REDUNDANCY IN QUANTISERS FOR NOISY CHANNELS

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ABSTRACT

This paper presents a study of the nature of optimal quantiser/dequantiser pairs for use with binary symmetric channels with moderate to high error rates. It is well known that some codewords of such quantisers are never output, because there is always another codeword with lower expected distortion. This paper investigates how high the bit error rate must be for this phenomenon to occur. It also investigates the nature of optimal quantisers for extremely noisy channels, in which the error probability is almost 1/2.

1. INTRODUCTION

Quantisation is an important aspect of signal processing for multimedia signals such as audio and video, forming a key component of many compression techniques. Traditionally quantisation schemes are optimised for use with noiseless channels, which do not introduce errors into the digital data. However with the increased use of highly noisy channels in such areas as mobile communications, channel noise can no longer be ignored. Much work has already been done on quantisation for use with noisy channels [1–9], although work has primarily focused on vector quantisation [1–5] rather than scalar quantisation, which is discussed here. A major contribution was made by Farvardin [3, 8], who presented an algorithm for finding locally optimal quantisers for a range of channel conditions. In [8], it was found that as the error rate of a binary symmetric channel increases, some of the codewords are never output by an optimal quantiser, i.e., the expected distortion is always lower when a different codeword is transmitted. These are known as redundant codewords. This result was extended to the vector case in [3]. This paper will use a powerful optimisation technique to investigate the bit error rate at which this first occurs for a range of quantiser resolutions, transmitting over binary symmetric channels. In particular, the result has previously only been demonstrated for comparatively high bit error rates, but results in this paper indicate that for realistic quantisers with 8 bits/sample, the phenomenon occurs at quite modest error rates of under 10^{-5}. Section 2 introduces some notation and describes the algorithm used in [8], and Section 3 describes the extended algorithm used in this paper. Simulation results are presented in Section 4. Section 5 describes the nature of quantisers in the presence of very high bit error rates.

2. NOTATION AND PRIOR WORK

In [8], Farvardin proposed a generalisation of the Lloyd-Max algorithm [10] to the case of noisy channels. This algorithm will be described using the following notation. Let x be the input to the quantiser. Let U be the output of the quantiser for input x. Let V be the channel output when U is transmitted over a noisy channel, V is the output. When V is input to the dequantiser, the output (reconstruction level) is $\hat{x}$. These are all random variables, but U is a deterministic function of x, and $\hat{x}$ is a deterministic function of V. Let u_i, v_i and $\hat{x}_i$be instances of U and V and $\hat{x}$ respectively. Let $E[\hat{x}|u_i] = E[\hat{x}|U = u_i]$ be the expected output given that codeword u_i is transmitted, and define other expectations analogously. The algorithm iteratively finds $\hat{x}_i$ and the encoding thresholds, t_i. Each iteration can be stated as follows [8]:

1. Calculate $E[\hat{x}|u_i]$ and $E[\hat{x}^2|u_i]$ for each i.
2. Renumber the codewords and reconstruction levels such that $E[\hat{x}|u_i] \le E[\hat{x}|u_j]$ if $i < j$.
3. Calculate thresholds $t_i = \min_{j>i}(T_{ij})$, where

$$T_{ij} = \frac{E[\hat{x}^2|u_j] - E[\hat{x}^2|u_i]}{E[\hat{x}|u_j] - E[\hat{x}|u_i]} \tag{1}$$

   If $j > i + 1$, the codewords u_k, $i < k < j$, are unused
4. Calculate $\hat{x}_i = E[x|v_j]$

3. OPTIMISATION TECHNIQUE

The algorithm described in [8] uses the generalised Lloyd-Max algorithm to generate locally optimal quantisers. This study uses a more powerful technique, based on the observation that, for a given codeword assignment, the design of quantisers for smooth input distributions is comparatively free of local minima. The task is thus to find the codeword assignment which minimises the expected coding error. An n bit quantiser has $(2^n)!$ possible codeword assignments, making the optimisation very difficult for moderate n. The task is simplified by using symmetry: two codeword assignments are equivalent if one can be obtained from the other by reordering the bits or by inverting the ith bit in every codeword, giving around $(2^n)!/n!2^n$ distinct orderings. From these, an ordering was sought using an extension of pseudo-gray coding for vector quantiser codeword assignment [6]. This takes a "seed" codeword assignment, and evaluates all perturbations obtained by interchanging pairs of codewords. If a better assignment is found, the procedure begins again with that as the seed. The algorithm used here has three main differences. First, for each codeword assignment the entire quantiser is re-computed. Second, once all of the perturbations are tested, the procedure is not only repeated if a better assignment was found, but instead it is repeated on the best k assignments found so far (typically $k = 5$).

