the GTAC prototype. However, the analysis of Davison and Birk (2006) would suggest that similar reductions should be possible for larger turbines, but may not be possible for smaller configurations.

2.3. Reduced order model validation

In order to validate this approximate model, both the full transient and reduced order models are simulated for a sudden increase in fuel flow rate and constant bleed valve position. The size and speed of the fuel valve position transient has been chosen to be representative of a severe actuator change that may occur in the controlled system. Fig. 3 shows the maximum percentage error in stagnation pressure, stagnation temperature and velocity across the gas path domain at all instances in the trace. The peak error occurs in the burner exit temperature trace, and is, at worst, less than 3%. Similarly, Fig. 4 shows the percentage error in the speed and wall temperatures between the reference and reduced models. The maximum error for these traces are all fractions of 1%. In exchange for this small degradation in model accuracy, the number of states in the transient model is reduced from 32 to 4. Additionally, the model stiffness has been reduced to a point where efficient solution of the model ODEs can be achieved with a fixed-step solver.

Given the relative scale of the shaft and thermal inertias in Table 1, it may also be possible to treat the wall temperature as

### Table 1
Representative time constants for GTAC dynamics.

<table>
<thead>
<tr>
<th>Component</th>
<th>Inertia term $\times$ reference time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intake</td>
<td>$1.8 \times 10^{-3}$</td>
</tr>
<tr>
<td>Compressor</td>
<td>$6.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>Bleed tee</td>
<td>0</td>
</tr>
<tr>
<td>Comp. air duct</td>
<td>$8.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>Comb. exp</td>
<td>$2.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>Burner</td>
<td>$1.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>Comb. convergence</td>
<td>$4.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>Turbine</td>
<td>$4.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>Exhaust duct</td>
<td>$2.5 \times 10^{-4}$</td>
</tr>
<tr>
<td>Shaft</td>
<td>$4.7 \times 10^{-2}$</td>
</tr>
<tr>
<td>Convergence wall</td>
<td>17.9</td>
</tr>
<tr>
<td>Turb. housing</td>
<td>21.5</td>
</tr>
<tr>
<td>Exhaust wall</td>
<td>42.7</td>
</tr>
</tbody>
</table>

To interpret (14)–(15) in terms of two separate time scales, $\chi$ is set to 0, resulting in the algebraic equation for the boundary layer system,

$$0 = g^* (M_1, x, u, md).$$

The system of algebraic equations $g^*$ possesses a single degree of freedom. Consequently, it is possible to nominate any single component of $z$ as the iterative variable with which to solve for the roots of $g^*$. The inlet Mach number $M_1$ is chosen for this purpose

$$0 = g^* (M_1, x, u, md),$$

and the solution of this equation for $M_1$ is denoted as $M_1^*(x, u, md)$.

The methodology presented above may be applied to gas turbines of any scale, although the specific model simplifications applicable via the time scale separation approach will be dependent on the relative geometry of the system in question, as will the representative heat transfer term $\dot{Q}$ in (10). Previous analysis by Davison and Birk (2006) suggests that the potential level of model
constant for sufficiently short periods without significant loss of accuracy. To examine this, the previous actuator transient is repeated with the wall temperatures held constant at their initial values. The maximum absolute percentage errors are plotted in Figs. 5 and 6. Comparing these figures with Figs. 3 and 4 shows that this approximation has not appreciably changed the error during the initial (shaft dynamics) stage of the transient. From these results, we can conclude that the discrepancy between the full transient model and reduced model with constant wall temperatures does not become significant until some time after the dynamics of interest have already equilibrated. Given that this is the case, it is proposed that the wall temperatures may be treated as constant, measured disturbances, provided that the period of interest following the initiation of a system transient is similar to the time scales associated with the shaft dynamics.

Using this proposed simplification, an appropriate reduced order model approximation of (1)-(3) is given by

\[
\frac{d\tilde{\omega}}{dt} = \frac{f_1(M_t, \tilde{\omega}, \mathbf{u}, \mathbf{md})}{r},
\]

\[
0 = g^*(M_t, \tilde{\omega}, \mathbf{u}, \mathbf{md}),
\]

where

\[
\mathbf{md} = \left[ \mathbf{md}, T_w \right].
\]

This model is suitable for model predictive control, and will constitute part of the constraint set for the nonlinear optimisation problem presented in the next section.
3. NMPC development

