Stability and Robustness Conditions using Frequency Dependent Half Planes

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***Abstract*— This paper presents a sufficient condition that establishes closed loop stability for linear time invariant dynamical systems with transfer functions that are analytic in the open right half complex plane. The condition is suitable for analyzing a large class of highly complex, possibly interconnected, systems. The result is based on bounding Nyquist curves by using frequency dependent half planes. It provides (usually non-trivial) robustness guarantees for the provably stable systems and generalizes to the multidimensional case using matrix field of values. Concrete examples illustrate the applications of the condition. From our condition, it is easy to derive a relaxed version of the classical result that the interconnection of a positive real and strictly positive real linear system under feedback is closed loop stable.**

I. Introduction

In the modeling and analysis of complex systems, it is useful to capture system properties of importance for the application with sufficient accuracy but abstract away irrelevant details, and also to select an analysis technique that matches the complexity of the model without being too conservative. Although the forgiving nature of feedback often allows the use of simple models to analyze closed loop properties, simple models may not always be available. This is the case, for example, when a system is intended to operate over a wide range of conditions, exhibits complex dynamics over a wide range of time scales, and has a bandwidth that is sensitive to parameter changes and is consequently not explicitly known *a priori*. For such scenarios we have to rely on methods that hide the "details", at the price of somewhat conservative results.

The dynamics of interconnections of even very simple dynamical systems tends to be very complex. They are inherently hard to analyze due to the decentralized structure and typically requires a fairly sophisticated machinery. Analysis and synthesis of such systems has lately attracted much interest; see [3], [8], [11], [13] and references therein. This line of work, however, does not explicitly address scalability. Here, scalability means that a class of interconnection structures can be shown to be stable, without requiring a new centralized analysis whenever an agent is added or removed, which subsequently alters the system equilibrium. Such scalability is required by many applications, such as data networks, flocking phenomena and power networks, but has received relatively little attention [9], [10], [12]. The results in the present paper sprung from an analysis [16] of Internet congestion control: a distributed, highly heterogeneous and continually changing system and a domain where scalability is essential. A centralized analysis, which would require distributing global information whenever a computer or router connects or disconnects, is simply infeasible. The common approach in this type of work has been to work in the frequency domain and use the Nyquist criterion to establish closed loop stability. Due to the complexity, this typically involves taking a convex hull of some aspect of the Nyquist curve [12], [9], potentially introducing a significant degree of conservativeness.

This paper proposes a less conservative stability criterion, in which the loop gain is decomposed into two factors, which must satisfy two half-plane inclusion conditions. This criterion, which can involve a frequency-*in*dependent half plane, implicitly bounds the Nyquist curve by a set of frequency-*de*pendent half planes whose union does not include the set $(-\infty, -1]$. Frequency dependent half plane conditions have previously been used in the context of interconnected systems [12], [15]; similarly, the convex hull conditions for stability in [9], [10], [18], need only hold pointwise over frequency (cf. the conditions in [6]). Considering half planes per frequency reduces conservatism, at the price of greater complexity in the stability test. The approach taken here circumvents the frequency dependence in the half planes. The complexity in the stability condition is transformed from finding infinitely many bounding half planes, into finding a suitable factorization of the loop gain.

Our criterion is related to notions of dissipativity and passivity [20], which have recently been applied to communication systems [21]. They have also long been applied to process control engineering, which also requires that closed loop stability be maintained in a multiloop control system when any subset of the control loops is detuned or even turned off; see [22] and references therein. In that context, the notion of scalable stability is referred to as decentralized unconditional stability.

The paper is organized as follows. Preliminary concepts and notation are introduced in Section II. Section III then presents the main results: a stability condition with a following robustnesses corollary. In Section IV, concrete examples including both SISO and MIMO systems are presented to illustrate our main results.

II. Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, and let $\mathbb{C}_+$ be the open right half plane, $\bar{\mathbb{C}}_+$ the

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closed right half plane and $\mathbb{C}_\mathrm{u}$ be the open upper half plane. Let $\mathbb{Z}^+$ be the set of non-negative integers.