The third difference between the algorithm used and pseudo-gray coding is that a much richer range of perturbations is used, rather than merely interchanging pairs of codewords. As well as these small perturbations, it also includes larger perturbations which still retain much of the codeword ordering information. The actual perturbation used is as follows: From the set $\{0,\ldots n-1\}$ of bit positions, select two disjoint subsets, s_1 and s_2. Select a particular pattern of 1s and 0s, p_1, for the bits in s_1. For all codewords such that the bits in positions s_1 match the pattern p_1, invert the bits in positions s_2. This causes the new codeword assignment to have similar bit patterns to the previous. For example, natural binary code may be perturbed to folded binary code by selecting $s_1 = \{n-1\}$ to be the most significant bit, p_1 to be 1, and $s_2 = \{0,\ldots n-2\}$ to be all of the bits except the most significant.

The overall algorithm can be summarised as follows:

1 Initialise a set of "best" codeword assignments, P, to contain only the natural binary code
2 **while** there is a codeword assignment in P which has not been used as a seed
  2.1 Select the unchecked codeword assignment, $a \in P$, with lowest MSE as the seed
  2.2 **foreach** perturbed codeword assignment, a'
    2.2.1 Perform iterations of the modified Lloyd-Max algorithm until the improvement in MSE drops below a threshold
    2.2.2 If the MSE is less than the current kth best, add the new codeword assignment to P and if P now contains more than k elements, delete the worst.
  endfor
endwhile

4. RESULTS

4.1. Empty encoding regions

In order to determine the smallest BER for which redundant codewords occur, quantisers were designed for uniform and Gaussian quantisers with bit rates ranging from 2 to 5 bits per sample. Figure 1 shows the smallest BER at which the optimal codeword ordering is different from the ordering for infinitesimal error rate. For infinitesimal error, uniform quantisers always have a natural binary codeword ordering. Gaussian quantisers have a folded binary ordering up to 4 bits/sample, but not for higher rates. Figure 2 shows the smallest BER for which one the codewords is actually redundant. Clearly the bit rate at which these occur varies exponentially with the resolution of the quantiser. This indicates that Farvardin's approach is necessary even at the very modest error rates found in real communication systems. If the trends shown in these figures continue, then an 8-bit quantiser designed optimally would have redundant codewords at error rates as low as 3×10^{-6}. It is also worth noting that, although the actual error rate at which the thresholds merge depends on the input distribution, the decrease in the error rate as the quantiser resolution increases is similar for both uniform and Gaussian quantisers. It is hazardous to draw firm conclusions from such limited data, but the super-exponential computational requirements make finding the optimal quantiser for a 6-bit quantiser intractable, and the phenomenon does not occur for 2-bit quantisers.

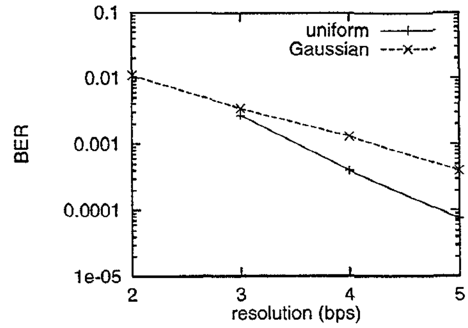

Figure 1. BER for first change of optimal codeword assignment as a function of quantiser resolution

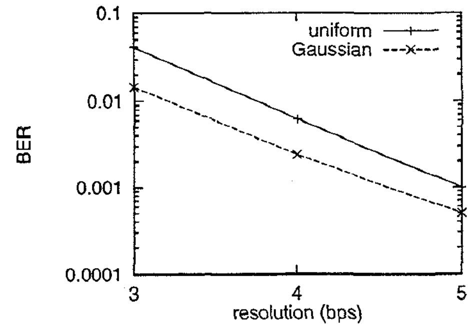

Figure 2. BER for first reduction in the number of thresholds as a function of quantiser resolution