The control objective for the GTAC prototype is to regulate delivery pressure \( P_{\text{td}} \) to some prescribed level, whilst tracking a time-varying demand for bleed air flow \( m_{\text{d}} \). To improve the fuel efficiency of the prototype, a secondary goal is the ability to return the GTAC to a lower power (or ‘idle’) operating condition when no demand for compressed air exists. The two plant outputs are functions of the plant state and actuator positions

\[
\begin{bmatrix}
  P_{\text{td}}, m_{\text{d}}
\end{bmatrix} = \begin{bmatrix}
y(\cdot, \cdot, \cdot)
\end{bmatrix},
\]

and the reference output is

\[
y_r(\cdot, \cdot, \cdot) = \begin{bmatrix} P_{\text{td}}^*, m_{\text{d}}^* \end{bmatrix}
\]

As the experimental setup does not include any intermediate storage of compressed air, the only setpoint information available to the controller is the instantaneous demand for compressed air at the point of consumption. Consequently, no future trajectory information is available at runtime, and the following assumption is made:

**Assumption 2.** The demand for compressed air flow rate and pressure at the current sampling instant \( t_i \) remains constant for the entire prediction horizon,

\[
y_r(\cdot, \cdot, \cdot) = \begin{bmatrix} y_r(\cdot, \cdot, \cdot) \end{bmatrix}, \quad t_i \in \{ t_i, t_i + t_i + N \}
\]

where \( N \) is the number of sampling periods in the prediction horizon, and the subscript convention

\[
y_r(\cdot, \cdot, \cdot) = y_r(\cdot, \cdot, \cdot)
\]

represents the value of a parameter at time \( t_i \).

The cost associated with deviations from the reference trajectory is expressed in a weighted sum of squares form

\[
C(\hat{o}(\cdot), M(\cdot), \hat{u}(\cdot), \hat{d}(\cdot), \hat{y}(\cdot)) = \begin{bmatrix} y_r(\cdot, \cdot, \cdot) - y_r(\cdot, \cdot, \cdot) \end{bmatrix} \begin{bmatrix} W(\cdot, \cdot, \cdot) \end{bmatrix} \begin{bmatrix} y_r(\cdot, \cdot, \cdot) - y_r(\cdot, \cdot, \cdot) \end{bmatrix}
\]

where

\[
W(\cdot, \cdot, \cdot) = \begin{bmatrix}
  W_p & 0 \\
  0 & W_{\text{m}^2}
\end{bmatrix}
\]

Evaluating this cost at each sampling instant in the prediction horizon leads to the discrete objective function shown in the control optimisation problem (30).

The discrete control trajectory \( \hat{u} \) is determined by the solution of an optimal control problem subject to the following assumptions:

**Assumption 3.** The acquisition of the current state measurement, execution of the NMPC algorithm and issue of new control signals occurs instantaneously at the time of sampling.

**Remark 2.** This assumption is made to simplify the formulation and stability analyses in the following sections of the paper. The effect of this assumption on the performance and stability of the controller was investigated in simulation and shown to have only a small effect on the performance of the controller for the sampling periods considered.

**Assumption 4.** The measured disturbances do not change from their initial value \( \hat{d}(\cdot) \) for the duration of the prediction horizon.

**Remark 3.** The measured disturbance vector is composed of ambient conditions, the current thermodynamic state of the fuel, and the wall temperatures. The worst case changes in these values are expected to be both small and slow. As any functional dependence on \( \hat{d}(\cdot) \) does not change for the remainder of this paper, it will not be explicitly included from this point forward, for notational convenience.

The proposed NMPC control law for a plant satisfying Assumptions 1–4 is

\[
u = \vartheta(\hat{o}(\cdot, \cdot, \cdot), y_r(\cdot, \cdot, \cdot)),
\]

where \( \vartheta(\hat{o}(\cdot, \cdot, \cdot), y_r(\cdot, \cdot, \cdot)) \) corresponds to the first element, \( u_r(\cdot, \cdot, \cdot) \), of the control trajectory \( \hat{u}(\cdot, \cdot, \cdot) \), which is obtained by solving

\[
\arg\min_{\hat{u}, \hat{y}, \vartheta} J = \sum_{i = 1}^{i + N + 1} C(\hat{o}_i, M_{i, j}, \hat{u}_i, y_r(\cdot, \cdot, \cdot)) + C(\hat{o}_i, M_{i, j}, \hat{u}_i, y_r(\cdot, \cdot, \cdot), \hat{u}_i, y_r(\cdot, \cdot, \cdot)), \quad \vartheta(\cdot, \cdot, \cdot) = \hat{u}_i,
\]

subject to:

\[
\begin{align*}
\frac{d\hat{o}_i}{dt} &= f_i(M_i, \hat{o}_i, u_r(\cdot, \cdot, \cdot)), \\
0 &= g^*(M_i, \hat{o}_i, u_r(\cdot, \cdot, \cdot)), \\
\hat{u}_i &= \hat{u}_i, \\
H(\hat{o}_i, u_r(\cdot, \cdot, \cdot)) &\geq 0.
\end{align*}
\]