Let $\sigma(A)$ denote the spectrum of a square matrix $A \in \mathbb{C}^{n\times n}$. If A is Hermitian, we further use $\overline{\lambda}(A)$ and $\underline{\lambda}(A)$ to denote its largest and smallest eigenvalue respectively. The *field of values* (also called the numerical range) of a matrix $A \in \mathbb{C}^{n\times n}$ is the set

$$F(A) := \{x^* A x \,|\, x \in \mathbb{C}^n, x^* x = 1\}\,.$$

We will use the properties [4]: $\sigma(A) \subset F(A)$, $\sigma(AB^{-1}) \subset F(A)/F(B)$ for $0 \notin F(B)$, and $\mathrm{Re}\,(F(A)) = F((A + A^*)/2)$ where A^* is the conjugate transpose of A.

Negative feedback is always implicitly assumed if not stated otherwise. The gain and phase margins are defined as in [14]: The *gain margin* (GM) of a stable feedback system with return ratio $L(s)$ is defined as $\mathrm{GM} = 1/|L(j\omega_{180})|$, where the phase cross-over frequency ω_{180} is where the Nyquist curve of $L(j\omega)$ crosses the negative real axis between -1 and 0. The *phase margin* (PM) is defined as $\mathrm{PM} = \arg(L(j\omega_c)) + 180°$, where the gain crossover frequency ω_c is where $L(j\omega)$ first crosses 1 from above, i.e., $|L(j\omega_c)| = 1$.

All angles are interpreted as equivalence classes modulo 2π. Thus, for example, $-3\pi/2 \in (0, \pi)$.

III. Main Result

We will now present the main contributions of the paper. We will start by deriving conditions on the individual parts of a factorization of the loop gain that are sufficient for closed loop stability. Then we will show how fulfilling stronger requirements also provides robustness guarantees.

A. Stability

In the following, we consider closed loop stability of a (negative) feedback system with open loop transfer function denoted $L(s)$.

Theorem 1: Assume $L(s) \in \mathbb{C}^{M\times M}$ is a transfer function analytic everywhere in the closed right half plane $\bar{\mathbb{C}}_+$ except possibly at a finite number N of points $j\omega_i$, $\omega_i \in \mathbb{R}$, $i = 1, \ldots, N$ along the imaginary axis, and with a factorization

$$L(s) = Q(s)P^{-1}(s).$$

Let

$$C_\theta = \{z \,|\, \mathrm{Im}\left(ze^{-j\theta}\right) < 0\}$$

(i.e., an open complex half plane with angle θ and which touches the origin), and let $\alpha \in [0, 1]$.

A system with open loop transfer function $L(s)$ is then closed loop stable if for all $\omega \geq 0, \omega \neq \omega_i, i = 1, \ldots, N$, there exists a $\theta \in \mathbb{R}$ such that both

$$F(P(j\omega)) \subseteq C_\theta, \tag{1}$$

$$F(Q(j\omega)) + \alpha F(P(j\omega)) \subseteq C_\theta \tag{2}$$

and either $N = 0$ or for all $i = 1, \ldots, N$

$$(-\infty, -1] \cap \sigma(L(s_i)) = \emptyset$$

when s_i traverses an infinitesimal semi-circle into the right half plane around $j\omega_i$.

Proof: Define the set of angles

$$\psi(\omega) = \theta - \arg(F(P(j\omega)))$$

and let the open cone to the right of all of the lines that pass through $-\alpha + j0$ with angles in $\psi(\omega)$ be denoted by

$$C(\omega) = \{z \,|\, \arg(z + \alpha) - \psi(\omega) \subset (-\pi, 0)\}. \tag{3}$$

The subset condition (1) implies that $\psi(\omega) \subset (0, \pi)$ (modulo 2π) for all $\omega \geq 0, \omega \neq \omega_i$; it follows that

$$(-\infty, -1] \cap \bigcup_{\omega\geq 0; \omega\neq\omega_i} C(\omega) = \emptyset.$$

