Understanding XCP: Equilibrium and Fairness

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Abstract—We prove that the XCP equilibrium solves a constrained max-min fairness problem by identifying it with the unique solution of a hierarchy of optimization problems, namely those solved by max-min fair allocation, but solved by XCP under an additional constraint. This constraint is due to the "bandwidth shuffling" necessary to obtain fairness. We describe an algorithm to compute this equilibrium and derive a lower and upper bound on link utilization. While XCP reduces to max-min allocation at a single link, its behavior in a network can be very different. We illustrate that the additional constraint can cause flows to receive an arbitrarily small fraction of their max-min fair allocations. We confirm these results using ns2 simulations.

Index Terms—Congestion control, max-min, optimization.

I. INTRODUCTION

TCP congestion control [1] has prevented severe congestion while the Internet has undergone explosive growth. However, as the network continues to scale in size and capacity, the algorithm is struggling with sensitivity to routine packet loss, slow convergence and low throughput [2], [3]. This has motivated several recent enhancements [4]–[10]. (See [6] for further references.) Of these, XCP [9], [10] has received much attention [11]–[15] and is especially suitable for private networks where its need for explicit communication between the traffic sources and the network is less of a deployment barrier than in the current Internet. Unlike proposals that set the flow rates according to the *sum* of congestion measures at the links of their paths, XCP sets them according to the *minimum* "available capacity" in their paths. This has the same flavor as MaxNet [16], [17], which sets flow rates according to the *maximum* of congestion measures in their paths.

The stability of XCP has been studied when all round-trip times (RTTs) are equal [9], [11] and when they differ [13]. Specific networks can also be shown to be stable [18]. However, that work focuses on very simple topologies. This paper takes the complementary approach [19], [20] of reverse engineering XCP to understand its *multi-link equilibrium* properties.

Although packet networks seldom achieve equilibrium, equilibrium analysis indicates the best-case fairness and capacity properties of a protocol. Moreover, dynamic analysis typically starts with a linearization around the equilibrium, which requires a clear characterization such as is presented here. We do not consider misbehaving nodes, as studied in [14], or errors in capacity estimates, as studied in [11].

A deterministic fluid model of a general XCP network with multiple links and multiple flows is presented in Section II. Section III analyzes the equilibrium rates of XCP and shows that all queues are empty in equilibrium We prove the existence and uniqueness of XCP equilibrium rates by identifying them with the unique solution to a hierarchy of optimization problems. This is the same set of problems solved by the standard max-min fair allocation [21], but XCP solves them under an additional constraint. While XCP reduces to max-min allocation at a single link, its behavior in a network can be very different. We describe an algorithm to compute this equilibrium and derive upper and lower bounds on link utilization.

In Section IV, we use these bounds to investigate the impact of the choice of protocol parameters on link utilization under the additional constraint. We show that flows can receive an arbitrarily small fraction of their max-min fair allocations. Specifically, with a max-min fair allocation, as long as a link is a bottleneck for some (not necessarily all) flows that pass through it, it will be fully utilized. Under XCP, this is no longer true: When the majority of flows using a link are bottlenecked at other links, the remaining flows at that link may not fully use the residual bandwidth. With the parameters specified in [9], however, link utilization is at least 80% at any link. XCP has a "shuffling parameter" $\gamma > 0$ to prevent the network from settling into an unfair state [9]. We show that, given any network topology, we can choose γ sufficiently small so that the resulting allocation is close to max-min fairness. For any fixed $\gamma > 0$, however, there are topologies in which some flow rates can be far away from their max-min allocations.

These properties and the accuracy of our algorithm are verified by ns2 simulations in Section V. We conclude in Section VI with limitations of this work.

II. MODEL

Consider a network with L links shared by N flows. Sources are indexed by $i = 1, \ldots, N$, links by $l = 1, \ldots, L$ and packets by k. Let R be the $L \times N$ routing matrix: $R_{li} = 1$ if flow i uses link l, and 0 otherwise. Let $L(i)$ be the set of links in the path of flow i, and $I(l)$ be the set of flows that use link l

$$L(i) := \{l \,|\, R_{li} = 1\} \quad I(l) := \{i \,|\, R_{li} = 1\}.$$

Note that $l \in L(i) \Leftrightarrow i \in I(l)$.

We will present a continuous-time fluid model of XCP. For flows i, define the following variables:

- $w_i(t)$: window size at time t, in packets;
- τ_i: round-trip propagation (and fixed processing) delay;
- $T_i(t)$: round-trip time (RTT) at time t;
- $x_i(t) := w_i(t)/T_i(t)$: flow rate at time t.

For links l, define the following variables:

- c_l: capacity, in packets/s.
- $b_l(t)$: backlog at time t, in packets.
- $y_l(t) := \sum_i R_{li}x_i(t)$: aggregate input rate at link l at time t. In equilibrium, we sometimes write $y_l(x)$ to emphasize the dependence on equilibrium rates x.

An XCP link divides time into control intervals of duration d_l, which is also used as a time scaling parameter. Nominally, d_l is the mean RTT of all the flows at link l [9]. We assume d_l to be constant in time, which is reasonable as we will show that there is no queueing delay in equilibrium.

To simplify notation, we assume all packets have size 1 unit. We use "flow" and "source" interchangeably.

A. XCP Description

We now summarize the XCP algorithm. See [9] for a detailed description. The equilibrium is independent of feedback delay, and hence such delay is not modeled.

For each packet, XCP generates a feedback signal prescribing a change in window size. Let $\tilde{H}_{lk}(t)$ be the feedback generated by link l for packet k at time t. The acknowledgment for packet k received by its source contains in its header the smallest feedback $\min_l \tilde{H}_{lk}(t)$ generated by links along its path. The source adds this quantity to its current window size.[1] We now describe how to compute the feedback.

Let

$$\phi_l(t) = \alpha d_l(c_l - y_l(t)) - \beta b_l(t)$$

where $\alpha, \beta > 0$ are constants, c_l is the link capacity, $y_l(t)$ is the aggregate input rate, and $b_l(t)$ is the backlog at time t. Let $\phi_l^+(t) = \max(\phi_l(t), 0)$ and $\phi_l^-(t) = \max(-\phi_l(t), 0)$. The feedback on the kth packet at link l is

$$\tilde{H}_{lk}(t) = \tilde{p}_{lk}(t) - \tilde{n}_{lk}(t)$$

where $\tilde{p}_{lk}(t)$ and $\tilde{n}_{lk}(t)$ are the increase and decrease components, respectively

$$\tilde{p}_{lk}(t) = (h_l(t) + \phi_l^+(t)) \frac{\tilde{T}_k(t)}{d_l} \frac{\tilde{T}_k(t)/\tilde{w}_k(t)}{\sum_{j=1}^{K_l(t)} \tilde{T}_j(t)/\tilde{w}_j(t)} \quad (1)$$

$$\tilde{n}_{lk}(t) = (h_l(t) + \phi_l^-(t)) \frac{\tilde{T}_k(t)}{d_l} \frac{1}{K_l(t)} \quad (2)$$

where $\tilde{T}_k(t)$ and $\tilde{w}_k(t)$ are the round-trip time and window size, respectively, of the flow that transmitted packet k, and $K_l(t)$ is the total number of packets seen by link l over the time interval $(t - d_l, t]$. Here

$$h_l(t) = \max(0, \gamma d_l y_l(t) - |\phi_l(t)|)$$

is a "traffic shuffling" term with $\gamma \geq 0$ a constant. (Note that we are using the definition of γ from the Appendix of [9], which differs by a factor of d_l from that used in the corresponding equation in [9].)

B. Dynamic Model

We now translate the per-packet feedback $\tilde{H}_{lk}(t)$ into per-flow feedback. Let $H_{li}(t)$ be the feedback generated by link l for flow i at time t. In general, a quantity with a tilde ($\tilde{\ }$) pertains to a packet while the corresponding variable without a tilde pertains to a flow.

Substituting $\tilde{x}_k(t) = \tilde{w}_k(t)/\tilde{T}_k(t)$ in (1) gives

$$\tilde{p}_{lk}(t) = \frac{\tilde{T}_k(t)}{\tilde{x}_k(t)} \frac{h_l(t) + \phi_l^+(t)}{d_l \sum_{i=j}^{K_l(t)} 1/\tilde{x}_j(t)}. \quad (3)$$

Recall $K_l(t)$ is the total number of packets arriving at link l in period $(t - d_l, t]$. For simplicity, we assume that

$$K_l(t) = y_l(t)d_l = d_l \sum_i R_{li}x_i(t).$$

Of these packets, we assume that $R_{li}x_i(t)d_l$ packets are from flow i. Hence

$$\sum_{j=1}^{K_l(t)} \frac{1}{\tilde{x}_j(t)} = \sum_{i=1}^{N} R_{li}x_i(t)d_l \cdot \frac{1}{x_i(t)} = N_l d_l$$

Thus, the feedback (3) yields feedback to flow i

$$p_{li}(t) = \frac{T_i(t)}{d_l^2} \frac{h_l(t) + \phi_l^+(t)}{N_l x_i(t)}$$

per packet. Using $K_l(t) = y_l(t)d_l$ again, the per-packet feedback (2) becomes

$$n_{li}(t) = \frac{T_i(t)}{d_l^2} \frac{h_l(t) + \phi_l^-(t)}{y_l(t)}.$$

The feedback *per packet* to flow i from link l is then

$$H_{li}(t) = \frac{T_i(t)}{d_l^2} \left(\frac{h_l(t) + \phi_l^+(t)}{N_l x_i(t)} - \frac{h_l(t) + \phi_l^-(t)}{y_l(t)} \right).$$

If flow i does not use link l, then set $H_{li}(t) = \infty$.

Let $H_i(t) = \min_{l \in L(i)} H_{li}(t)$ be the minimum feedback along i's path. Since source i receives $x_i(t)$ feedback packets per unit time (assuming every packet carries control information and is acknowledged), its window evolves according to

$$\dot{w}_i(t) = x_i(t) \cdot H_i(t).$$

Substituting $x_i(t) = w_i(t)/T_i(t)$, we have

$$\dot{w}_i(t) = w_i(t) \min_{l \in L(i)} \left(\frac{h_l(t) + \phi_l^+(t)}{d_l^2 N_l x_i(t)} - \frac{h_l(t) + \phi_l^-(t)}{d_l^2 y_l(t)} \right).$$

Remark: The pseudo code in [9] contains additional "residual" terms. These terms, modeled in [20], have no effect on the equilibrium if the average rate of flows bottlenecked

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[1]In practice, the window size has a lower bound of 1 packet, but for notational simplicity, we ignore this.

at upstream links is significant. Otherwise, the resulting link utilization is slightly increased (by around 4% in Scenario 1 of Section V). Since these terms hardly affect the equilibrium, we ignore them in the analysis in this paper.