For the GTAC NMPC application, the inequality constraints

\[
H(\hat{o}_i, u_r(\cdot, \cdot, \cdot)) \geq 0
\]

include surge margin, maximum and minimum shaft speeds, maximum turbine inlet temperature, and actuator limits. For this controller, a zero-order-hold is arbitrarily selected to define the control trajectory between sampling instances.

3.1. Nominal stability

This section describes the conditions for which (29) is nominally, asymptotically stable for a constant \( y_r(\cdot, \cdot, \cdot) \). The first requirement is the existence of a unique and feasible equilibrium point corresponding to \( y_r(\cdot, \cdot, \cdot) \):

**Assumption 5.** For all setpoints \( (y_r(\cdot, \cdot, \cdot)) \) of interest, there exists a unique \( \hat{o}(y_r(\cdot, \cdot, \cdot), u_r(\cdot, \cdot, \cdot)) \), such that

\[
y_r(\hat{o}(y_r(\cdot, \cdot, \cdot), u_r(\cdot, \cdot, \cdot))) = y_r(\cdot, \cdot, \cdot)
\]

\[
f_i(M_i(\hat{o}(y_r(\cdot, \cdot, \cdot), u_r(\cdot, \cdot, \cdot)), \hat{o}(y_r(\cdot, \cdot, \cdot), u_r(\cdot, \cdot, \cdot)))) = 0.
\]

\[
H(\hat{o}(y_r(\cdot, \cdot, \cdot), u_r(\cdot, \cdot, \cdot))) \geq 0.
\]

Then, the nominal stability of (29) can be proven by demonstrating the stability of this equilibrium point for the discrete system

\[
\begin{align*}
\hat{o}({\hat{t}_{i+1}}) &= \varphi(\hat{t}_{i+1}, \hat{t}_i, \hat{o}(\hat{t}_i), \hat{y}_i) \vartheta(\hat{o}(\hat{t}_i), y_r(\cdot, \cdot, \cdot)), \\
\vartheta(\hat{o}(\hat{t}_i), y_r(\cdot, \cdot, \cdot)) &= \hat{u}_i,
\end{align*}
\]

where

\[
\varphi(\hat{t} + \hat{t}_i, \hat{t}_i, \hat{o}(\hat{t}_i), \hat{y}_i)
\]

is the solution of the plant model for \( \hat{t} \) at some time increment \( \hat{t} > 0 \) after the initial time \( \hat{t}_i \), given the initial state \( \hat{o}(\hat{t}_i) \) and some input trajectory \( \hat{u}(\cdot, \cdot, \cdot) \).

Before stating the region of attraction for this set point, several definitions are required. Let

\[
\begin{align*}
\hat{\mathcal{U}}(\hat{o} + \delta \hat{o}, y_r(\cdot, \cdot, \cdot)) &= \begin{bmatrix} u_1, u_{r1}, ..., u_{rN-1} \end{bmatrix}
\end{align*}
\]

be the optimal solution of the constrained nonlinear optimisation problem (30) at time \( \hat{t}_i \) for some initial deviation from the setpoint.
and for convenience define
\[ \hat{\xi}_{i+N} = \varphi(\hat{\xi}_{i+N}, \hat{\xi}_i, \hat{\omega} + \delta \hat{\omega}, \mathcal{U}^\alpha(\hat{\omega} + \delta \hat{\omega}, \hat{y}_i)), \] (38)
as the terminal state having applied \( \mathcal{U}^\alpha \). In addition to the optimal solution \( (37) \), derivation of the stability conditions also requires the introduction of a suboptimal solution
\[ \mathcal{U}_{\text{aux}} = \left[ u_{i+1}^1, \ldots, u_{i+N-1}^i, \kappa(\hat{\xi}_{i+N} - \hat{\omega}) \right], \] (39)
for a successor time \( \hat{t}_{i+1} \), where
\[ \kappa(\delta \hat{\omega}_{i+N}) = \bar{u} + \left[ k_{\text{aux}}^1, k_{\text{aux}}^2 \delta \hat{\omega}_{i+N}, \quad \bar{\epsilon} \in [\bar{\hat{t}}_{i+N}, \bar{\hat{t}}_{i+N+1}] \right], \] (40)
is a simple proportional control law with gains \( k_{\text{aux}} \).