4.2. Thresholds

Figures 3 and 4 show the actual thresholds for the optimal quantisers for 2, 3 and 4 bit/sample qantisers, for unit variance uniform and Guassian sources respectively. Because both of the pdfs considered are entirely symmetrical about zero, the optimal quantiser is generally also symmetric. However, it can be seen that these quantisers are often asymmetric near the points where the thresholds merge, that is, where the number of redundant codewords increases. For example, this occurs at $\epsilon = 0.061$ for the 3 bit/sample uniform quantiser, and $\epsilon = 0.013$ for the 3 bit/sample Gaussian quantiser, where the number of redundant codewords is odd. It also occurs for $0.003 < \epsilon < 0.01$ for the 3 bit/sample Gaussian quantiser, even though the number of redundant codewords is even, and a symmetric quantiser would be expected. It has been verified by exhaustive search that these results are not simply the result of finding a local optimum.

In general the largest and smallest thresholds converge monotonically to zero. However, in the case of the 4-bit Gaussian quantiser, these thresholds diverge for $0.01 < \epsilon < 0.03$. This effect was found in [11] to occur frequently in optimal encoders for fixed decoders, and is due to the particular ordering of the codewords. For these error rates, the codeword for the smallest reconstruction level is 0000. However, the codewords 0100 and 1000 both correspond to comparatively large reconstruction levels (11th smallest and 13th smallest). Because the codeword 0000 can easily be corrupted into either of these, the actual input must be quite negative to justify transmitting such a "dangerous" codeword. As ϵ, and hence the "danger", increases, the lowest threshold diverges from zero.

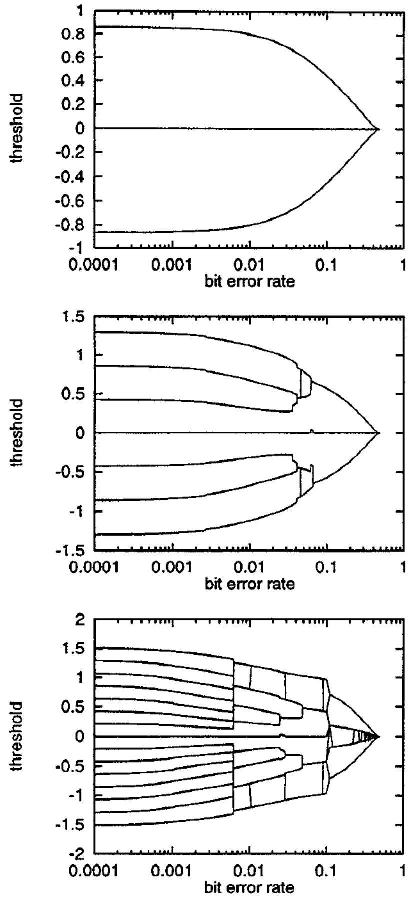

Figure 3. Thresholds vs BER for a unit variance uniform source

4.3. Quality of results

The results obtained by the optimisation procedure of section 3 are compared in Tables 1 and 2 with those obtained by the simpler approach of [8]. Clearly the results presented here are better, particularly for the higher resolution quantisers, which are plagued by local optima. This indicates that the technique used in this study has avoided most of the local minima, and gives confidence that the quantisers of Figures 3 and 4 are likely to be the actual optimal quantisers.

5. HIGH ERROR RATES

It is also informative to look at the nature of an optimal quantiser in the presence of very high levels of noise. If some codewords become redundant at moderate error rates, it may seem reasonable to assume that as the bit error rate approaches $\epsilon = 0.5$ all quantisers will revert to two-level quantisers. However this is not the case, as seen in Figures 3 and 4. Two and three bit quantisers both reduce to four-level quantisers, and four bit quantisers only reduce to five level quantisers. In this section, a heuristic investigation of this will be presented, and it will be argued that, for $\epsilon \approx 0.5$, an n-bit quantiser (with $n > 2$) degenerates to $n + 1$ levels, and an approximation to those levels will be derived. The resulting quantiser represents most of the possible inputs with one of two codewords, but in a small range around zero, other codewords will be used. These codewords are coded in unary, rather than binary. That is, the ith codeword