Since by hypothesis $(-\infty, -1] \cap \sigma(L(s_i)) = \emptyset$ when circumventing the points on the imaginary axis where $L(s)$ fails to be analytic, invoking the generalized Nyquist criterion [14] yields that the system is closed loop stable if

$$\sigma(L(j\omega)) \subseteq C(\omega),$$

for all $\omega \geq 0, \omega \neq \omega_i$. This is equivalent to

$$\arg(\sigma(L(j\omega)) + \alpha) - \psi(\omega) \subseteq (-\pi, 0). \tag{4}$$

Note that (1) implies $0 \notin F(P(j\omega))$ and thus [4]:

$$\sigma(L(j\omega)) = \sigma\left(Q(j\omega)P^{-1}(j\omega)\right) \subset \frac{F(Q(j\omega))}{F(P(j\omega))}.$$

It follows since

$$\arg(\sigma(L(j\omega)) + \alpha) \subset \arg\left(\frac{F(Q(j\omega))}{F(P(j\omega))} + \alpha\right),$$

that (4) is satisfied if

$$\begin{aligned}
&\arg(F(Q(j\omega))/F(P(j\omega)) + \alpha) - \psi(\omega)\\
&= \arg(F(Q(j\omega)) + \alpha F(P(j\omega))) - \arg(F(P(j\omega))) - \psi(\omega)\\
&= \arg(F(Q(j\omega)) + \alpha F(P(j\omega))) - \theta \subseteq (-\pi, 0),
\end{aligned} \tag{5}$$

which is equivalent to (2). ■

Remark 2: The result is independent of whether a left or right factorization is used; $L(s) = P^{-1}(s)Q(s)$ may be preferred. Note, however, that different choices of $Q(s)$ and $P(s)$ may yield predictions with different degree of conservativeness.

Remark 3: There are two forms of frequency dependence in Theorem 1. In the simplest case, θ is fixed and the $C(\omega)$ are (intersections of) frequency dependent half planes. If $F(P(j\omega))$, $\omega \geq 0$, does not reside in a single half plane C_θ, then θ may also vary with ω so that the conditions on P and Q depend on frequency-dependent half planes.

The use of Theorem 1 is illustrated in Section IV by the means of a series of examples.

B. Robustness

By putting stronger sector requirements on $P(s)$, we will now show how bounds on the gain and phase margin and the peak of the sensitivity function $S(s) = (1 + L(s))^{-1}$ can be achieved when $\alpha < 1$. Note that in the multidimensional case analogous margins are of limited use since they are only

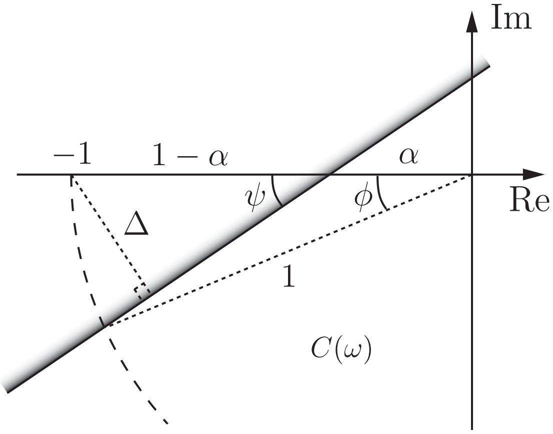

Fig. 1. Geometric illustration of robustness bounds.

valid for a simultaneous parameter change in all of the loops. This section will therefore consider only scalar systems.

Corollary 4: Assume there exists a $\psi_{\min} \in (0, \pi/2)$ such that for all $\omega \geq 0, \omega \neq \omega_i, i = 1 \ldots, N$, there exists a θ satisfying condition (2) and also

$$P(j\omega) \in C_{\psi_{\min},\theta} \subset C_\theta, \tag{6}$$

where

$$C_{\psi_{\min},\theta} = \{z \mid \arg(z) - \theta \in (-\pi + \psi_{\min}, -\psi_{\min}]\}\,.$$