**Lemma 1.** For a plant satisfying Assumptions 1–4, a set point \( y \), satisfying Assumption 5, and any \( \gamma < \in \) minimum eigenvalue of \( W \), the closed loop system \( (35) \) is nominally, asymptotically stable within a region of attraction \( \Omega_{\text{aux},\hat{\omega}} \), where \( \Omega_{\text{aux},\hat{\omega}} \) is the largest set \( \Omega_{\text{aux},\hat{\omega}} \subset \Omega_{\text{aux},\hat{\omega}} \subset \mathbb{R} \), such that
\[ \delta \hat{\omega}_0 \leq 0 \leq \delta \hat{\omega}_n, \]
and \( \forall \delta \hat{\omega}_n \in [\delta \hat{\omega}_0, \delta \hat{\omega}_1] \), a feasible solution to the optimisation problem exists, and
\[ -\gamma (\varphi + \delta \hat{\omega}, \bar{u}^i)^\top \varphi(\hat{\omega} + \delta \hat{\omega}, \bar{u}^i) - C(\hat{\xi}_{i+N}, u_{i+N-1}^i, y_i) 
+ C(\varphi^*, \kappa(\hat{\xi}_{i+N} - \hat{\omega}), y_i) + C(\hat{\xi}_{i+N}, \kappa(\hat{\xi}_{i+N} - \hat{\omega}), y_i) \leq 0, \] (41)
where
\[ \varphi^* = \varphi(\hat{\xi}_{i+N}, \hat{\xi}_{i+N}^i, \hat{\xi}_{i+N}^i, \kappa(\hat{\xi}_{i+N} - \hat{\omega})). \] (42)

**Proof.** The derivation of this condition follows the same conceptual path as presented in Rawlings and Mayne (2009) for the nominal asymptotic stability of constrained systems. Consider the criteria for a Lyapunov function \( \mathcal{V} \), specified by Theorem B.12 in Rawlings and Mayne (2009),
\[ \mathcal{V}(\delta \hat{\omega}) \geq P_1[|\delta \hat{\omega}|], \] (43)
\[ \mathcal{V}(\delta \hat{\omega}) \leq P_2[|\delta \hat{\omega}|], \] (44)
\[ \mathcal{V}(\varphi(\hat{\xi}_{i+N}, \hat{\xi}_i, \bar{\omega}(\hat{\omega}, \hat{y}_i)) - \hat{\omega}) - \mathcal{V}(\delta \hat{\omega}) \leq -P_3[|\delta \hat{\omega}|], \] (45)
where \( P_1, P_2 \) and \( P_3 \) are all \( \mathcal{K}_\infty \) functions of \( |\delta \hat{\omega}| \). The proposed Lyapunov function is the objective function from \( (30) \). As the cost function \( \mathcal{J} \geq 0 \) and the feasible operating range of \( \delta \hat{\omega} \) is finite, satisfaction of \( (43) \) and \( (44) \) is trivial.

As in Rawlings and Mayne (2009), satisfaction of \( (45) \) is demonstrated by comparing the value of \( \mathcal{J} \) using both \( (37) \) and \( (39) \), resulting in the condition specified by \( (41) \). Obtaining this result requires that the suboptimal control trajectory \( (39) \) always incurs a cost equal to or, greater than, its optimal equivalent. For this to be guaranteed, \( (39) \) must be feasible. Consequently, determining \( \Omega_{\text{aux},\hat{\omega}} \) for a given setpoint requires three steps:

1. Determine the largest connected set \( \Omega_{\text{aux}} \subset \mathbb{R} \), including the equilibrium point, for which the proportional control law \( (40) \) results in a feasible trajectory.
2. Determine the largest connected set \( \Omega_{\text{aux},\hat{\omega}} \subset \mathbb{R} \), including the equilibrium point, for which applying \( (37) \) will result in a terminal state within \( \Omega_{\text{aux}} \).
3. Determine the largest connected set \( \Omega_{\text{aux},\hat{\omega}} \subset \Omega_{\text{aux},\hat{\omega}} \), including the equilibrium point, for which \( (41) \) is satisfied.

Then, \( \Omega_{\text{aux},\hat{\omega}} \) represents the largest connected set for which conditions \( (43)–(45) \) are satisfied, and therefore according to Theorem B.12 in Rawlings and Mayne (2009), the region of attraction for which \( (29) \) is nominally, asymptotically stable.