In summary, an XCP network is described by the following set of equations:

$$\dot{w}_i(t) = w_i(t) \min_{l \in L(i)} F_{li}(t) \tag{4a}$$

$$\dot{b}_l(t) = \begin{cases} y_l(t) - c_l, & \text{if } b_l(t) > 0 \\ \max(y_l(t) - c_l, 0), & \text{if } b_l(t) = 0 \end{cases} \tag{4b}$$

where

$$F_{li}(t) = \frac{1}{d_l^2} \left(\frac{h_l(t) + \phi_l^+(t)}{N_l x_i(t)} - \frac{h_l(t) + \phi_l^-(t)}{y_l(t)} \right) \tag{5a}$$

$$\phi_l(t) = \alpha d_l(c_l - y_l(t)) - \beta b_l(t) \tag{5b}$$

$$h_l(t) = \max(\gamma d_l y_l(t) - |\phi_l(t)|, 0) \tag{5c}$$

$$x_i(t) = \frac{w_i(t)}{T_i(t)} \tag{5d}$$

$$y_l(t) = \sum_i R_{li} x_i(t) \tag{5e}$$

$$T_i(t) = \tau_i + \sum_l R_{li} \frac{b_l(t)}{c_l}. \tag{5f}$$

Here, $\alpha > 0, \beta \geq 0, \gamma \geq 0$ are constants, and $\phi_l^+(t) = \max(\phi_l(t), 0), \phi_l^-(t) = \max(-\phi_l(t), 0)$. Standard XCP uses $\alpha = 0.4, \beta = 0.226$ and $\gamma = 0.1$. We will study the behavior of the general model, which includes this as a special case. As we will see below, the qualitative properties, such as existence and uniqueness of equilibrium rates, do not depend on specific values of these parameters (as long as $\gamma > 0$).

III. EQUILIBRIUM RATES

This section characterizes the equilibrium of XCP and describes an algorithm to compute it; the next considers the implications of these results on utilization and fairness.

Equations (4)–(5) describe the evolution of the window vector $w(t) = (w_i(t), \text{for all } i)$ and the backlog vector $b(t) = (b_l(t), \text{for all } l)$. A pair of rate and backlog vectors (x, b), with window vector w given by $w_i = x_i(\tau_i + \sum_l R_{li} b_l/c_l)$, is said to be in *equilibrium* if both $\dot{w}(t) = 0$ and $\dot{b}(t) = 0$. To avoid technicalities, we only consider cases with $w_i > 0$; under a minor reformulation, it can be shown that $w_i = 0$ in equilibrium only if $\gamma = 0$, and all theorems and lemmas still hold.

We start by defining a bottleneck link and other notation for XCP equilibrium. In general, quantities without t dependence denote equilibrium quantities, e.g., w_i, T_i, x_i, F_{li}.

Definition 1: A link l is said to be a *bottleneck* for source i with respect to (w.r.t.) x if F_{li} is minimum among all the links that i uses, i.e., $F_{li} = \min_{m \in L(i)} F_{mi}$. In this case, source i is said to be *bottlenecked* at link l w.r.t. x.

By definition, every source i has a bottleneck. Lemma 1 below implies that $F_{li} = 0$ in equilibrium at bottleneck l.

We distinguish between links that are bottlenecks and those that are not. Let $L_1(i)$ be the set of links that are bottlenecks for source i w.r.t. a given equilibrium rate x

$$L_1(i) := \left\{ l \in L(i) \mid F_{li} = \min_{m \in L(i)} F_{mi} \right\}$$

and $L_0(i) := L(i) \setminus L_1(i)$ be the set of links in i's path that are not bottlenecks for source i w.r.t. x. We also distinguish between sources that are bottleneck locally and those that are not. Let $I_1(l)$ be the set of sources bottlenecked at link l w.r.t. a given equilibrium rate x

$$I_1(l) := \left\{ i \in I(l) \mid F_{li} = \min_{m \in L(i)} F_{mi} \right\}$$

and $I_0(l) := I(l) \setminus I_1(l)$ be the set of sources bottlenecked elsewhere. Let $N_l := |I(l)|$ be the number of sources at link $l, N_{l0} := |I_0(l)|$, and $N_{l1} := |I_1(l)|$. Let $\rho_l := N_{l0}/N_l$ be the fraction of flows through link l which are not bottlenecked at link l, and $\sigma_l := y_{l0}/c_l$ be the fraction of the link's capacity consumed by such flows. Note that while $L(i), I(l)$, and N_l depend only on R the routing matrix, $L_1(i), L_0(i), I_1(l), I_0(l), N_{l0}, N_{l1}, \rho_l$ and σ_l depend also on the equilibrium rate x through F_{li}.

From (4) and the definition of $I_0(l)$, we have the following.

Lemma 1: The rate and backlog vector (x, b) is in equilibrium if and only if

1) for all $l, y_l \leq c_l$ with equality if $b_l > 0$, and
2) for all $i, \min_{l \in L(i)} F_{li} = 0$.

Moreover,

3) if $i \in I_0(l)$ and $j \in I_1(l)$ then $F_{li} > 0$ and $F_{lj} = 0$;
4) if $I_1(l) \neq \emptyset$ then $h_l = 0$ implies $\phi_l = 0$.

Proof: Parts 1 to 3 are immediate. To see part 4, note that $h_l = 0$ implies

$$F_{li} = \frac{1}{d_l^2} \left(\frac{\phi_l^+}{N_l x_i} - \frac{\phi_l^-}{y_l} \right)$$

By part 2, $F_{li} = 0$ for all $i \in I_1(l)$. Since at most one of ϕ_l^+ and ϕ_l^- is nonzero, $\phi_l^+ = \phi_l^- = 0$, whence $\phi_l = 0$. ∎

A. The Need for Bandwidth Shuffling

Without bandwidth shuffling, XCP would have $\gamma = 0$, giving $h_l(t) = 0$ for all l and t, and $h_l = 0$ in equilibrium.

Theorem 1: Suppose $\gamma = 0$. Then, (x, b) with $x_i = w_i/\tau_i$ is an equilibrium if and only if

1) for all $l, y_l \leq c_l$ and $b_l = 0$, and
2) for all i, there exists $l \in L(i)$ with $y_l = c_l$.

Proof: The first condition in the theorem implies that for all $l, \phi_l \geq 0$. Combined with $h_l = 0$, this implies $F_{li} \geq 0$ for all i. The second condition then implies that for all $i, \min_{l \in L(i)} F_{li} = 0$. Hence, the conditions in the theorem are sufficient, by (5b) and the first half of Lemma 1.

For necessity, there are two cases. If $I_1(l) \neq \emptyset$ then $\phi_l = 0$ by Lemma 1 part 4, and (4b) implies $y_l = c_l$ and $b_l = 0$, since $y_l \leq c_l, b_l \geq 0$, and $\beta > 0$. Otherwise, $l \in \bigcap_i L_0(i)$ and $F_{li} > 0$ by definition of $L_0(i)$. This implies $\phi_l^+ > 0$, whence $y_l < c_l, b_l = 0$ in equilibrium. ∎

Remark: Without bandwidth shuffling, any (possibly unfair) boundary point of the set $\{x \mid Rx \leq c\}$ would be an equilibrium. This is why XCP uses $\gamma > 0$ [9].

The rest of the paper considers the more complicated case of $\gamma > 0$.

B. $\gamma > 0$ *Case: Definitions and Preliminaries*

This subsection provides a conceptually simple characterization which is used in the next subsection to prove the existence and uniqueness of XCP equilibrium. Following that, we provide an iterative algorithm to compute this equilibrium.

From (4)–(5) and Lemma 1, (x, b) is an XCP equilibrium if and only if it satisfies conditions 1 and 2 of Lemma 1.

By (5a), condition 2 becomes: for all i, for all $l \in L(i)$

$$x_i \leq \frac{y_l}{N_l} \frac{h_l + \phi_l^+}{h_l + \phi_l^-} =: r^l \quad (6)$$

with equality for some $l \in L(i)$. Hence for links l with $I_1(l) \neq \emptyset$, all flows $i \in I_1(l)$ that are bottlenecked at link l must have the *common* rate r^l. This has important implications as we will see below.

Several of the results will use the following technical lemma, which is proved in Appendix A.

Lemma 2: For all l
1) $x_i < x_j = r^l$ if $i \in I_0(l)$ and $j \in I_1(l)$;
2) $\sigma_l < \rho_l$ if $I_0(l) \neq \emptyset$;
3) $r^l \geq y_l/N_l$ with equality if and only if $I_0(l) = \emptyset$;
4) $h_l > 0$ if $I_1(l) \neq \emptyset$;[2]
5) $y_l/c_l \geq \sigma_l$ with equality if and only if $I_1(l) = \emptyset$.

Unlike in the $\gamma = 0$ case, we characterize the equilibrium backlogs and rates separately. The following result says that the equilibrium queue under XCP is zero. This originates from the definition of ϕ_l in (5b), which is nonnegative in equilibrium. The same property is used in REM [22] to drive the queue to zero, or more generally, to a target value.

Theorem 2: In equilibrium, $b_l = 0$ and $\phi_l \geq 0$ for all l.

Proof: Links can be of three types: a) $I_1(l) \neq \emptyset, I_0(l) = \emptyset$; b) $I_1(l) = \emptyset, I_0(l) \neq \emptyset$; and c) $I_1(l) \neq \emptyset, I_0(l) \neq \emptyset$. Each of these will be considered in turn.

Type a) links l are bottlenecks for all flows passing through them, i.e., (6) holds with equality for all $i \in I(l)$. Since all flows have common rate r^l, $y_l = N_l r^l$, whence equality in (6) implies $\phi_l^+ = \phi_l^-$. Thus, $\phi_l = 0$, and (5b) implies $y_l = c_l$ and $b_l = 0$, i.e., they share the link capacity fully and equally, with no queueing delay.

Type b) links are not bottlenecks for any of the flows they carry. Hence, for all $i \in I(l)$,

$$x_i < \frac{y_l}{N_l} \frac{h_l + \phi_l^+}{h_l + \phi_l^-}$$

Multiplying both sides by R_{li} and summing over i, we have

$$y_l < \frac{y_l}{N_l} \frac{h_l + \phi_l^+}{h_l + \phi_l^-} \cdot \sum_i R_{li}.$$

Hence

$$\frac{h_l + \phi_l^+}{h_l + \phi_l^-} > 1.$$

Since both numerator and denominator are positive, $\phi_l^+ > \phi_l^-$. This implies $\phi_l > 0$ whence $y_l < c_l$ and $b_l = 0$.