Then the following bounds hold:

$$\|S\|_\infty = \sup_{\omega\geq 0} |S(j\omega)| \leq \frac{1}{(1-\alpha)\sin(\psi_{\min})}, \tag{7}$$

$$\begin{aligned} \mathrm{PM} &\geq \psi_{\min} - \arcsin(\alpha \sin\psi_{\min}) && \text{(8a)}\\ &\geq (1-\alpha)\psi_{\min}. && \text{(8b)}\end{aligned}$$

Proof: For all $\omega = \omega_i$, $i = 1, \ldots, N$, trivially (7) is fulfilled since $\lim_{\omega\to\omega_i} |S(j\omega)| = 0$, and by definition $\omega_c \neq \omega_i$. Now consider $\omega \geq 0, \omega \neq \omega_i$. The restriction $P(j\omega) \in C_{\psi_{\min},\theta}$ implies that $\psi(\omega) \in [\psi_{\min}, \pi - \psi_{\min})$, that is the angle of the half planes $C(\omega)$ containing $L(j\omega)$. Consider first the case $\psi \in (\psi_{\min}, \pi/2]$. The shortest distance between the line defining the half plane $C(\omega)$ and -1 is simply

$$\Delta = (1-\alpha)\sin(\psi),$$

as shown in Fig. 1. Since $S^{-1}(j\omega)$ by definition correspond to the line from $-1 + j0$ to $L(j\omega)$, we have

$$1/|S(j\omega)| \geq \Delta \geq (1-\alpha)\sin(\psi_{\min}),$$

and thus (7). The case $\psi \in (\pi/2, \pi - \psi_{\min})$ is equivalent by symmetry.

The angle ϕ can be computed by applying the sine rule to the triangle shown in Fig. 1 containing the origin and the point $-\alpha + j0$. This gives

$$\frac{\sin(\psi-\phi)}{\alpha} = \sin(\psi),$$

and since $\psi - \phi < \pi/2$ from the geometry, this gives

$$\begin{aligned} \phi &= \psi - \arcsin(\alpha\sin(\psi)) && \text{(9a)}\\ &\geq \psi(1-\alpha), && \text{(9b)}\end{aligned}$$

where the inequality uses $\alpha\sin(\psi) \leq \sin(\alpha\psi)$ and $\arcsin(\sin(\alpha\psi)) \leq \alpha\psi$ for $\alpha \in [0,1]$ and $\psi \in [0,\pi]$. Both of the right hand sides of (9) are non-decreasing in ψ, since $\frac{d}{d\psi}\arcsin(\alpha\sin(\psi)) \leq 1$ for the parameter values of interest, which establishes (8). ■

Remark 5: The phase margin bound (8) still holds if the sector condition on $P(s)$ is relaxed to $P(j\omega) \in C_{\tilde{\psi}_{\min}}$ with

$$C_{\tilde{\psi}_{\min}} = \{z \mid \arg(z) - \theta \in (-\pi, -\psi_{\min}], \psi_{\min} \in (0, \pi/2)\}\,.$$

Note that $C_{\psi_{\min}} \subset C_{\tilde{\psi}_{\min}} \subset C_\theta$.

Remark 6: It follows directly from the definition of the gain margin, that any scalar system satisfying Theorem 1 must have gain margin $\mathrm{GM} \geq 1/\alpha$.

Remark 7: By the circle criterion [7], it is possible to derive sector conditions similar to (6) that guarantee stability when sector bounded static nonlinearities are present in the loop.

IV. EXAMPLES

The results of Section III will now be illustrated by considering specific examples, from a very simple example to a very general interconnected system.

The first serves as an introduction to Theorem 1 and also demonstrates that it is capable of producing tight bounds. The second, involving heterogeneous delays, demonstrates the robustness results. The third is a realistic example, modeling congestion control on a single Internet link. The final example demonstrates a relationship between the new criterion and passivity.

A. Simple Example: Delay and two poles

Consider the closed loop system with loop gain

$$L(s) = k\frac{e^{-s\tau}}{s(s+a)} \tag{10}$$

where $a \geq 0$ and $\tau \geq 0$ are given, and where $k > 0$ is a constant control gain to be determined such that the system is stable. Clearly k cannot be arbitrarily large since the phase of the system decreases linearly with increasing frequency due to the delay present in the loop.