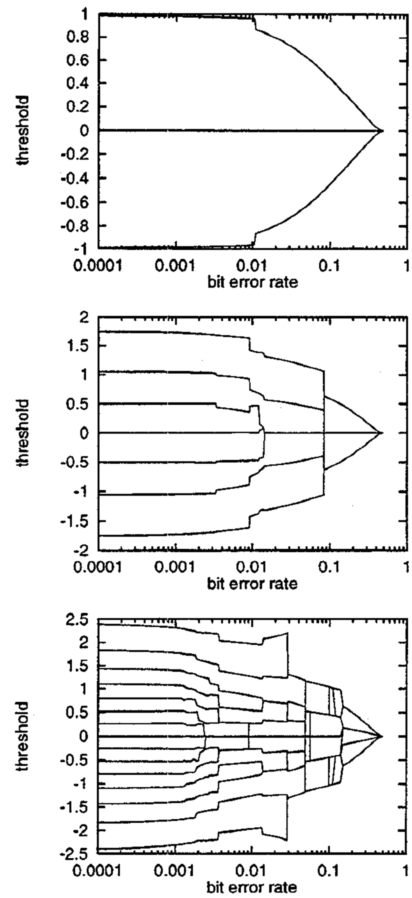

Figure 4. Thresholds vs BER for a unit variance Gaussian source

used is i 1s followed by $n - i$ 0s. This derivation is far from rigorous, but provides some insight into the nature of these quantisers.

Consider an optimal quantiser for input drawn from a symmetric pdf ($p(x) = p(-x)$) for a bit error rate of $\epsilon = 1/2 - \delta$ with $\delta << 1$. Let $\hat{x}_0$ be the smallest reconstruction level, and $\hat{x}_{N-1}$, $N = 2^n$, be the largest. Because of the high error rate, the received codeword will give very little information about the input, and the minimum expected mean square error (MEMSE) will be obtained when all of the reconstruction levels and all of the thresholds are approximately zero. Hence, $\Pr(U = u_0) = \Pr(U = u_{N-1}) \approx 1/2$, and the encoding will be dominated by these two codewords. Without loss of generality, let the first codeword be $u_0 = 0$. For very high error rates, the MEMSE will be obtained when $u_{N-1} = N - 1$.

Let $E = E[x|x > 0]$, and denote the Hamming weight of codeword u_j, the number of 1 bits in it, by $H(u_j)$.

Then the expected value of the input given that a codeword v_j was received will be given by

$$E[x|v_j] \approx \Pr(u_0|v_j)(-E) + \Pr(u_{N-1}|v_j)(E) + \sum_{k \neq 0, N-1} \Pr(u_k|v_j) E[x|u_k]$$

where $\Pr(u_k|v_j)$ is shorthand for $\Pr(U = u_k|V = v_j)$, the probability that codeword u_k was transmitted given that

n	0.005		0.010		0.050		0.100	
	[8]	new	[8]	new	[8]	new	[8]	new
2	8.52	8.52	7.88	7.88	5.20	5.20	3.63	3.63
3	12.04	12.06	10.50	10.59	6.47	6.77	4.67	4.67
4	14.15	14.73	12.30	13.03	7.81	8.27	5.60	5.87

Table 1. SNR (dB) of quantisers for a Gaussian quantiser designed by the method of Section 3 (denoted 'new') compared with those of [8] for rates of $n = 2, 3, 4$ bits/sample, and bit error rates of 0.005, 0.01, 0.05 and 0.1.

n	0.005		0.010		0.050		0.100	
	[8]	new	[8]	new	[8]	new	[8]	new
2	10.93	10.92	10.06	10.06	6.51	6.51	4.50	4.50
3	14.68	14.71	13.00	13.04	8.12	8.12	5.87	5.87
4	17.02	17.08	14.87	15.31	9.74	10.18	7.03	7.09

Table 2. SNR (dB) of quantisers for a Uniform quantiser designed by the method of Section 3 (denoted 'new') compared with those of [8] for rates of $n = 2, 3, 4$ bits/sample, and bit error rates of 0.005, 0.01, 0.05 and 0.1.