### 3.2. Integral action

Steady-state mismatch between the reduced order model and the GTAC prototype exists due to the imperfection of the steady-state modelling, and the thermal storage reduction described in Section 2. In a stable system, this may cause convergence to an equilibrium point of the closed-loop system
\[ \hat{y}_{\text{meas}}(\hat{\omega}(\hat{y}_i, \bar{u}(\hat{y}_i))) \neq \hat{y}_i. \] (46)
To address this, integral feedback is added to the controller by adjusting the reference trajectory,
\[ y_{\text{ref}} = \hat{y}_i - E_{\text{int}} = \left[ \begin{array}{c} y_{\text{det}, \text{int}} \cr m_{\text{det}, \text{int}} \end{array} \right], \] (47)
where
\[ E_{\text{int}} = \left[ \begin{array}{c} E_{p, \text{int}} \cr E_{m, \text{int}} \end{array} \right]. \] (48)
As sensor measurements are available to calculate both of the system outputs, the accumulation functions for \( E_{\text{int}} \) are
\[ E_{p,i+1} = \left[ \begin{array}{c} E_{p,i} + k_{i,p} \Delta p_i \cr E_{p,i} \end{array} \right], \] (49)
\[ E_{m,i+1} = \left[ \begin{array}{c} E_{m,i} + k_{i,m} \Delta m_i \cr E_{m,i} \end{array} \right], \] (50)
where
\[ \Delta p_i = p_{\text{int}, \text{thres}} - p_{\text{int}, \text{thres}}, \] (51)
\[ \Delta m_i = m_{\text{int}, \text{thres}} - m_{\text{int}, \text{thres}}. \] (52)
This approach, previously used in Al Seyab (2006), ensures that the constraints are treated consistently, whilst having the output trajectory shift towards the desired trajectory. The addition of integral action to the NMPC is shown in Fig. 7. The tuning of \( p_{\text{int}, \text{thres}} \) and \( m_{\text{int}, \text{thres}} \) is influenced by conflicting design requirements. Larger threshold values improve the range of steady-state errors which the integral can act on, while smaller values avoid unnecessary integrator wind-up and allow for more aggressive tuning of the integral gains.

![Fig. 7. Controller block diagram.](image-url)
The choice of sign in (47)–(52) depends on the closed loop system responding consistently to reference adjustment. This requirement is formally stated in the following assumption:

**Assumption 6.** For all candidate values of $y_{int}$

$$\frac{\partial p_{int,meas}}{\partial p_{int,int}} > 0,$$

$$\frac{\partial m_{int,meas}}{\partial m_{int,int}} > 0.$$ (53)

**Proposition 1.** For a stable, closed-loop system satisfying Assumption 6, there exists a pair of thresholds $(p_{int,thres}^*, m_{int,thres}^*)$, and a pair of positive gains $(k_{tp}^*(p_{int,thres}), k_{tm}^*(m_{int,thres}))$, such that for all

$$p_{int,thres} \in \left(p_{int,thres}^*, \infty\right),$$

$$m_{int,thres} \in \left(m_{int,thres}^*, \infty\right),$$

$$k_{tp}(p_{int,thres}) \in (0, k_{tp}^*(p_{int,thres})),$$

$$k_{tm}(m_{int,thres}) \in (0, k_{tm}^*(m_{int,thres}^*)),$$

the closed-loop system with integral tracking (20), (21), (29), (30), (47)–(50), causes $y_{meas} \to y$ asymptotically.

4. Experimental demonstration of control system

4.1. Implementation

To assess the performance of the NMPC developed in the previous section, the controller was implemented on the GTAC prototype (Fig. 8) using a dSpace DS1006 embedded control platform. The DS1006 processor board is composed of a 4-core x64 processor, with 1 Gb of memory. The optimal control problem (30) is solved online using the direct multiple shooting algorithm of Bock et al. (2007), adapted to preserve the nonlinearity of the optimisation problem. This implementation utilises a single core of the DS1006. Naturally, there are alternative approaches in the literature to solving the NMPC problem including orthogonal collocation (e.g. Kim et al., 2013), although the approach adopted here was found to be effective for the implementation considered. As the internal model of the GTAC lacks fidelity at lower compressor pressure ratios, the minimum fuel flow maintains the GTAC at a pressure ratio of approximately 2:1. The low pressure (or idle) condition is then set by the resting position of the bleed flow throttle valve, and the lower constraint on fuel valve position.