Type c) links are bottleneck links for some but not all of the flows using them. From (6), we have

$$\frac{h_l + \phi_l^+}{h_l + \phi_l^-} = \frac{r^l}{y_l/N_l} > 1$$

where the inequality follows from Lemma 2. As for type b) links, this implies $\phi_l > 0$, $y_l < c_l$ and $b_l = 0$. ∎

We next characterize XCP's equilibrium rates. Let

$$g_l(x) := \frac{\gamma y_l^2}{N_l[(\gamma + \alpha)y_l - \alpha c_l]}$$

where $y_l = \sum_i R_{li} x_i$. Since $g_l(x)$ depends on x only through y_l, we will abuse notation and also write $g_l(y_l)$ or $g_l(y_l(x))$. Define the *feasible set* of source rates x to be

$$X_0 := \{x \in \mathbb{R}_+^N \mid g_l(y_l) < 0 \text{ or } x_i \leq g_l(y_l), \ \forall l, i \in I(l)\} \quad (7)$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. We will later show that the XCP equilibrium must be in X_0. Note that $x \in X_0$ implies

$$Rx \leq c.$$

To see this, multiply both sides of the inequalities in (7) by R_{li} and sum over i to get

$$\frac{\gamma y_l^2}{(\gamma + \alpha)y_l - \alpha c_l} \notin \left[0, \sum_i R_{li} x_i\right) = [0, y_l)$$

whence $y_l \leq c_l$. The converse may not be true, i.e., X_0 may be a strict subset of $\{x \mid Rx \leq c\}$.

Intuitively, g_l exceeds the equal-share rate y_l/N_l by a factor that increases as the link underutilization increases and as the bandwidth shuffling decreases. It arises since the link tries to shuffle bandwidth from flows it controls to ones it does not. Thus, X_0 restricts the utilization of links which are not bottlenecks for all flows passing through them.

Our main result is to prove the existence and uniqueness of XCP equilibrium in a general network, and that this equilibrium solves a *constrained* max-min fairness problem.

Definition 2: A rate vector $x^* \in X_0$ is *constrained max-min fair* if for any other feasible $x \in X_0$, $x_i > x_i^*$ implies that there is a j with $x_j < x_j^*$ and $x_j^* \leq x_i^*$.

Intuitively, a constrained max-min fair vector x^* is such that it is not possible to increase a component x_i^* without reducing another smaller or equal component x_j^*. This differs from standard max-min fairness only in that the feasible set X_0 is a subset of $\{x \mid Rx \leq c\}$ [23]. This restriction has important ramifications, as we will see in the next section.

We will prove constructively that the unique XCP equilibrium is constrained max-min fair by identifying it with the solution of

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[2]$h_l > 0$ implies $y_l/c_l > \alpha/(\alpha + \gamma)$. Theorem 6 provides a tighter lower bound.

a hierarchy of optimization problems over the feasible set X_0: it maximizes the smallest source rates in X_0, and then maximizes the second smallest rates over all rates that solve the first problem, and so on. These problems are defined inductively, following the idea of [24].

Let $L_0 = \emptyset$ and $I_0 = \emptyset$. The sets (L_0, I_0, X_0) define the first problem $\mathbf{P}_1$, whose solution is described by the sets (L_1, I_1, X_1). These sets in turn define the second problem $\mathbf{P}_2$, and so on. To simplify notation, let

$$\bar{L}_n := \bigcup_{m \leq n} L_m \quad \bar{I}_n := \bigcup_{m \leq n} I_m.$$

Given sets $(X_0, L_0, I_0), \ldots, (X_{n-1}, L_{n-1}, I_{n-1})$, if $\bar{I}_{n-1}$ contains all flows, then we stop. Otherwise, we define problem $\mathbf{P}_n$ and its solution $L_n, I_n, X_n, n \geq 1$, as follows:

$$\mathbf{P}_n : \max_{x \in X_{n-1}} \min_{i \notin \bar{I}_{n-1}} x_i. \tag{8}$$

Let

$$r_n := \min_{l \notin \bar{L}_{n-1}} \max_{x \in X_{n-1}} g_l(x) \tag{9}$$

$$L_n := \{\text{minimizing } l \text{ in } (9)\} \tag{10}$$

$$I_n := \bigcup_{l \in L_n} I(l) \setminus \bar{I}_{n-1} \tag{11}$$

$$X_n := \left\{ x \in X_{n-1} \middle| x_i \begin{cases} = r_n, & \forall i \in I_n \\ > r_n, & \forall i \notin \bar{I}_n \end{cases} \right\}. \tag{12}$$

A few important properties are immediate from these definitions. First, the rates r_n are monotonic

$$\min_l \frac{c_l}{N_l} = r_1 < r_2 < \cdots < r_n. \tag{13}$$

Second, L_n and I_n are nonempty; moreover, they are disjoint from $\bar{L}_{n-1}$ and $\bar{I}_{n-1}$, respectively. Hence, $\bar{I}_n$ will eventually contain all the flows and there are only a finite number of problems $\mathbf{P}_n$. Finally, X_n are *strictly* nested

$$X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n.$$

Indeed it will become clear that X_n is exactly the set of solutions to problem $\mathbf{P}_n$, i.e., X_1 is the set of feasible rates $x \in X_0$ whose smallest rates are maximized, X_2 is a subset of X_1 whose second smallest rates are also maximized, and so on. We prove below that if $\mathbf{P}_{n^*}$ is the last problem, then X_{n^*} is a singleton that solves all problems $\mathbf{P}_1, \ldots, \mathbf{P}_{n^*}$.

To contrast XCP equilibrium with the standard max-min fair allocation, we derive a "bottleneck" characterization that is analogous to that for max-min fairness; see the beginning of Section IV.

Lemma 3: Let x be an XCP equilibrium rate vector. Link l is a bottleneck for source $i \in I(l)$ w.r.t. x if and only if
  1) $x_i = g_l(x)$, and
  2) $x_i \geq x_j$ for all $j \in I(l)$.
  Proof: Suppose link l is a bottleneck link for source i w.r.t. equilibrium x. Then, Lemma 1-2 implies that $F_{li} = 0$, i.e., equality holds in (6). Since $\phi_l \geq 0$ by Theorem 2 and $h_l > 0$ by Lemma 2, (5c) becomes $h_l = \gamma d_l y_l - \phi_l$. Thus from (6)

$$x_i = r^l = \frac{y_l}{N_l} \frac{\gamma d_l y_l}{\gamma d_l y_l - \phi_l} = g_l(x) \tag{14}$$

proving the first condition. Condition (6) then implies the second condition.

Conversely, suppose the two conditions are satisfied. If $h_l = 0$, then $F_{li} = 0$ from (5a) and Lemma 1-4. Lemma 1-2 then implies F_{li} is the minimum among links in source i's path, i.e., link l is a bottleneck. On the other hand, if $h_l > 0$, then, as above, $\phi_l \geq 0$ and $h_l = \gamma d_l y_l - \phi_l$. Then, $x_i = g_l(x)$ is equivalent to $F_{li} = 0$, proving that l is a bottleneck. $\blacksquare$

Motivated by this lemma, we call link l *nonbottleneck w.r.t.* x if either $g_l(x) < 0$ or $x_i < g_l(x)$ for all $i \in I(l)$.

C. $\gamma > 0$ *Case: Main Result*

We are now ready to state the main result.

Theorem 3: The problems $\mathbf{P}_n$ are well-defined and have a unique solution. Moreover, the following are equivalent:
  1) x^* is an XCP equilibrium.
  2) x^* is the unique rate vector that solves all the problems $\mathbf{P}_n$.
  3) x^* is constrained max-min fair.
  4) $x^* \in X_0$ and every flow has a bottleneck w.r.t. x^*, i.e., for all i, there is an $l \in L(i)$ such that $x_i^* = g_l(x^*)$ and $x_i^* \geq x_j^*$ for all $j \in I(l)$.

The remainder of this subsection is a proof of this theorem. It will use lemmas, whose proofs are relegated to Appendix B. We start with a simple observation that greatly simplifies the solution of $\mathbf{P}_n$.

Lemma 4: Suppose X_n is nonempty. The maximization in (9) can be taken over $x \in X_n$ that have equal x_i for $i \notin \bar{I}_n$.

In view of Lemma 4, we can replace X_n in (12), for $n \geq 1$, by their subsets

$$\hat{X}_n := \left\{ x \in X_{n-1} \middle| \exists \epsilon > 0 : x_i = \begin{cases} r_n, & i \in I_n \\ r_n + \epsilon, & i \notin \bar{I}_n \end{cases} \right\} \tag{15}$$

and use them instead of X_{n-1} in computing r_n:

$$r_n := \min_{l \notin \bar{L}_{n-1}} \max_{x \in \hat{X}_{n-1}} g_l(x).$$

This greatly reduces the complexity of (9) from maximizing over n-vectors $x \in X_n$ to over a scalar $\epsilon > 0$.

Denote an $x \in \hat{X}_n$ by $x(\epsilon; n)$, with

$$x_i(\epsilon; n) = \begin{cases} r_m, & i \in I_m, m \leq n \\ r_n + \epsilon, & i \notin \bar{I}_n \end{cases} \tag{16}$$

and let $x(0; n) := \lim_{\epsilon \to 0} x(\epsilon; n)$. Note that $x(0; n), n \geq 1$, is not in X_n according to definition (12), though it is in X_0. We will see in Lemma 6 below that $x(0, n)$ plays an important role in the proof of Theorem 3. The vector $x(\epsilon; n)$ induces link flows

$$\begin{aligned} y_l(\epsilon; n) &= \sum_i R_{li} x_i(\epsilon; n) \\ &= \sum_{m=1}^{n} r_m \sum_{i \in I_m} R_{li} + r_n \sum_{i \notin \bar{I}_n} R_{li} + \epsilon \sum_{i \notin \bar{I}_n} R_{li}. \end{aligned} \tag{17}$$

This motivates the following main technical lemma.

Lemma 5: Given any scalars $z \geq 0, \eta > 0$, and $s \geq 0$, define

$$\hat{g}_l(\epsilon) := \hat{g}_l(\epsilon; z, \eta) := g_l(z + \eta\epsilon)$$
$$= \frac{\gamma(z + \eta\epsilon)^2}{N_l((\gamma + \alpha)(z + \eta\epsilon) - \alpha c_l)} \quad (18)$$

for some $N_l \geq 1, \alpha > 0, \gamma > 0$ and $c_l > 0$.

1) If either $\hat{g}_l(0) < 0$ or $s \leq \hat{g}_l(0)$, then there exists a unique $\epsilon_l \geq 0$ such that $s + \eta\epsilon_l = \hat{g}_l(\epsilon_l)$, where $\epsilon_l = 0$ if and only if $s = \hat{g}_l(0)$.
2) Moreover, over $\{\epsilon \mid \hat{g}_l(\epsilon) > 0\}$, $s + \eta\epsilon \leq \hat{g}_l(\epsilon)$ if and only if $\epsilon \leq \epsilon_l$.