v_j was received. The last term will be neglected since $\Pr(u_k|v_j) \approx 0$ for $k \neq 0, N-1$. Let $i = H(v_j)$. Using Bayes' rule, the fact that $\Pr(u_0) \approx 1/2$, and the fact that $\Pr(v_j) \approx (\Pr(v_j|u_0) + \Pr(v_j|u_{N-1}))/2$ gives

$$\begin{aligned}\Pr(u_0|v_j) &\approx \frac{(1-\epsilon)^{n-i}\epsilon^i}{(1-\epsilon)^{n-i}\epsilon^i + (1-\epsilon)^i\epsilon^{n-i}} \\ &= \frac{(1+2\delta)^{n-2i}}{(1+2\delta)^{n-2i} + (1-2\delta)^{n-2i}} \\ &\approx 1/2 + (n-2i)\delta\end{aligned}$$

and similarly $\Pr(u_{N-1}|v_j) \approx 1/2 - (n-2i)\delta$. Thus

$$\hat{x}_j = E[x|v_j] \approx 2(2i-n)\delta E \qquad (2)$$

Some interesting conclusions can be drawn from equation 2. Firstly, the reconstruction levels only are independent of the shape of the input distribution, and only depend on the distribution through the quantity $E = E[x|x > 0]$. This is largely because all of the thresholds converge to zero for high error rates. A second interesting observation is that the spacing between the levels is independent of the rate of the quantiser, n. That is, increasing the number of bits used to code each sample increases the dynamic range of the quantiser but does not increase its resolution.

Equation 2 shows that there are $n+1$ distinct and evenly spaced groups of closely spaced reconstruction levels, depending on the Hamming weight of the received codeword. Similarly, the expected reconstruction level, $\hat{x}$, given that symbol u_k is transmitted will depend primarily on $H(u_k)$. The thresholds can then be obtained by equation 1. Since $E[\hat{x}|u_k] \approx E[\hat{x}|u_j]$ when $H(u_k) = H(u_j)$, the denomitator becomes very small, and these thresholds are likely to be indeterminate. However, when $H(u_k) = H(u_j) + 1$, $E[\hat{x}|u_k] \neq E[\hat{x}|u_j]$, the threshold between u_j and u_k will be well formed.

This provides some explanation for the empirical observation that there are n distinct thresholds separating $n+1$ distinct encoding regions for arbitrarily high bit error rates.

6. CONCLUSION

It has been shown that channel error rate at which quantiser codewords first become redundant decreases exponentially with the data rate (bits/sample). A heuristic analysis of quantisers for extremely noisy channels has also been presented, which shows that the optimal quantiser using n bits is an $n+1$ level quantiser, and the actual quantiser is independent of the shape of the input distribution.

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REFERENCES

[1] L. L. H. Andrew and M. Palaniswami, "A study on the effect of neighbourhood functions for noise robust image vector quantisation," in *Proc. Int. Conf. Neural Networks*, (Orlando, FL), 1994, pp. 4159–4162.

[2] D.-M. Chiang and L. C. Potter, "Graph covering index assignment in vector quantisation for noisy channels," *Electron. Lett.*, vol. 31, pp. 1550–1552, Aug. 1995.

[3] N. Farvardin, "A study of vector quantization for noisy channels," *IEEE Trans. Inform. Theory*, vol. 36, pp. 799–809, July 1990.

[4] D. Miller and K. Rose, "Combined source/channel vector quantization using deterministic annealing," *IEEE Trans. Commun.*, vol. 42, pp. 347–356, Feb. 1994.

[5] H.-S. Wu and J. Barba, "Index allocation in vector quantisation for noisy channels," *Electron. Lett.*, vol. 29, pp. 1317–1318, July 1993.

[6] K. A. Zeger and A. Gersho, "Pseudo-gray coding," *IEEE Trans. Commun.*, vol. 38, pp. 2147–2158, Dec. 1990.

[7] M. Y. Cheung and J. Vaisey, "A comparison of scalar quantization strategies for noisy channel data transmission," *IEEE Trans. Commun.*, vol. 43, pp. 738–742, Mar. 1995.

[8] N. Farvardin and V. Vaishampayan, "Optimal quantizer design for noisy channels: An approach to combined source channel coding," *IEEE Trans. Inform. Theory*, vol. 33, pp. 827–838, Nov. 1987.

[9] A. K. Khandani, "Optimal source-to-channel assignment," *Electron. Lett.*, vol. 33, pp. 126–127, Jan. 1997.

[10] S. P. Lloyd, "Least squares quantization in PCM," *IEEE Trans. Inform. Theory*, vol. 28, pp. 129–137, Mar. 1982.

[11] L. L. H. Andrew and M. S. Iswahyudi, "Optimal reconstruction from a fixed quantiser over a noisy channel," in *Proc. Asia Pacific Conf. Commun.*, (Sydney, Australia), 1997, To appear.