The system is analytic everywhere in the closed right half plane except at $s = 0$ and thus Theorem 1 is applicable. Since $\lim_{\epsilon\to 0}\arg\left(L(\epsilon e^{j\phi})\right) = -\phi$ it follows that when s traverses an infinitesimal semi-circle into the right half plane around the origin, $\epsilon e^{j\phi}$, the Nyquist curve of $L(s)$ remains in $\bar{\mathbb{C}}_+$. One possible factorization of $L(s) = Q(s)P^{-1}(s)$ is

$$\begin{aligned} Q(s) &= k\frac{e^{-s\tau}}{s}, && \text{(11)}\\ P(s) &= s + a. && \text{(12)}\end{aligned}$$

Since $P(j\omega)$ is in the interior of the first quadrant of the complex plane, two natural candidates for θ in Theorem 1 are $\theta = \pi/2$, corresponding to $C_{\pi/2} = \mathbb{C}_+$, and $\theta = \pi$

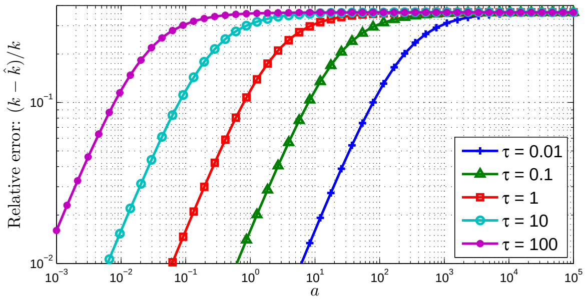

Fig. 2. Log-log plot of the relative looseness of Theorem 1 for transfer function (10) for various α's and τ's.

corresponding to $C_\pi = \mathbb{C}_\text{u}$. Both contain the Nyquist curve of $P(j\omega)$. However, since

$$\lim_{\omega\to 0^+} \arg\left(Q(j\omega) + P(j\omega)\right) = -\pi/2,$$

there exist $\omega > 0$ where

$$Q(j\omega) + P(j\omega) \notin C_\pi = \mathbb{C}_\text{u},$$

which means that $\theta = \pi$ does not satisfy the axioms of Theorem 1.[1] Instead, we continue with $\theta = \pi/2$ ($C_\theta = \mathbb{C}^+$).

The Nyquist curve satisfies $P(j\omega) \in \mathbb{C}_+$ for all frequencies. Therefore, by Theorem 1, the closed loop system is provably stable if for $\omega > 0$

$$\text{Re}\left(Q(j\omega)\right) > -\text{Re}\left(P(j\omega)\right) = -a.$$

This means that we can completely ignore the complexity in the imaginary part of $Q(j\omega) + P(j\omega)$. Since

$$-\text{Re}\left(Q(j\omega)\right) = k\frac{\sin(\omega\tau)}{\omega} = k\tau\frac{\sin(\omega\tau)}{\omega\tau} \le k\tau,$$

a sufficient condition for closed loop stability is that

$$k \le a/\tau.$$

Since calculating the phase cross-over frequency requires solving a transcendental equation, the maximum possible stable k (i.e., the gain margin of $L(s)/k$) cannot be expressed analytically. This means that any analytical stability condition must introduce some degree of conservativeness. By computing the gain margin numerically for different parameters a and τ and comparing it with the largest stable $\hat{k} = a/\tau$ we have predicted, we get an estimate on how conservative our method is for this particular example. The relative gain looseness $(k - \hat{k})/k$ as a function of the pole location a for different values of the delay τ is plotted in a log-log plot in Fig. 2. The worst case relative error seems to be about $35\,\%$ which occur for large a. However, the error monotonically decreases when a and τ decrease, and Theorem 1 thus is arbitrarily tight. To establish that tightness formally, note that

$$\begin{aligned}\arg(L(j\omega_{180})) &= -\pi - \omega_{180}\tau + \arctan\left(a/\omega_{180}\right)\\ &= -\pi - \omega_{180}\tau + \frac{a}{\omega_{180}} + O\left(\left(\frac{a}{\omega_{180}}\right)^2\right).\end{aligned}$$