For later reference, we will denote the mapping from (z, η, s) to the unique ϵ_l in Lemma 5 by

$$G_l(z, \eta, s) = \epsilon_l. \quad (19)$$

This function is used in the rest of the proof.

Lemma 5 implies that if link l is a bottleneck for some source i with respect to an $x \in X_0$, then the rate of source i cannot be increased without violating the feasibility constraint in (7). For instance, let $n \geq 1$ be such that $l \in L_n$. Setting $z = y_l(0; n - 1), s = r_{n-1}$ and $\eta = \sum_{i \notin \bar{I}_{n-1}} R_{li}$ gives $\epsilon_l = r_n - r_{n-1}$ and l is a bottleneck for all $i \in I_n$ w.r.t. $x(\epsilon_l; n - 1)$. Lemma 5-2 then implies that rates greater than r_n are infeasible at link l.

The next lemma implies that all links $l \in L_n$ are bottlenecks w.r.t. all $x \in \hat{X}_n$, and all links $l \notin \bar{L}_n$ are nonbottlenecks w.r.t. $x(0; n)$. In particular, this implies that X_n are nonempty.

Lemma 6: For each $n \geq 1$,

1) if $l \in L_n$, then $x_i = g_l(x)$ for all $i \in I_n$ w.r.t. all $x \in \hat{X}_n$.
2) if $l \notin \bar{L}_n$, then either $g_l(x(0; n)) < 0$ or $x_i(0; n) < g_l(x(0; n))$ for all $i \in I(l)$.

Lemmas 5 and 6 suggest the following abstract algorithm to compute the solution of problems $\mathbf{P}_n$. An analogous concrete algorithm to find the XCP equilibrium will be given in the Subsection III-D. At the start of iteration $n + 1$, all links $l \in L_m, m \leq n$, are bottlenecks for some sources $i \in I_m$ w.r.t. all $x \in X_m$. A source $i \in \bar{I}_n$ passes through at least one bottleneck $l \in \bar{L}_n$, and hence its rate cannot be raised further without violating the constraint in (7). All links $l \notin \bar{L}_n$ are nonbottleneck links w.r.t. $x(0; n)$ defined in (16). Sources $i \notin \bar{I}_n$ pass through only these nonbottlenecks, and hence their rates can be increased further, starting from r_n. At each nonbottleneck link l, nonbottlenecked sources $i \notin \bar{L}_n$ can raise their rates to (19), (16), and (17)

$$r_n + G_l\left(\sum_{m=1}^{n} r_m \sum_{i \in I_m} R_{li} + r_n \sum_{i \notin \bar{I}_n} R_{li}, \sum_{i \notin \bar{I}_n} R_{li}, r_n\right)$$

to make l a bottleneck. The smallest of these rates, smallest over $l \notin \bar{L}_n$, is r_{n+1}. It is assigned to all previously nonbottlenecked sources going through the new bottleneck links and is the optimal objective value for problem $\mathbf{P}_n$. These new bottleneck links are collected into L_{n+1}, the newly bottlenecked sources into I_{n+1} and their rates into $\hat{X}_{n+1}$ (or X_{n+1}). The

other nonminimizing links remain nonbottleneck w.r.t. the new rates $x \in \hat{X}_{n+1}$, and the cycle repeats, until all sources are assigned their bottleneck rates.

The solution of each problem $\mathbf{P}_n$ fixes the components $x_i, i \in I_n$, to be rate r_n, until all components have been assigned. Hence, if $\mathbf{P}_{n^*}$ is the last problem, then $X_{n^*} = \{x^*\}$ is a singleton.

The above discussion is summarized in the following lemma, which justifies Theorem 3.

Lemma 7: The problems $\mathbf{P}_n$ are well defined. X_n is exactly the set of solutions to problem $\mathbf{P}_n$. There is a unique solution to the hierarchy of problems.

We now prove Theorem 3.

Proof (Theorem 3): Lemma 7 implies that $\mathbf{P}_n$ are well defined and have a unique solution. It is clear that characterizations 2 and 3 are equivalent, i.e., x^* is the unique solution to the hierarchy of problems $\mathbf{P}_n$ if and only if it is constrained max-min fair. We will first prove the equivalence of characterizations 3 and 4, and then that of 1 and 4. We will use the equivalent definition of bottleneck links in Lemma 3.

Equivalence of characterizations 3 and 4: We will prove that x^* is constrained max-min fair if and only if both $x^* \in X_0$ and every flow i has a bottleneck link w.r.t. x^*, i.e., for all i, there is an $l \in L(i)$ such that $x_i^* = g_l(x)$ and $x_i^* \geq x_j^*$ for all $j \in I(l)$. The proof follows the same approach as the corresponding result for standard max-min fairness; see [23]. The difference is in the use of Lemma 5 because of the more complicated feasible set X_0.

Suppose $x^* \in X_0$ and every flow i has a bottleneck link w.r.t. x^*. If x^* is not constrained max-min fair, then there exists another $x \in X_0$ such that $x_i > x_i^*$ for some i, and if $x_j^* \leq x_i^*$ then $x_j \geq x_j^*$. We will derive a contradiction. Let $l \in L(i)$ be a bottleneck for i w.r.t. x^*. Then, $x_i^* \geq x_j^*$ for all $j \in I(l)$, and hence $\epsilon_j := x_j - x_j^* \geq 0$ for all $j \in I(l)$, with $\epsilon_i > 0$. Write the link flow $y_l(x)$ due to rates x in terms of ϵ_j and the link flow $y_l(x^*)$ due to x^*

$$y_l(x) = \sum_j R_{lj}(x_j^* + \epsilon_j)$$
$$= y_l(x^*) + \sum_j R_{lj}\epsilon_j.$$

Let the scalar ϵ be the average ϵ_j

$$\epsilon = \frac{\sum_j R_{lj}\epsilon_j}{\sum_j R_{lj}} > 0.$$

Then, the rate vector $\hat{x}$ defined by $\hat{x}_j = x_j^* + \epsilon$ if $j \in I(l)$ and $\hat{x}_j = x_j^*$ otherwise induces the same flow rate at link l as x does

$$y_l(\hat{x}) = y_l(x^*) + \epsilon \cdot \sum_j R_{lj} = y_l(x). \quad (20)$$

Since $\epsilon \leq \max_j \epsilon_j$, and x is feasible, we must have, for all $j \in I(l)$

$$\hat{x}_j \leq \max_{\hat{j} \in I(l)} x_{\hat{j}} \leq g_l(y_l(x))$$
$$= g_l(y_l(\hat{x})) \quad (21)$$

where the last equality follows from (20). Hence, $\hat{x}$ is also feasible. Since link l is a bottleneck for source i w.r.t. x^*

$$x_i^* = g_l(y_l(x^*)) > 0 \tag{22}$$

and so applying Lemma 5-1 with $z = y_l(x^*)$, $s = x_i^*$ and $\eta = \sum_j R_{lj}$ gives $\epsilon_l = 0$. Since $\hat{x}_j = x_j^* + \epsilon$ for all $j \in I(l)$, with $\epsilon > \epsilon_l$, Lemma 5-2 and (20) imply that

$$\hat{x}_i > g_l(y_l(\hat{x}))$$

contradicting (21). Hence, x^* is constrained max-min fair.

Conversely, let $x^* \in X_0$ be constrained max-min fair. If there is a source i that has no bottleneck link w.r.t. x^*, then for all $l \in L(i)$, either $g_l(y_l(x^*)) < 0$ or $x_i^* < g_l(y_l(x^*))$ for all $i \in I(l)$. Lemma 5 then implies that there exists a unique scalar $\epsilon_i > 0$, given by

$$\epsilon_i = \min_{l \in L(i)} G_l(y_l(x^*), 1, x_i^*)$$

such that $x \in X_0$, given by $x_i = x_i^* + \epsilon_i$ and $x_j = x_j^*$ for $j \neq i$, strictly increases component i without having to reduce other components j, contradicting the fact that x^* is constrained max-min fair.

Equivalence of characterizations 1 and 4: We will prove that a vector $x^* \in \mathbb{R}_+^N$ is an XCP equilibrium if and only if $x^* \in X_0$ and every flow i has a bottleneck link w.r.t. x^*.

The discussion at the beginning of Section III-B shows that x^* is an XCP equilibrium if and only if, for all i, (6) holds for all $l \in L(i)$, with equality for some $l \in L(i)$. This, with (14), establishes $x^* \in X_0$. As observed after Definition 1, every flow has a bottleneck by definition.

To show characterization 4 implies characterization 1, it suffices to show that the characterization in Lemma 3 implies statements 1 and 2 of Lemma 1. The discussion after (7), and setting $b = 0$, establishes statement 1. This shows $\phi_l^- = 0$ for all l. If $x_i \leq g_l(x)$ then (5c) and (5a) give $F_{li} \geq 0$, with equality when $x_i = g_l(x)$. Otherwise, $g_l(x) < 0$ giving $h = 0$ and, by (5a), $F_{li} \geq 0$. ∎

D. Algorithm for Computing Equilibrium

The equilibrium rates of XCP can be found using an algorithm analogous to that of [23] for max-min fairness. However, because the constraint on the link throughput in (6) depends on the aggregate flow rate through h_l and ϕ_l, some extra bookkeeping is required.

In [20], an approximation for y_l was given in terms of σ_l and ρ_l. Here, we use the exact expression from [19].

Theorem 4: The utilization of a bottleneck l satisfies

$$\frac{y_l}{c_l} = \frac{\alpha + (\gamma + \alpha)\sigma_l + \sqrt{(\alpha - (\gamma + \alpha)\sigma_l)^2 + 4\alpha\gamma\sigma_l(1 - \rho_l)}}{2(\gamma\rho_l + \alpha)}. \tag{23}$$

The rates of all sources $i \in I_1(l)$ bottlenecked at l satisfy

$$r^l = \frac{c_l}{N_l} \times \frac{[\Xi_l - \gamma\sigma_l\rho_l] + \sqrt{[\Xi_l + \gamma\sigma_l\rho_l]^2 - 4\alpha\gamma\sigma_l(\rho_l - \sigma_l)}}{2(1 - \rho_l)(\gamma\rho_l + \alpha)} \tag{24}$$

where

$$\Xi_l = (\gamma\sigma_l + \alpha)(1 - \sigma_l) - \gamma\sigma_l(\rho_l - \sigma_l). \tag{25}$$

Proof: Substituting $r^l = (y_l - y_{l0})/(N_l - N_{l0})$ into (14) and solving the resulting quadratic equation gives

$$y_l = \frac{\alpha c_l + (\gamma + \alpha)y_{l0} \pm \sqrt{(\alpha c_l + (\gamma + \alpha)y_{l0})^2 - K}}{2(\gamma N_{l0}/N_l + \alpha)} \tag{26}$$

where $K = 4\alpha c_l y_{l0}(\gamma N_{l0}/N_l + \alpha)$. By Lemma 8 in Appendix A, only the larger solution of (26) satisfies part 5 of Lemma 2 and is a valid equilibrium. Rearranging the term in the square root gives (23).