The proposed controller parameters for the GTAC NMPC are collated in Table 2. These parameters were tuned in simulation, with Lemma 1 used to determine nominal stability. The sample period $t_s$ and prediction horizon $t_p$ were selected to balance controller performance, stability and the computation time requirements. As the higher priority for the controller is to regulate the pressure, the cost function weightings are set such that

$$W_p = 20,$$

and these values are held constant across throughout the prediction horizon. It can be shown numerically that the proposed controller parameters produce a nominally stable controller, where the corresponding region of attraction encompasses the entire feasible range of shaft speeds.

![Fig. 8. GTAC prototype.](image)
4.2. Double set-point change

In these tests, the response speed of the GTAC is measured when changing from the idle pressure and bleed condition, to the desired delivery pressure (300 kPa for the GTAC prototype) and a single commanded bleed flow rate. Figs. 9 and 10 show the output traces for the GTAC responding to commanded bleed flow rates of 0.06 and 0.1 kg/s respectively.

These figures include two different measures of the bleed mass flow rate. The first is calculated entirely from the available sensor measurements, and shows an unacceptably large overshoot. The reason for this apparent overshoot can be determined from Fig. 11, which shows the traces from those sensors that contribute to the bleed flow calculation, and in particular the wide-band exhaust gas oxygen sensor (UEGO) sensor and fuel mass flow rate sensors. When the command is issued to the fuel solenoid valve, the effect of the sudden increase in fuel flow appears in the \( \lambda \) trace quickly, due to the fast response of the UEGO sensor. However, the Coriolis mass flow metre used to measure fuel introduces a greater lag than the UEGO introduces for \( \lambda \). As a result, a sudden decrease in \( \lambda \) must result in the controller inferring a sudden decrease in turbine mass flow rate, leading to over-estimation of \( m_{bl,meas} \). The alternative trace shown in Figs. 9–10 is calculated assuming that the steady-state throttle valve correlation, and the assumption of quasi-steady gas path dynamics are both accurate. Whilst this calculation is not a viable alternative for use with the integral control, as it uses assumptions that are themselves sources of steady-state error, it does suggest that the actual bleed flow rate

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**Fig. 9.** 0.06 kg/s Dual set-point change. Experiment (- -), reference (- -).

**Fig. 10.** 0.10 kg/s Dual set-point change. Experiment (- -), reference (- -).

**Fig. 11.** Components of sensed bleed flow, for 0.06 kg/s dual set-point change.
has substantially less overshoot than the measured values indicate.

One of the motivations for applying model predictive control to gas turbines is the inherent handling of constraints. To investigate this particular benefit, the state constraint margins for the 0.06 kg/s transient are plotted in Fig. 12. In this case, the turbine inlet temperature and surge margin constraints are active for the initial phase of the transient. Results from the 0.08 (not shown) and 0.1 kg/s transients also demonstrated that the turbine inlet temperature constraint was active, but the surge margin constraint was not. This is due to the increased deviation from the bleed flow set-point, which leads to an optimal control trajectory that naturally exists on the normal operation side of the surge line.

4.3. Pressure regulation

The next tests are used to evaluate the controlled prototype’s performance for regulation of delivery pressure whilst tracking a time-varying reference for bleed flow rate. For this test, the compressed air demand is imposed in the form of a square wave. Although simulation results suggested that the controller and plant should be able to handle set-point changes at rates of at least 0.5 Hz, the previously identified errors in the bleed mass flow measurement prevent the integral feedback aspect of the controller from functioning in the presence of shorter dwell times.

Fig. 13 shows the resulting trace where the bleed flow rate set-point alternates between the minimum, or idle flow rate, and nominally 10 s bursts of demand for compressed air bleed flow. For this trace, the worst case variations from the reference pressure are approximately 10–12 kPa and occur when returning to the low demand state. For cases where demand is increasing, the worst supply pressure drop is 7–8 kPa. These results are encouraging, as even the best results achieved in simulation included deviations of ± 5 kPa. Fig. 13 again includes both methods of bleed flow estimation, and while the sensed trace confirms that the integral action is working correctly, the valve model bleed trace is more valuable for assessing the transient response. Close examination of model based trace shows that for increases in bleed flow demand, the time between the set point change and the GTAC achieving the new flow target is less than 0.5 s. This result is also consistent with simulation, and strongly suggests that with better bleed flow measurement, it should be possible to respond to set-point changes considerably faster than the current method permits.