Thus

$$\omega_{180}^2 = \frac{a}{\tau} + O\left(\frac{a^2}{\omega_{180}\tau}\right) = \frac{a}{\tau} + O\left(\sqrt{\frac{a^3}{\tau}}\right).$$

Then

$$\begin{aligned}L(j\omega_{180}) &= \frac{k\left(\exp(-j(\omega_{180}\tau + a/\omega_{180})) + O\left(\left(\frac{a}{\omega_{180}}\right)^2\right)\right)}{(j\omega_{180})^2}\\ &= -\frac{k\left(1 + O(a\tau)\right)}{\frac{a}{\tau} + O\left(\sqrt{a^3/\tau}\right)} = -\frac{k\tau}{a(1 + O(\sqrt{a\tau}))}.\end{aligned}$$

This shows that $(k - \hat{k})/k = O(\sqrt{a\tau})$, in keeping with the slope of 2 in Fig. 2.

B. Multiple time delays

In the next example, the pole $-a$ in the loop gain of the previous example is replaced with a sum of exponentials parametrized by τ_i. The open loop transfer function is then

$$L(s) = k\frac{e^{-s\tau}}{s\left(a + \sum_{i=1}^N e^{-s\tau_i}\right)}. \qquad (13)$$

This system is considerably more complex to analyze using standard tools because of the transcendental denominator. However, if $a > N$ then the system is analytic everywhere in $\bar{\mathbb{C}}_+$, except at the origin, and this system fits well into the framework derived in Section III.

First note that the behavior of the system when encircling the origin is analogous to the previous example. Next, consider the factorization with $Q(s)$ given by (11) and

$$P(s) = a + \sum_{i=1}^N e^{-s\tau_i}.$$

Since $a > N$ by assumption,

$$\text{Re}\left(P(j\omega)\right) \ge a - N > 0,$$

and therefore the half plane $\mathbb{C}_+$ contains $P(j\omega)$. Analogously to the previous example, the system thus is closed loop stable provided

$$k \le \frac{a - N}{\tau}.$$

Consider now the robustness. Writing

$$P(j\omega) = \sum_{i=1}^N \frac{1}{N}\left(a + Ne^{-j\omega\tau_i}\right),$$

which has the form of a convex sum over points located on the circle centered at a with radius N, shows that $P(j\omega)$ is

[1] It is still possible to use the technique to check that $L(j\omega)$ does not intersect $(-\infty, -1]$ for some frequencies.

contained in the closed disk centered in a with radius N. From basic trigonometric identities,

$$\arg\left(P(j\omega)\right) \in [-\pi/2 + \psi_{\min}, \pi/2 - \psi_{\min}],$$

where $\psi_{\min} = \arccos\left(N/a\right) \in (0, \pi/2)$. Corollary 4 is thus directly applicable and yields bounds on the phase margin and the peak of the sensitivity function. To achieve robustness, we must relax the previous stability condition. Let $\alpha \in (0,1)$ and choose

$$k \le \alpha \frac{a-N}{\tau}$$

which implies a gain margin $\mathrm{GM} \ge 1/\alpha$.

In the case that $N/a = 1/\sqrt{2}$ and $\alpha = 1/2$, the bound (8b) gives a phase margin $\mathrm{PM} \ge 22.5°$. Furthermore, we have from Corollary 4 that

$$\|S\|_\infty \le \frac{2}{\sin\left(\arccos(N/a)\right)} = \frac{2}{\sqrt{1 - N^2/a^2}}.$$

The bounds are illustrated in Fig. 3. Fulfilling the gain margin bound corresponds to crossing the imaginary axis right of $-\alpha$. Fulfilling the phase margin bound corresponds to not entering the unit circle (the solid black curved line) above the solid black line. (The dash-dotted line corresponds to the more accurate bound (8a).) Fulfilling the bound on the peak of the sensitivity function corresponds to not entering the disk defined by the circle shown by the dashed black line. The gray dots corresponds to Nyquist curves of 1000 realizations of $L(j\omega)$ where N is chosen to be a random integer between 1 and 10, and τ and τ_i, $i = 1, \ldots, N$, are chosen uniformly in the interval $[0, 10]$. The parameter $a = N\sqrt{2}$ (to keep the ratio N/a fixed) and the gain is set to $k = \alpha(a-N)/\tau$. From the figure it is apparent that all bounds predicted are satisfied. Our predictions seem correct but slightly conservative for this case.