To obtain (24), instead substitute $y_l = (N_l - N_{l0})r^l + y_{l0}$ into (14), giving

$$A_l \left(\frac{r^l N_l}{c_l}\right)^2 + B_l \frac{r^l N_l}{c_l} + C_l = 0 \tag{27}$$

where

$$\begin{aligned} A_l &= (1 - \rho_l)(\gamma\rho_l + \alpha) \\ B_l &= 2\gamma\sigma_l\rho_l - \gamma\sigma_l - \alpha(1 - \sigma_l) \\ C_l &= -\gamma\sigma_l^2. \end{aligned}$$

Since y_l is increasing in r^l, it is again only the larger root that represents the XCP equilibrium. Thus

$$\frac{r^l N_l}{c_l} = \frac{[\Xi_l - \gamma\sigma_l\rho_l] + \sqrt{[\Xi_l - \gamma\sigma_l\rho_l]^2 - 4A_l C_l}}{2(1 - \rho_l)(\gamma\rho_l + \alpha)}$$

where Ξ_l is given in (25). Rearranging the expression in the square root gives (24). ∎

Note that the right-hand side of (24) depends on the rate vector x through ρ_l and σ_l. Hence, it is not an explicit formula for the throughput of a general flow. However, it says that the common "bottleneck" rate at each link l depends on the rate vector x only through y_{l0} and N_{l0} that are bottlenecked elsewhere. These are source rates smaller than the "bottleneck" rate at link l by Lemma 1. This motivates an algorithm similar to the max-min algorithm of [23] that calculates the throughput x_i of each flow in increasing order, without the need for recourse to simulation.

1   Set $\bar{I}_0 \leftarrow \emptyset$, $\bar{L}_0 \leftarrow \emptyset$, $\sigma_l(0) \leftarrow 0$, $\rho_l(0) \leftarrow 0$ for all l, $n \leftarrow 1$

2   **repeat**

    2.1 For each link, $l \notin L_{n-1}$ find $r^l(n)$ from (24) using $\sigma_l(n-1)$ and $\rho_l(n-1)$ from rates already allocated

    2.2 Set $r_n \leftarrow \min_j r^j(n)$

    2.3 Set $L_n \leftarrow \{l : r^l(n) = r_n\}$

    2.4 **foreach** $l \in L_n$

       2.4.1 Set $r^l \leftarrow r_n$

       2.4.2 For each flow $i \in I(l) \setminus \bar{I}_n$, set $x_i \leftarrow r^l$

      endfor

    2.5 Set $I_n \leftarrow \bigcup_{j \in L_n} I(j) \setminus \bar{I}_n$

2.6 Set $\bar{I}_n \leftarrow \bar{I}_{n-1} \cup I_n$
2.7 Set $\bar{L}_n \leftarrow \bar{L}_{n-1} \cup L_n$
2.8 **foreach** $l \in \bigcup_{i \notin \bar{I}_n} L(i)$
    2.8.1 Set $\sigma_l(n) \leftarrow \sigma_l(n-1) + \sum_{i \in I_n} R_{li} x_i / c_l$
    2.8.2 Set $\rho_l(n) \leftarrow \rho_l(n-1) + \sum_{i \in I_n} R_{li} / N_l$
    endfor
2.9 Set $n \leftarrow n + 1$
until $\bar{I}_n = \{\text{all flows}\}$

This solves each of the optimization problems, $\mathbf{P}_n$, in turn. The key is that, by keeping track of the used capacity of each link, $\sigma_l(n)$ and $\rho_l(n)$, it can compute the maximization in (9) in closed form. For each l, the values $\sigma_l(n)$ and $\rho_l(n)$ vary during the algorithm. For the algorithm to be correct, they must have the right values when link l is the minimum in step 2.2. This occurs as long as the link rates are allocated in increasing order, as is guaranteed by the following theorem, proved in Appendix C.

Theorem 5: The above algorithm calculates the correct equilibrium rates of XCP.

If $\gamma = 0$, then (24) reduces to $r^l = (c_l - y_{l0})/(N_l - N_{l0})$, and hence the algorithm reduces to the algorithm in [23] to compute the max-min fair allocation. This suggests that, given any topology specified by the routing matrix R and link capacity vector c, one can choose $\gamma > 0$ to be sufficiently small so that the equilibrium of (4) is close to max-min fair. On the other hand, with small γ, the convergence of individual rates to fairness can be very slow. We will return to this point in the following section.

IV. UTILIZATION AND FAIRNESS

In this section, we discuss some implications of the results in Section III on link utilization and fairness of the equilibrium rates. Theorem 3 shows that XCP equilibrium is constrained max-min fair. It is instructive to compare the XCP equilibrium with the (standard) max-min fair allocation and a class of algorithms proposed in [24].

It is proved in [24] that a (standard) max-min fair rate vector x^* is the unique solution of the same hierarchy of problems $\mathbf{P}_n$ (8)–(12) defined in Section III, except that the feasible set X_0 in (7) is replaced with the superset

$$\bar{X}_0 := \left\{ x \in \mathbb{R}_+^N \mid Rx \le c \right\}. \tag{28}$$

The key feature that results from this much simpler feasible set $\bar{X}_0$ is that the bottleneck links under a max-min fair allocation are all fully utilized. Indeed, a rate vector $x^* \in \bar{X}_0$ is max-min fair if and only if, for every source i, there is a link $l \in L(i)$ in its path such that [23]
  1) $y_l(x^*) = c_l$
  2) $x_i^* \ge x_j^*$ for all $j \in I(l)$.
From Theorem 3, condition 1 is replaced with the fixed point equation $x_i^* = g_l(y_l(x^*))$ for XCP equilibrium. This more complex condition has several implications.

First, it precludes the much simpler proof techniques used in [24] to show the max-min fair vector is the unique solution of the problems $\mathbf{P}_n$.

Second, the algorithm in Section III-D to compute the constrained max-min fair rate vector is more complex than the (centralized) one for the max-min fair vector [24], [23].

Third, and most importantly, the XCP equilibrium can underutilize link capacities and deviate by an arbitrarily large factor from the max-min fair allocation, as we illustrate below.

Max-min fairness is generalized in [24] by restricting the feasible set to a (strict) subset of $\bar{X}_0$ in (28). Like XCP, the restriction is specified as additional constraints on source rates x_i and link flows y_l. Like XCP, explicit feedback is required: Each link l feeds back the spare capacity $c_l - y_l(t)$ to sources that go through this link. Sources adjust their individual rates based on feedback on its path in a way that is distributed, yet avoids overshoot. MaxNet [16] also allows explicit control of link utilization and fairness by separate link and source parameters.

We now illustrate the effect of the additional constraint (7) in XCP on link utilization and fairness.

As we explained in the proof of Theorem 2, there are three types of links. The first type are bottlenecks for all the flows that go through that link. All links of this type, such as all $l \in L_1$ in problem $\mathbf{P}_1$, are fully utilized, $y_l = c_l$. The second type are bottlenecks for none of the flows that go through that link. They are underutilized, $y_l < c_l$, because the flow rates going through the link are constrained elsewhere. The third type are bottlenecks for some, but not all, of the flows that go through the link. In contrast to the standard max-min fair allocation, these links are also underutilized, $y_l < c_l$. We can bound the utilization of these partial bottlenecks.

Theorem 6: If $l \in L_1(i)$ for some i, then

$$\frac{\alpha}{\gamma \rho_l + \alpha} \le \frac{y_l}{c_l} \le 1 - \frac{\gamma \sigma_l (\rho_l - \sigma_l)}{\gamma \rho_l + \alpha}.$$

Proof: Noting that $\rho_l < 1$ (and that $2(\gamma \rho_l + \alpha) > 0$), removing the last term from the square root in (23) gives the lower bound

$$\frac{y_l}{c_l} \ge \frac{\alpha + (\gamma + \alpha)\sigma_l + |(\alpha - (\gamma + \alpha)\sigma_l)|}{2(\gamma \rho_l + \alpha)}$$
$$\ge \frac{\alpha}{\gamma \rho_l + \alpha} \tag{29}$$

where the second inequality is an equality if $\sigma_l \le \alpha/(\gamma + \alpha)$.

To derive the upper bound, first note that $\rho_l \ge \sigma_l$ from Lemma 2-2. Since $2(1 - \rho_l)(\gamma \rho_l + \alpha) > 0$ and $\Xi \ge 0$, removing the last term from the square root of (24) yields

$$\frac{r^l N_l}{c_l} \le \frac{1 - \sigma_l}{1 - \rho_l} - \frac{\gamma \sigma_l (\rho_l - \sigma_l)}{(1 - \rho_l)(\gamma \rho_l + \alpha)}. \tag{30}$$

Multiplying both sides by $(1 - \rho_l)$ and adding σ_l lead to the upper bound on utilization. ∎

Substituting either $\gamma = 0$ or $\rho_l = \sigma_l$ into either the exact expressions (23) and (24) or the upper and lower bounds (29) and (30) gives full utilization as in the max-min case: $y_l = c_l$ and $r^l = c_l(1 - \sigma_l)/(N_l(1 - \rho_l))$. This shows that XCP could be made to approach max-min fairness if the bandwidth shuffling were reduced.

On the other hand, link utilization could be arbitrarily low if α and γ had been chosen poorly. With the values suggested in [9] however the utilization is at least 80%. Consider a network of two links. Link 1 has $c_1 = 1$ and carries N_1 flows, while link 2 has $c_2 = 1 + \gamma/\alpha$ and carries $N_2 = N_1 + 1$ flows, consisting of all the traffic on link 1 plus one other flow. As $N_1 \to \infty$ we

get $\rho_2 \to 1$. This gives $\sigma_2 = \alpha/(\gamma + \alpha)$ in the limit. Thus, both terms in the square root of (23) go to zero, whence (29) becomes tight, and $y_2/c_2 \to 0$ as $\gamma/\alpha \to \infty$. However, with $\alpha = 0.4$ and $\gamma = 0.1$ [9], for which positive stability results exist, (23) gives $y_l/c_l = 0.8$.