5. Conclusions

This paper presented the development and experimental implementation of an online, fully nonlinear model predictive controller (NMPC) for a gas turbine. The internal model of the NMPC was first generated by replacing the fastest system dynamics with quasi-steady approximations. This analysis showed that treating all of the elements in the gas path in this manner had only a minor impact on simulation accuracy. The model order was further reduced by noting that the wall temperature dynamics are associated with a much longer time scale than the dynamics of interest for control of the GTAC prototype, and that wall temperatures could be assumed constant over likely prediction horizons. Using this process, the previously validated component level gas turbine model (Wiese et al., 2013b) was successfully reduced to a single dynamic state representing the shaft speed.
The paper then presented the formulation of a MIMO control optimisation problem intended to minimise output tracking error subject to limits inherent in gas turbines. The constraint set included the range of the control actuators, the valid range of the internal model, and safety limits such as the compressor surge margin and maximum turbine inlet temperature. This formulation included derivation of the conditions sufficient to prove nominal asymptotic stability of the closed-loop system. The resulting NMPC algorithm was applied to the prototype gas turbine, and controller parameters were selected that result in the region of attraction incorporating the entire feasible operating range of the prototype. To address plant-model mismatch, the optimisation problem was augmented with integral action. The augmented control problem was solved using a technique that maintained the nonlinearity of the core model.

Controller performance was then evaluated through representative tracking and regulation experiments. This highlighted the advantages of the proposed controller’s constraint handling and disturbance rejection, with the controller being shown to balance both compressor surge and turbine inlet temperature limits while minimising response times. More broadly, these experiments demonstrated that a nonlinear model predictive controller can be successfully deployed on a gas turbine, in real time, without the need for any linearisation.

Appendix A. Transient model equations

The component level model developed and validated in Wiese et al. (2013b) is an extension of a model framework derived in Badmus et al. (1995), which begins with the one-dimensional conservation equations in the form

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho v A) = 0,$$

$$\frac{\partial}{\partial t}(\rho v A) + \frac{\partial}{\partial x}\left[ \rho(v^2 + p)A \right] = p \frac{\partial A}{\partial x} + \rho A(F_3 + F_w),$$

$$\frac{\partial}{\partial t}\left[ \rho \left( e + \frac{v^2}{2} \right) A \right] + \frac{\partial}{\partial x}\left[ \rho v \left( e + \frac{p}{\rho} + \frac{v^2}{2} \right) A \right] = AQ + \rho A F_w,$$

where $F_3$, $F_w$, and $Q$ represent the forcing terms due to body forces, friction and heat transfer respectively. The model assumes a calorically perfect working fluid of constant composition, with an entropy of zero at the reference temperature ($T$) and pressure ($p$). These equations are recast in terms of non-dimensional time, $\bar{t} = t/\bar{t}$, pressure, $\bar{p} = \ln(p/\bar{p})$, entropy, $\bar{s} = s/C_p$, and Mach number

$$\epsilon\left( \begin{array}{c} M_k = \frac{M_k}{\bar{M}_k} \\ \bar{p}_k = \frac{p_k}{\bar{p}_k} \\ \bar{s}_k = \frac{s_k}{\bar{s}_k} \end{array} \right) = \Xi(M, \bar{r}) \left[ \begin{array}{c} M_k \\ \bar{p}_k \\ \bar{s}_k \end{array} \right] + Y(M, \bar{r}) \left[ \begin{array}{c} \bar{Q} \\ \bar{F}_3 \\ \bar{F}_w \end{array} \right]$$

$$+ 2 \left[ \begin{array}{c} \zeta_1(M, \bar{r}) \\ \zeta_2(M, \bar{r}) \end{array} \right] \frac{\partial A}{\partial x} + \epsilon \left[ \begin{array}{c} 0 \\ \bar{r} - 1 \end{array} \right] \frac{\partial A}{\partial x},$$

where $\bar{t}$ is the reference time scale, the gas path inertia parameter, $\epsilon$, is defined as

$$\epsilon = L \exp \left( -\frac{1}{2} \left( \bar{s} + \frac{\bar{r} - 1}{\bar{r}} \right) \bar{p}_k \right),$$

and $\Xi$, $Y$ and $\zeta$ are matrices of influence coefficients as derived in Badmus et al. (1995).

To convert (A.4) to ordinary differential equations for simulation, the total simulation domain is divided on a component-by-component basis. Fig. 2 shows the components that comprise the GTAC transient simulation model. Simulation element boundaries are based on the steady-state model characteristics identified in Wiese et al. (2013a).