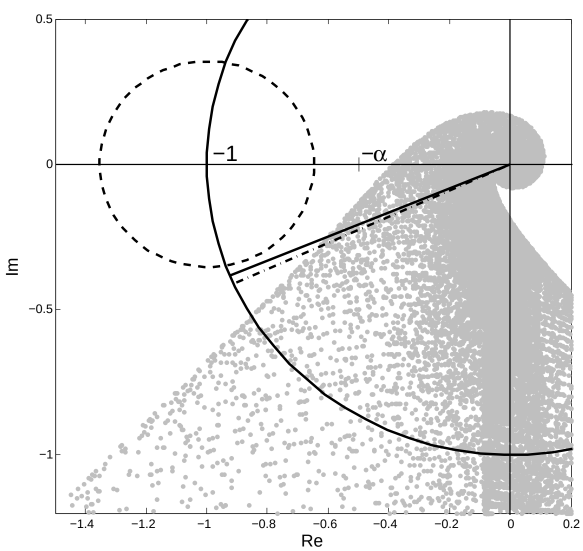

Fig. 3. Graphical illustration of robustness bounds for (13). See text.

C. Internet Congestion Control

We next illustrate the usefulness of our results by studying the closed loop stability of a highly complex transfer function that appears in Internet congestion control.

Consider the following open loop transfer function:

$$L(s) = \frac{1/c}{x_c/c + \sum_i \frac{x_i}{c}\frac{1}{1-\exp(-s\tau_i)}} \sum_i \frac{e^{-s\tau_i}}{1 - e^{-s\tau_i}} W_i(s), \tag{14}$$

where $c > 0$, $x_c \in [0, c]$, $x_i \in [0, c]$, $\tau_i > 0$, and the conservation law

$$\sum_i x_i + x_c = c \tag{15}$$

holds. This transfer function derives from [5] (see also [1]) where the general version of the model is shown to accurately model the equilibrium dynamics of a communication network where sources apply window based congestion control (such as TCP [23]). In particular (14) models the specific case of a single bottleneck link network where the ith source updates its window size according to $W_i(s)$ using the observed queuing delay as input signal, which is considered to reflect the amount of congestion in the network. The model is thus suitable for studying, for example, FAST TCP [19] and TCP Vegas [2]. The parameter c corresponds to the capacity of the link, x_c models the amount of uncontrolled traffic (UDP traffic and short lived TCP flows), and x_i and τ_i are the ith source's equilibrium sending rate and experienced loop (round trip) delay respectively. The conservation law (15) corresponds to a fully-utilized link: at the equilibrium point, the traffic flowing into the link equals the capacity. We remark that results from analyzing FAST TCP [17] suggests that applying standard low order Padé approximations of the exponential functions is not sufficient to study scalable stability of the closed loop system. This is due to the periodic narrow resonance peaks introduced by the multiple $1 - e^{-s\tau_i}$ terms in the denominator that are challenging to capture over a sufficiently wide frequency interval for arbitrary parameters.

Apparently $L(s)$ is quite a non-standard transfer function, and even assuming the simplest possible window control, a constant gain $W_i(s) = k_i$, stability analysis seems quite cumbersome at a first glance. However, it can be shown [5] that $L(s)$ is analytic in $\bar{\mathbb{C}}_+$ if $W_i(s)$ is asymptotically stable as well for all i. Thus, under the assumption of a static gain window update, Theorem 1 is applicable, it is also very useful since it "hides" much of the complexity of the model.