Similarly, a given flow may obtain an arbitrarily small proportion of its max-min fair bandwidth for *any* $\alpha > 0$ and $\gamma > 0$. The ratio of the upper bound on XCP bandwidth (30) to the max-min fair bandwidth, $r^{l,\mathrm{mm}} = c_l(1 - \sigma_l)/(N_l(1 - \rho_l))$, is minimized with respect to σ_l when $\rho_l = 2\sigma_l - \sigma_l^2$. Substituting this value into (24) and dividing by $r^{l,\mathrm{mm}}$ gives

$$\frac{r^l}{r^{l,\mathrm{mm}}} = \frac{C - \frac{D}{1-\sigma_l} + \sqrt{\left[C + \frac{D}{1-\sigma_l}\right]^2 - \frac{E}{1-\sigma_l}}}{2\left(\gamma\left(2\sigma_l - \sigma_l^2\right) + \alpha\right)} \quad (31)$$

where

$$C = \Xi_l/(1 - \sigma_l) = \gamma\sigma_l(1 - \sigma_l) + \alpha$$
$$D = \gamma\sigma_l^2(2 - \sigma_l)$$
$$E = 4\gamma\sigma_l^2\alpha.$$

Thus

$$\frac{r^l}{r^{l,\mathrm{mm}}} = \frac{C - \frac{D}{1-\sigma_l} + \frac{D}{1-\sigma_l}\sqrt{1 + \frac{(1-\sigma_l)^2}{D^2}\left(C^2 + \frac{2CD - E}{1-\sigma_l}\right)}}{2\left(\gamma\left(2\sigma_l - \sigma_l^2\right) + \alpha\right)}$$

Applying the identity $\sqrt{1 + x} \le 1 + x/2$, for $x \ge -1$, gives

$$\frac{r^l}{r^{l,\mathrm{mm}}} \le \frac{C + \frac{1-\sigma_l}{2D}C^2 + \frac{2CD - E}{2D}}{2(\gamma\sigma_l(2 - \sigma_l) + \alpha)}. \quad (32)$$

In the limit as $\sigma_l \to 1$, the right-hand side tends to 0 for any $\gamma \neq 0$. This demonstrates that, for any nonzero amount of bandwidth shuffling, XCP can be arbitrarily unfair for some topology.

Hence, although the equilibrium of (4) converges to max-min as $\gamma \to 0$, this convergence is not uniform with respect to topology. In other words, given any topology specified by (R, c), we can choose γ sufficiently small so that the resulting allocation is close to max-min fairness. However, for any fixed $\gamma > 0$, such as 0.1 used by XCP, there are topologies in which some source rates can be far away from their max-min allocations.

This behavior can be exhibited by a simple two-link network: One link has capacity 1 and carries n^2 flows, while the other carries $n^2 - 1$ of those same flows and has capacity $(n-1)/n$. This network has $\sigma_2 = (n-1)/n$ and $\rho_2 = (n^2 - 1)/n^2 = 2\sigma_2 - \sigma_2^2$. Hence, $\sigma_2 \to 1$ as $n \to \infty$ and $r^2/r^{2,\mathrm{mm}} \to 0$.

These asymptotic results will be illustrated and confirmed by simulation in the following section.

V. SIMULATION RESULTS

This section presents simulation results using the implementation available from [9] for NS-2 [25]. These results verify the accuracy of our algorithm in Section III-D and confirm our qualitative discussion in Section IV on the utilization and fairness properties of XCP.
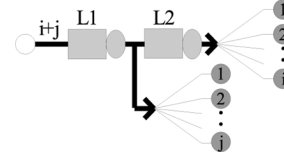

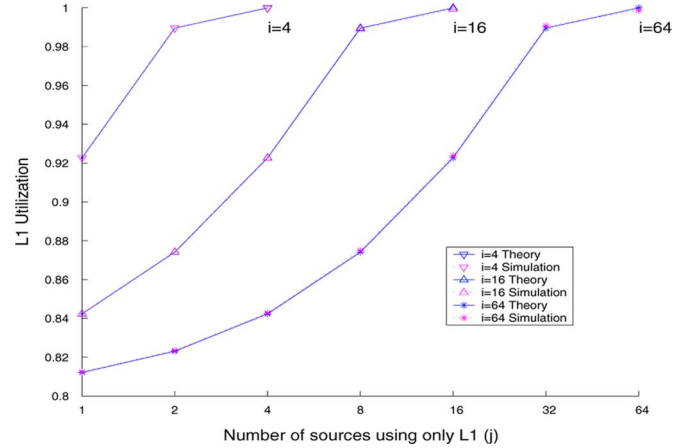Fig. 1. Topology for Scenarios 1 and 2.

Fig. 2. Scenario 1: Utilization.

All sources always have packets to send. Links are randomly assigned delays between 5 and 500 ms, uniformly on a log scale. Results for different realisations were indistinguishable. The XCP default parameters $\alpha = 0.4, \beta = 0.226$ and $\gamma = 0.1$ are used. Although the analysis neglects the "residual" terms, the simulations include them. However, as remarked in Section II-B, they have minimal impact on equilibrium properties, yielding a good match between theory and simulation. Although XCP has been found to be unstable in specific circumstances [26], all of the simulations here exhibited stable dynamics.

The topology used for Scenarios 1 and 2 is shown in Fig. 1 and consists of two links, with $i + j$ sources traversing link L1 and j sources traversing L2.

Using protocols that have nonempty queues in eqilibrium, flows in each direction are coupled through changes in the RTT and a process known as ACK compression. Since XCP maintains empty queues at eqilibrium, it is sufficient to consider unidirectional flows, which we do here for clarity.

Scenario 1 investigates the utilization of L1 as the number of sources traversing L1 and L2 is changed. In the experiment $i \ge j$, with $c_1 = 155$ Mbps and $c_2 = 100$ Mbps. The utilization of L1 for a range of i and j is shown in Fig. 2. A max-min fair allocation would result in a full utilization of L1 for all i and j combinations. However, as the number of sources bottlenecked at L2 increases, XCP's utilization of L1 decreases.

Since XCP's "residual" terms depend on feedback from upstream nodes, the equilibrium rates depend on the order in which links are traversed. If the direction of flow in this network were reversed, then the utilization would be 0%–4% higher than for the case considered and than the theoretical predictions.

Scenario 2 demonstrates that XCP can be arbitrarily unfair for some topology. Let $c_1 = 155$ Mbps, $c_2 = c_1(n - 1)/n, i =$
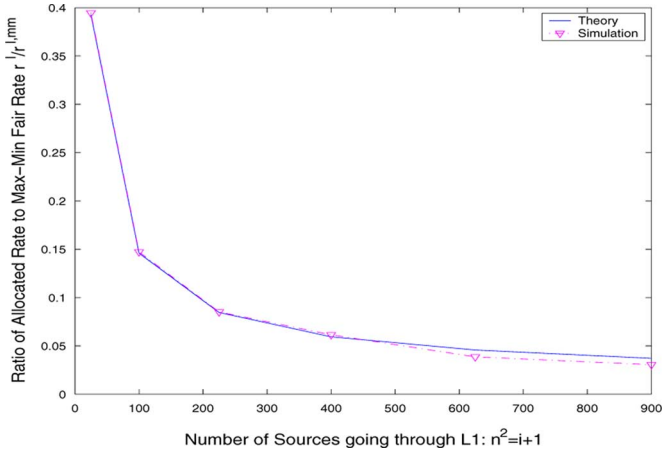
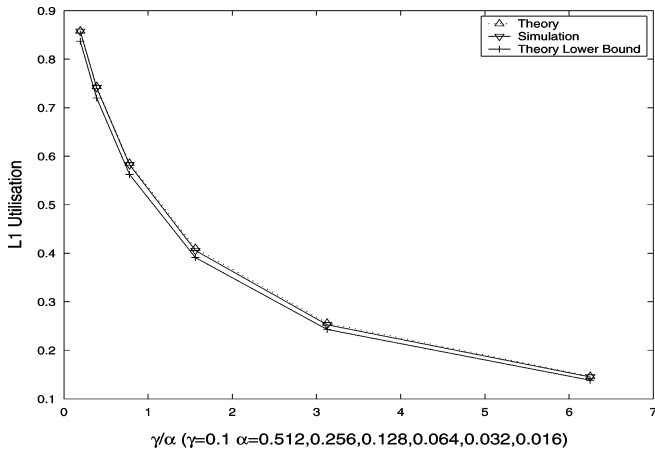Fig. 3. Scenario 2: Unfairness, $r^2/r^{2,\mathrm{mm}} \to 0$ as $n \to \infty$.

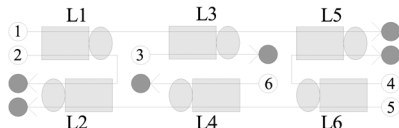Fig. 4. Scenario 3: Utilization of L1 as function of γ/α.

Fig. 5. Scenario 3 topology.

$n^2 - 1$ and $j = 1$. The ratio of the rate of the source traversing only L1 to the max-min fair rate is plotted in Fig. 3. Indeed the unfairness increases with the number of sources in the network, confirming the theory.

Scenario 3 studies XCP with nonstandard parameters. It verifies that $y_2/c_2 \to 0$ as $\gamma/\alpha \to \infty$. We set $c_2 = 200$ Mbps, $c_1 = c_2(1 + \gamma/\alpha)$, $i = 256$ and $j = 1$. The parameter α is varied from 0.512 to 0.016 and the utilisation of L1 as a function of γ/α, as well as the lower bound from (29), are plotted in Fig. 4.

Scenario 4 tests the rate allocation algorithm for a more complicated topology as shown in 5. The link capacities in Mbps are $c_1 = 10, c_3 = 8, c_4 = 8, c_5 = 7, c_6 = 6$ and c_2 is varied in this experiment. Delay is set to $d_l = 10$ ms. The source rates are plotted in Fig. 6. There is a good agreement between the predicted and measured rates even though the lower bandwidth delay product makes the fluid flow approximation more questionable.
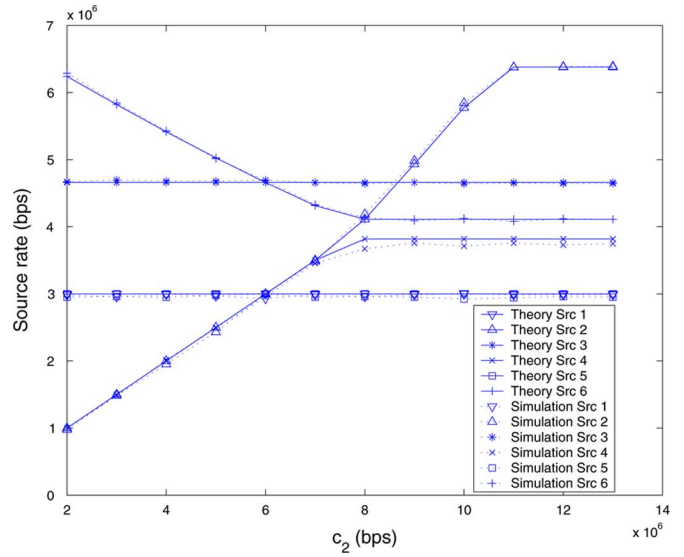

Fig. 6. Scenario 4: Throughputs.