The non-dimensional equations for each gas path component are

$$\epsilon\left( \begin{array}{c} \bar{p}_{k-1}, \bar{s}_{k-1}, M_k \end{array} \right) \frac{d}{dt} \left[ \begin{array}{c} M_k \\ \bar{p}_k \\ \bar{s}_k \end{array} \right] = \epsilon \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] \frac{\partial A_k}{\partial t} + \Xi(M, \bar{r}) \left[ \begin{array}{c} M_k - M_{k-1} \\ \bar{p}_k - \bar{p}_{k-1} \\ \bar{s}_k - \bar{s}_{k-1} \end{array} \right]$$

$$+ Y(M, \bar{r}) \left[ \begin{array}{c} \bar{Q} \\ \bar{F}_3 \\ \bar{F}_w \end{array} \right] + \epsilon \left[ \begin{array}{c} 0, \zeta_1(M, \bar{r}) \end{array} \right] \left[ \begin{array}{c} 0 \\ \bar{A}_k - \bar{A}_{k-1} \end{array} \right],$$

where subscripts $(k - 1)$ and $k$ indicate the upstream and downstream interfaces of an element respectively. $M_k$, $\bar{p}_{k-1}$, and $\bar{s}_{k-1}$ are treated as inputs to a given element, as are the effect of time varying area and the source terms $\bar{Q}$, $\bar{F}_3$, and $\bar{F}_w$. To identify the unknown forcing terms, Badmus et al. (1995) defined three new functions $\kappa_1, \kappa_2$, and $\kappa_3$ such that

$$\epsilon\left( \begin{array}{c} \bar{p}_{k-1}, \bar{s}_{k-1}, M_k \end{array} \right) \frac{d}{dt} \left[ \begin{array}{c} M_k \\ \bar{p}_k \\ \bar{s}_k \end{array} \right] = \epsilon \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] \frac{\partial A_k}{\partial t} + \Xi(M, \bar{r}) \left[ \begin{array}{c} M_k - M_{k-1} \\ \bar{p}_k - \bar{p}_{k-1} \\ \bar{s}_k - \bar{s}_{k-1} \end{array} \right] - \left[ \begin{array}{c} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{array} \right].$$

At steady-state, the map functions $\kappa_1$, $\kappa_2$, and $\kappa_3$ must satisfy

$$\left[ \begin{array}{c} M_k - M_{k-1} \\ \bar{p}_k - \bar{p}_{k-1} \\ \bar{s}_k - \bar{s}_{k-1} \end{array} \right] = \left[ \begin{array}{c} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{array} \right].$$

Therefore, steady-state characteristic data is used to calibrate these mapping functions for the chosen application. Derivations of $\kappa_1$, $\kappa_2$ and $\kappa_3$ specific to components of the GTAC prototype are presented in Wiese et al. (2013b).

In Section 2, (A.7)–(A.8) is denoted as

$$E = g(z, T_{\infty}, \omega, u, md),$$

where $u$ denotes the actuator positions (fuel, bleed valve), and the gas path boundary conditions and the thermodynamic state of the fuel are treated as measured disturbances $md$.

The shaft dynamics use Newton’s second law

$$\frac{d\omega}{dt} = (\tau_{\text{turb}} - \tau_c),$$

where the turbine and compressor torques are both expressed as instantaneous functions of the current states, inputs and measured disturbances. This dependency is denoted in Section 2 as

$$\int \frac{d\omega}{dt} = f_j(z, T_{\infty}, \omega, md).$$

In Wiese et al. (2013b), lumped parameter heat transfer models are implemented for the combustor convergence, turbine housing
...and first section of the exhaust. These models include radiative and convective heat transfer with both the hot gas and the ambient air, and assume negligible conduction effects

\[
C_w \frac{dT_w}{dt} = h_{int} A_{int} (T_1 - T_w) + \alpha A_{int} (T_1^k - T_w^k) - h_{ext} A_{ext} (T_w - T_{ext}) - \alpha A_{ext} (T_w^k - T_{ext}^k),
\]

(A.12)

where

\[
C_w = \rho_v V_v C_{p,v},
\]

is the lumped capacitance of each wall component, as identified experimentally in Wiese et al. (2013b). In Section 2, instances of (A.12) for the convergence, turbine housing and exhaust are represented in the vector form

\[
C_w \frac{dT_w}{dt} = f_w(z, T_w, md),
\]

(A.14)

where \(C_w\) is the vector of lumped capacitance values \(C_w\).

References