A suitable factorization $L(s) = Q(s)/P(s)$ is

$$Q(s) = \frac{1}{c}\sum_i \frac{e^{-s\tau_i}}{1 - e^{-s\tau_i}} k_i,$$

$$P(s) = \frac{x_c}{c} + \sum_i \frac{x_i}{c}\frac{1}{1 - e^{-s\tau_i}}.$$

A key observation is that $1/(1 - e^{-j\omega\tau_i})$ lies on the line $\mathrm{Re}(z) = 1/2$. In particular it monotonically traces the line from $1/2 - j\infty$ to $1/2 + j\infty$ when ω increases from $n2\pi/\tau_i$ to $(n+1)2\pi/\tau_i$, where $n \in \mathbb{Z}^+$. Fortunately we do not have to consider the complexity in the imaginary part explicitly as demonstrated next. Since $\mathrm{Re}\left(1/(1-e^{-j\omega\tau_i})\right) = 1/2$,

$$\mathrm{Re}\left(P(j\omega)\right) = \frac{x_c}{c} + \sum_i \frac{x_i}{c}\mathrm{Re}\left(\frac{1}{1-e^{-j\omega\tau_i}}\right) = \frac{1}{2} + \frac{x_c}{2c} \geq \frac{1}{2},$$

whence $P(j\omega) \in \mathbb{C}_+$. Theorem 1 then yields that any combination of k_i's such that

$$\mathrm{Re}\left(Q(j\omega)\right) > -\frac{1}{2}$$

provides closed loop stability. Now, since

$$\mathrm{Re}\left(Q(j\omega)\right) = \frac{1}{c}\sum_i k_i \mathrm{Re}\left(-\frac{1}{1-e^{j\omega\tau_i}}\right) = -\frac{1}{2}\sum_i \frac{k_i}{c}$$

and $\sum_i x_i \leq c$ it follows that if each source scales the window gain k_i to be proportional to but smaller than its sending rate, $k_i < x_i$, the closed loop system is stable. This information is available locally at each source and need not be communicated over the network explicitly. Note that, in this case, the step of factoring the loop gain reflects the physical structure of the system.

D. Relation to passivity

Consider the multidimensional open loop system $L(s) = Q(s)P^{-1}(s)$ and assume for simplicity that $L(s), Q(s)$ and $P(s)$ are analytic everywhere in $\bar{\mathbb{C}}_+$, Assume further that, for all ω, $P(j\omega) + P^*(j\omega)$ is positive definite, or equivalently $\underline{\lambda}(P(j\omega) + P^*(j\omega)) > 0$. That is, $P(\cdot)$ is strictly positive real or strictly passive [20]. Then, $F(P(j\omega)) \subset \mathbb{C}_+$, since, for any square matrix A, [4]

$$\mathrm{Re}(F(2A)) = F(A+A^*) = [\underline{\lambda}(A+A^*), \overline{\lambda}(A+A^*)]. \quad (16)$$

By Theorem 1, the system is closed loop stable if

$$F(P(j\omega)) + F(Q(j\omega)) \subset \mathbb{C}_+,$$

or equivalently if

$$\underline{\lambda}(P(j\omega) + P^*(j\omega)) + \underline{\lambda}(Q(j\omega) + Q^*(j\omega)) > 0. \quad (17)$$

Clearly, closed loop stability is achieved if $\underline{\lambda}(Q(j\omega) + Q^*(j\omega)) \geq 0$, corresponding to $Q(s)$ being positive real (i.e., passive). It is well-known that interconnection of a positive real and strictly positive real linear system under feedback is closed loop stable [7]. Similarly, an interconnection of a strictly passive and a non-passive system can be stable if the *sum* of their "storage functions" [7] is decreasing. Condition (17) is analogous to this less restrictive form, and explicitly quantifies how far $Q(s)$ can be from being passive, in terms of how passive $P(s)$ is.

V. Conclusions

This paper has presented a sufficient condition that establishes closed loop stability for linear time invariant dynamical systems with transfer functions that are analytic in the open right complex half plane. The technique is based on bounding Nyquist curves by using frequency dependent half planes and is suitable for analyzing a large class of highly complex, possibly interconnected, systems. Robustness guarantees for the provably stable systems were also derived.

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