VI. CONCLUSION

We have presented a dynamic model of XCP and used it to completely characterize its equilibrium properties. We have shown that XCP clears the queues in equilibrium and has unique equilibrium rates that solve a constrained max-min fairness problem. The additional constraint under XCP can lead to unfairness for some network topologies. XCP gives a utilization of at least 80%, but a poor choice of α or γ could lead to arbitrarily low utilization. We have provided an algorithm to compute the equilibrium for general networks, and have presented simulation results to illustrate these findings.

An important question that we have not pursued is the dynamic properties of XCP, such as its stability. Even though the "residual" terms in the XCP code do not seem to affect equilibrium properties drastically, they may be important in determining its dynamic properties and, hence, should be taken into account in such an analysis. It is important to understand the stability of individual source rates $x_i(t)$, in addition to the aggregate rate $y_l(t)$, as studied in [9], in general networks in the presence of nonuniform delay. Since equilibrium queues are zero, the usual practice of linearizing around the equilibrium needs caution at the tightest bottlenecks that have zero queue yet full utilization.

APPENDIX A
VALID EQUILIBRIUM RATE

Proof (Lemma 2):

1) By Lemma 1, $F_{li} > 0$ and $F_{lj} = 0$. This implies

$$\frac{h_l + \phi_l^+}{N_l x_i} > \frac{h_l + \phi_l^-}{y_l} = \frac{h_l + \phi_l^+}{N_l x_j}.$$

2) If this were not the case, then the average rate of flows in $I_0(l)$ (and hence the rate of at least one such flow) would be greater than c_l/N_l. Part 1 would then require the total rate to exceed the capacity.

3) From part 1

$$N_l r^l = N_{l1} r^l + N_{l0} r^l$$
$$\geq N_{l1} r^l + y_{l0} = y_l$$

with equality if and only if $y_{l0} = 0$.

4) Otherwise, $h_l = 0$ and $\phi_l = 0$ since $I_1(l) \neq \emptyset$ (by Lemma 1, part 4). Then, $y_l = c_l$ and $b_l = 0$, whence $h_l = \max(\gamma d_l c_l - 0, 0) > 0$.

5) $\sigma_l = y_{l0}/c_l \leq y_l/c_l$ with equality if and only if $y_{l0} = y_l$. ∎

Lemma 8: The smaller solution to (26) does not give a valid equilibrium rate for link l.

Proof: For a rate, y_l, to be valid requires $y_l/c_l > \sigma_l$, by Lemma 2(5).

Let $\theta = (1 + \gamma/\alpha)\sigma_l$ and $\psi = (1 + \rho_l \gamma/\alpha)\sigma_l \leq \theta$. Then, dividing the numerator and denominator of (26) by α gives

$$\frac{y_l}{c_l} = \frac{1 + \theta \pm \sqrt{(1 + \theta)^2 - 4\psi}}{2\psi/\sigma_l}.$$

If the lower root were valid, then

$$1 + \theta - 2\psi > \sqrt{(1 + \theta)^2 - 4\psi}$$

whence

$$(1 + \theta)^2 - 4(1 + \theta)\psi + 4\psi^2 > (1 + \theta)^2 - 4\psi.$$

However, $0 \leq \psi \leq \theta$, which yields a contradiction. Thus, the lower root is not valid. ∎

APPENDIX B
PROPERTIES OF OPTIMIZATION PROBLEMS

Proof (Lemma 4): Note that $g_l(x) = g_l(y_l)$ depends on x only through y_l. Now, $x \in X_n$ are all of the form

$$x_i = \begin{cases} r_m, & i \in I_m, m \leq n \\ r_n + \epsilon_i, & i \notin \bar{I}_n \end{cases}$$

where $\epsilon_i > 0$. Write $x \in X_n$ as $x(\epsilon_i, i \notin \bar{I}_n)$. Hence, we can write y_l also as a function of $(\epsilon_i, i \notin \bar{I}_n)$

$$y_l(\epsilon_i, i \notin \bar{I}_n)$$
$$= \sum_{m=1}^{n} r_m \sum_{i \in I_m} R_{li} + \sum_{i \notin \bar{I}_n} R_{li}(r_n + \epsilon_i)$$
$$= \sum_{m=1}^{n-1} r_m \sum_{i \in I_m} R_{li} + r_n \sum_{i \notin \bar{I}_{n-1}} R_{li} + \sum_{i \notin \bar{I}_n} R_{li}\epsilon_i. \quad (33)$$

Given any $(\epsilon_i, i \notin \bar{I}_n)$, define the average ϵ by

$$\epsilon \cdot \sum_{i \notin \bar{I}_{n-1}} R_{li} = \sum_{i \notin \bar{I}_n} R_{li}\epsilon_i$$

and consider the vector $(\epsilon_i = \epsilon, i \notin \bar{I}_n)$ with equal components. From (33), this vector produces the same link flow y_l. Moreover, $x(\epsilon) := x(\epsilon_i = \epsilon, i \notin \bar{I}_n)$ defined by this vector also satisfies $x_i(\epsilon) \leq g_l(y_l)$ for all $i \in I(l)$, and hence is in X_n. This is

because for $i \in I_m, m \leq n, x_i(\epsilon) = r_m \leq g_l(y_l)$ since the original $x((\epsilon_i, i \notin \bar{I}_n))$ is in X_n. For $i \notin \bar{I}_n$

$$x_i(\epsilon) = r_n + \epsilon \leq r_n + \max_i \epsilon_i \leq g_l(y_l)$$

where the first inequality follows because ϵ is the average of ϵ_i, and the last inequality follows because the original $x((\epsilon_i, i \notin \bar{I}_n))$ is in X_n. Hence, if $(\epsilon_i^*, i \notin \bar{I}_n)$ achieves the maximum in (9), the vector $(\epsilon_i^* = \epsilon^*, i \notin \bar{I}_n)$ with a common value ϵ^* also achieves the maximum. ∎

Proof (Lemma 5): Define

$$f(\epsilon) := \hat{g}_l(\epsilon) - (s + \eta\epsilon)$$

We will show that there exists $\epsilon_l \geq 0$, with equality if and only if $s = \hat{g}_l(0)$, such that $f(\epsilon_l) = 0$ under the conditions given in the lemma. Consider the two cases separately.

Case 1: $s \leq \hat{g}_l(0)$. If $s = \hat{g}_l(0)$ then $f(0) = 0$ and $\epsilon_l = 0$. Otherwise, $f(0) > 0$ and it remains to show that there exists a suitable $\epsilon_l > 0$. Consider

$$\lim_{\epsilon \to \infty} f(\epsilon) = \lim_{\epsilon \to \infty} \hat{g}_l(\epsilon) - (s + \eta\epsilon)$$
$$= \lim_{\epsilon \to \infty} \frac{\gamma(z + \eta\epsilon)^2}{N_l((\gamma + \alpha)(z + \eta\epsilon) - \alpha c_l)} - (s + \eta\epsilon)$$
$$= \lim_{\epsilon \to \infty} \epsilon$$
$$\cdot \left(\lim_{\epsilon \to \infty} \frac{\gamma(z/\epsilon + \eta)^2}{N_l((\gamma + \alpha)(z/\epsilon + \eta) - \alpha c_l/\epsilon)} \right.$$
$$\left. - (s/\epsilon + \eta) \right)$$
$$= \lim_{\epsilon \to \infty} \epsilon \cdot \eta \left(\frac{1}{N_l} \frac{\gamma}{\gamma + \alpha} - 1 \right)$$
$$= -\infty$$

since $N_l \geq 1$. This implies that there exists ϵ_1 sufficiently large such that $f(\epsilon_1) < 0$. Since f is continuous on $[0, \epsilon_1]$, there exists an $\epsilon_l \in (0, \epsilon_1)$ such that $f(\epsilon_l) = 0$. This is illustrated in Fig. 7.

Case 2: $\hat{g}_l(0) < 0$. Then

$$z < \frac{\alpha c_l}{\gamma + \alpha}$$

and hence

$$\bar{\epsilon} := \frac{1}{\eta} \left(\frac{\alpha c_l}{\gamma + \alpha} - z \right) > 0.$$

Moreover, as $\epsilon \to \bar{\epsilon}$ from above, $\hat{g}_l(\epsilon) \to +\infty$. Since $\hat{g}_l(\epsilon)$ is continuous for $\epsilon > \bar{\epsilon}$, there exists $\epsilon_0 > \bar{\epsilon}$ such that $s + \eta\epsilon_0 < \hat{g}_l(\epsilon_0)$, i.e., $f(\epsilon_0) > 0$. The same argument as in Case 1 shows that there exists an ϵ_1 with $f(\epsilon_1) < 0$. Since $\hat{g}_l$ is continuous on $[\epsilon_0, \epsilon_1]$, there exists $\epsilon_l \in (\epsilon_0, \epsilon_1)$ where $f(\epsilon_l) = 0$.

We now prove that ϵ_l is unique. The argument also shows that over $\{\epsilon \mid \hat{g}_l(\epsilon) > 0\}$, $s + \eta\epsilon > \hat{g}_l(\epsilon)$ if and only if $\epsilon > \epsilon_l$. First note that $\bar{\epsilon}$ may be negative. They key observation, illustrated in Fig. 7, is that

1) for $\epsilon < \bar{\epsilon}$, $\hat{g}_l(\epsilon)$ is negative, concave, and approaches $-\infty$ as $\epsilon \to \bar{\epsilon}$ from below;
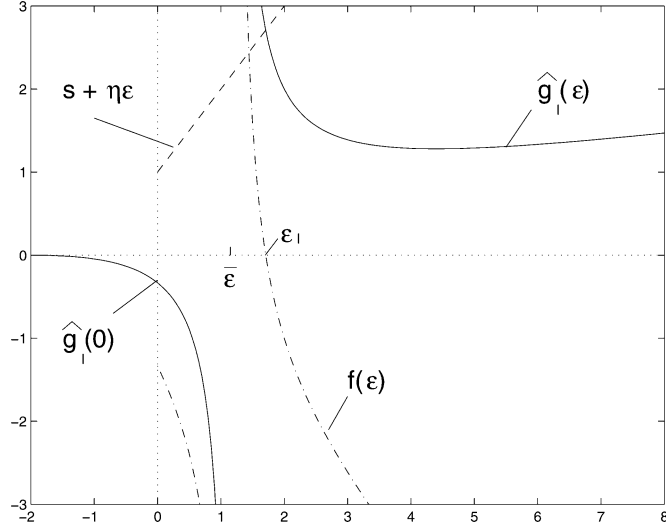
Fig. 7. $\hat{g}_l(\epsilon)$ in (18) and $f(\epsilon) = \hat{g}_l(\epsilon) - (s + \eta\epsilon)$.

2) for $\epsilon > \bar{\epsilon}$, $\hat{g}_l(\epsilon)$ is positive, convex, attains its minimum at

$$\epsilon = \frac{1}{\eta}\left(\frac{2\alpha c_l}{\gamma + \alpha} - z\right)$$

and approaches $+\infty$ as $\epsilon \to \bar{\epsilon}$ from above.

A moment of thought then convinces one that it suffices to show that $\hat{g}_l(\epsilon)$ grows less rapidly than $s + \eta\epsilon$, i.e.,

$$\frac{d\hat{g}_l}{d\epsilon} < \eta. \tag{34}$$

Letting $y_l = z + \eta\epsilon$, we have

$$\frac{d\hat{g}_l}{d\epsilon} = \frac{\eta}{N_l} \cdot \frac{\gamma y_l((\gamma + \alpha)y_l - 2\alpha c_l)}{((\gamma + \alpha)y_l - \alpha c_l)^2}.$$

Since $N_l \geq 1$, if (34) is violated, the above implies that

$$\frac{\gamma y_l((\gamma + \alpha)y_l - 2\alpha c_l)}{((\gamma + \alpha)y_l - \alpha c_l)^2} \geq 1$$

which implies

$$\alpha^2(y_l + c_l)^2 + \alpha\gamma y_l^2 \leq 0$$

which is a contradiction since $c_l, \alpha > 0$ and $\gamma, y_l \geq 0$. Hence, (34) must hold and $s + \eta\epsilon > g_l(\epsilon)$ if and only if $\epsilon > \epsilon_l$ whenever $g_l(\epsilon) > 0$. ∎

Proof (Lemma 6): The first assertion follows directly from the definitions of L_n, I_n and $\hat{X}_n$ in (10), (11) and (15), respectively. We will prove the second assertion by induction on n.

Base case $n = 1$: Fix any $x \in \hat{X}_1$.

Consider an $l \notin L_1 = \bar{L}_1$. By (13), $c_l/N_l > r_1$. Now

$$x_i(\epsilon; 1) = \begin{cases} r_1 & i \in I_1 \\ r_1 + \epsilon & i \notin I_1 \end{cases}$$

Hence

$$g_l(x(0; 1)) = \frac{\gamma(N_l r_1)^2}{N_l((\gamma + \alpha)N_l r_1 - \alpha c_l)}.$$

If

$$N_l r_1 < \frac{\alpha}{\gamma + \alpha}c_l$$

then $g_l(x(0; 1)) < 0$. Otherwise, $g_l(x(0; 1)) > 0$. We claim $x_i(0, 1) = r_1 < g_l(x(0; 1))$ for all $i \in I(l)$. If not, then

$$r_1 \geq \frac{\gamma(N_l r_1)^2}{N_l((\gamma + \alpha)N_l r_1 - \alpha c_l)}$$

yielding $N_l r_1 \geq c_l$, a contradiction. Hence, if $l \notin \bar{L}_1$, then either $g_l(x(0; 1)) < 0$ or $x_i(0; 1) < g_l(x(0; 1))$ for all $i \in I(l)$.

Induction hypothesis: Suppose the second assertion holds for n. We will prove it for $n + 1$.

Induction: Fix an $l \notin \bar{L}_{n+1}$. First note that $l \notin \bar{L}_n$. Consider all links $\hat{l} \notin \bar{L}_n$. By the induction hypothesis, either $g_{\hat{l}}(x(0; n)) < 0$ or $x_i(0; n) < g_{\hat{l}}(x(0; n))$ for all $i \in I(\hat{l})$. For an $i \notin \bar{I}_n$, we have from (10), for $\epsilon \geq 0$

$$x_i(\epsilon; n) = r_n + \epsilon$$
$$y_{\hat{l}}(\epsilon; n) = z_{\hat{l}} + \eta_{\hat{l}}\epsilon_{\hat{l}}$$

where, from (17)

$$z_{\hat{l}} := \sum_{m=1}^{n} r_m \sum_{i \in I_m} R_{\hat{l}i} + r_n \sum_{i \notin \bar{I}_n} R_{\hat{l}i}$$

$$\eta_{\hat{l}} := \sum_{i \notin \bar{I}_n} R_{\hat{l}i}.$$

From (18), $\hat{g}_{\hat{l}}(0) = g_{\hat{l}}(z_{\hat{l}}) = g_{\hat{l}}(x(0; n)) < 0$, and hence the induction hypothesis implies that either $\hat{g}_{\hat{l}}(0) < 0$ or $r_n < \hat{g}_{\hat{l}}(0)$. Lemma 5 then implies that, for each $\hat{l} \notin \bar{L}_n$, there exists a unique $\epsilon_{\hat{l}} > 0$ such that $r_n + \epsilon_{\hat{l}} = \hat{g}_{\hat{l}}(\epsilon_{\hat{l}}) = g_{\hat{l}}(z_{\hat{l}} + \eta_{\hat{l}}\epsilon_{\hat{l}})$. The minimum $g_{\hat{l}}(z_{\hat{l}} + \eta_{\hat{l}}\epsilon_{\hat{l}})$ over $\hat{l} \notin \bar{L}_n$ is r_{n+1}, and these minimizing $\hat{l}$ constitute L_{n+1}. All the sources $i \notin \bar{I}_n$ that go through a link in L_{n+1} are assigned the common rate r_{n+1}, and they are collected into X_{n+1}.

Since $l \notin \bar{L}_n$ and $l \notin L_{n+1}$, the corresponding ϵ_l satisfies

$$r_n + \epsilon_l = g_l(z_l + \eta_l\epsilon_l) \tag{35a}$$
$$r_n + \epsilon_l > r_{n+1} = r_n + \min_{\hat{l} \notin \bar{L}_n} \epsilon_{\hat{l}}. \tag{35b}$$

However, $x_i(\epsilon; n + 1) = r_{n+1} + \epsilon$ for all $i \notin \bar{I}_{n+1}$. Hence (35) and Lemma 5-2 (with $z = y_l(0; n), s = r_n, \eta = \sum_{i \notin \bar{I}_n} R_{li}$) imply that either $g_l(x(0; n + 1)) \leq 0$ or $x_i(0; n + 1) = r_n + \min_{\hat{l} \in \bar{L}_n} \epsilon_{\hat{l}} < \hat{g}_l(\min_{\hat{l} \in \bar{L}_n} \epsilon_{\hat{l}}) = g_l(x(0; n + 1))$. The proof is completed by noting that $g_l(x(0; n + 1)) \neq 0$, since $y_l \geq r_1 > 0$. ∎

APPENDIX C
CORRECTNESS OF RATE ALGORITHM

To show the correctness of the algorithm in Section III-D to find XCP's equilibrium rates, it suffices to show that the results satisfy the two conditions stated in Section III-B.

Each source i is assigned a rate in the same step as a particular link l. If r^l is chosen according to (24) with the true equilibrium

values of ρ_l and σ_l, then $F_{li} = 0$, since that is the condition from which (24) was derived. Similarly, (24) implies $y_l \leq c_l$, as established in the proof of Theorem 6.

The correctness of the algorithm can thus be established by showing that the true equilibrium values of σ_l and ρ_l are used when the final value r^l is calculated in step 2.4.1. For each link l, the values of σ_l and ρ_l depend only on network parameters and flows i with rates $x_i < r^l$, by Lemma 2-1. That is, if the rates selected by the algorithm are such that the rate r^l of each link l is greater than the rates of the flows flowing through l but bottlenecked elsewhere, then the rates must form an equilibrium of XCP. The theorem then results from the following lemma.

Lemma 9: For each l and n for which $r^l(n + 1)$ is defined, $r^l(n + 1) \geq r_n$. Moreover, if $\gamma < \alpha$, then for each l and n for which $r^l(n + 1)$ is defined, $r^l(n + 1) \geq r^l(n)$.

Proof: Consider an arbitrary link, l, and iteration, n. If l does not carry any flows in $\Delta(n)$, then $r^l(n+1) = r^l(n) > r_n$ as required. Consider now the case that l does carry a flow in $\Delta(n)$.

Let $x = r_n N_l/c_l$ be the rate of flows allocated in iteration n, normalized to link l. Let $\rho(u) = \rho_l(n) + u$ and $\sigma(u) = \sigma_l(n) + ux$. Then $\rho(0)$ and $\sigma(0)$ are the fraction of allocated flows and allocated capacity on link l before step 2.4.1 of iteration n, while $\rho(\delta)$ and $\sigma(\delta)$ are the values after the update, where $\delta = \rho_l(n+1) - \rho_l(n) > 0$.

Let $z(u)$ be the value of $r^l N_l/c_l$ calculated from (27) using $\rho(u)$ and $\sigma(u)$. Note that $z \geq 1$ by Lemma 2-3, since $c_l \geq y_l$. Moreover, $x < z(0)$, since if $z(0) \leq x$ then link l would have been an element of $M(n)$ and $r^l(n+1)$ would not be defined. To prove the lemma, it is sufficient to show that $z(\delta) \geq x$ and that if $\gamma > \alpha$ then $dz/du \geq 0$.

Below, the argument (u) will be dropped when no ambiguity can arise. Differentiating (27) with respect to u, and noting that $z(u)$ is the larger of the two solutions of (27), gives

$$0 = \frac{dA_l}{du}z^2 + 2zA_l\frac{dz}{du} + \frac{dB_l}{du}z + B_l\frac{dz}{du} + \frac{dC_l}{du}$$
$$= \sqrt{B_l^2 - 4A_lC_l}\frac{dz}{du} + \frac{dA_l}{du}z^2 + \frac{dB_l}{du}z + \frac{dC_l}{du}.$$

Now

$$\frac{dA_l}{du}z^2 + \frac{dB_l}{du}z + \frac{dC_l}{du}$$
$$= [\gamma(1 - \rho) - (\gamma\rho + \alpha)]z^2$$
$$+ [2\gamma\sigma - \gamma x(1 - \rho) + x(\gamma\rho + \alpha)]z - 2\gamma\sigma x$$
$$= (x - z)[z(\alpha - \gamma) + 2\gamma(z\rho - \sigma)]$$

giving

$$\frac{dz}{du} = (z - x)\frac{[z(\alpha - \gamma) + 2\gamma(z\rho - \sigma)]}{\sqrt{B_l^2 - 4A_lC_l}}.$$

If $\gamma < \alpha$, then $dz/du > 0$, and z remains greater than x. This is because $2\gamma(z\rho - \sigma) \geq 0$, since $z \geq 1$ and $0 \leq \sigma \leq \rho$. This establishes the second part of the lemma.

If $\gamma \geq \alpha$, then the right-hand side need not be positive. However, the second factor is bounded, and so z approaches x expo-

nentially as u increases, and so can never drop below x. In particular, $z(\delta) > x$, which establishes the first part of the lemma. ∎

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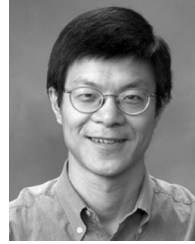
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