On Binary Output of Cellular Neural Networks

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1 Introduction

This letter presents an amended proof of an important result, the binary output property (BOP), in the theory of cellular neural networks (CNNs) [1, 2]. A CNN is a dynamical system of the form

$$\dot{\mathbf{x}} = -\mathbf{x} + \mathbf{A}\mathbf{y} + \mathbf{k}, \tag{1}$$

where $\mathbf{x}$ is the state vector, $\mathbf{y} = f(\mathbf{x})$ is the corresponding output vector, $\mathbf{k}$ represents the input and bias terms, and $\mathbf{A}$ is the constant feedback matrix. Typically f is the piecewise linear function $f_L(x) = (|x+1| - |x-1|)/2$.

The BOP is important in applications [3–5] and theory [6, 7], and may be stated thus:

Property 1 (BOP) *A convergent CNN with $a_{ii} > 1$ for all i will, for a given input,* $\mathbf{k}$*, converge to a state with $|y_i| = 1$ for all i for almost all initial conditions.*

The standard proofs of the BOP given in [1, 2] are unclear on the technical justification of one of the steps involved. Section 2 of this letter describes an example which demonstrates the weakness of existing proofs of the BOP, and Section 3 provides a more rigorous proof.

2 The Previous Approach

In [1] and [2] a proof of the BOP was presented which treats the system (1) as a set of uncoupled first order systems. It is then assumed that the dynamics of these uncoupled systems are the same as those of the single coupled system. It is not obvious that this is the case. In particular, in [2] it was stated that each individual uncoupled equation will settle in a state with $x_i \notin (-1, +1)$ for all initial conditions, $x_i(0)$, except one, with no reference to the initial conditions of the other equations, $x_j(0)$, $j \neq i$. However, in the following example, for any $x_i(0) \in (-1, +1)$ there exists an $x_j(0)$ such that the system will remain in a state with $x_i \in (-1, +1)$. Consider the system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2}$$

which is a CNN satisfying the requirement $a_{ii} > 1$. In the linear region $\mathbf{x} \in (-1, +1)^2$, $y_i = f_L(x_i) = x_i$, and so (2) can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}. \tag{3}$$

Clearly the solution of this will not diverge for any initial condition of the form $\mathbf{x} = (u, -u)^T$ with $u \in (-1, +1)$, and thus for any $x_1(0) \in (-1, +1)$, there exists an $x_2(0) = -x_1(0)$ for which $x_1(t) \in (-1, +1)$ for all t, as required.

The key to this example is the fact that there is a non-positive eigenvalue, $\lambda \leq 0$ of $\mathbf{A} - \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix. Thus it has a set of equilibria in $(-1, +1)$ which is of measure zero in $\mathbb{R}^2$, but whose projection onto the coordinate axes is of non-zero measure in $\mathbb{R}^1$. Note that because the set of equilibria is of measure zero in $\mathbb{R}^2$, this system does not violate the BOP, but merely casts doubt upon an aspect of the original proof, namely the decoupling of the system.

3 New Proof of the Binary Output Property

A more rigorous proof of the BOP can be obtained by considering the complete coupled system. As the CNN is convergent by hypothesis, it is sufficient to show that every equilibrium with $|x_i| < 1$ for some i is unstable. It can be shown that an equilibrium point will be unstable if there is at least one positive eigenvalue of $\mathbf{AD} - \mathbf{I}$ in the corresponding homogeneous linearised system,

$$\dot{\mathbf{x}} = (\mathbf{AD} - \mathbf{I})\mathbf{x}, \tag{4}$$

where $\mathbf{D} = \mathrm{diag}(f'(x_1), \ldots, f'(x_n))$ and $\mathbf{I}$ denotes the identity matrix. This has been shown to be the case when $\mathbf{A} - \mathbf{I}$ is diagonally dominant and has positive diagonal elements [8]. With the aid of the following theorem, it will be shown to hold for general $\mathbf{A}$ with $a_{ii} > 1$, so the BOP depends only on the diagonal elements, as stated in [1, 2].

Theorem 1 *Let α and β be positive real numbers. If $\mathbf{A}$ is an $n \times n$ matrix whose diagonal elements, a_{ii}, are real and satisfy $a_{ii} > \alpha$, and $\mathbf{D} = diag(\delta_1, \ldots, \delta_n)$ is a real diagonal matrix with $\delta_i = 0$ or $\delta_i \geq \beta$ for all i, and not all δ_i are zero, then $\mathbf{AD}$ has at least one eigenvalue λ with $Re(\lambda) > \alpha\beta$.*

Proof: The trace of $\mathbf{AD}$, $\sum_{i=1}^{n} a_{ii}\delta_i$, is equal to the sum of its eigenvalues. Each zero on the diagonal of $\mathbf{D}$ causes an entire column of $\mathbf{AD}$ to be zero, and thus corresponds to a zero eigenvalue. Thus the sum of the m non-zero eigenvalues is equal to the sum of the m non-zero diagonal elements of $\mathbf{AD}$. Since the non-zero elements $a_{ii}\delta_i$ are all greater than $\alpha\beta$, their sum must exceed $m\alpha\beta$, and so at least one of the eigenvalues must have real part greater than $\alpha\beta$. Thus $\mathbf{AD}$ has at least one eigenvalue λ with $\mathrm{Re}(\lambda) > \alpha\beta$ as required. □

With the aid of this theorem, the BOP may now be proved for $f = f_L$. In the linearised system (4), $\mathbf{D} = \mathrm{diag}(\delta_1, \ldots \delta_n)$ with $\delta_i = 1$ if $x_i \in (-1, +1)$ and 0 if $x \notin [-1, +1]$. Any linearisation with $x_i \in (-1, +1)$ for some i satisfies the conditions of Theorem 1 with $\alpha = \beta = 1$. Hence $\mathbf{AD}$ has at least one eigenvalue with real part greater than 1 and $\mathbf{AD} - \mathbf{I}$ will have a positive eigenvalue. Thus there will be no stable equilibrium in that linearisation. If $x_i = \pm 1$ for some i then there will be multiple linearisations possible. If $x_j \in (-1, +1)$ for some $j \neq i$, then all possible linearisations satisfy the conditions of Theorem 1, and none will have a stable solution, as required by the BOP. Alternatively if $x_j \notin (-1, +1)$ for all $j \neq i$ then $|y_j| = 1$ for all j as required by the BOP. This establishes the BOP when $f = f_L$.

4 Conclusion

This letter has pointed out a weakness in the standard proofs of the binary output property in CNN theory, and provided a more rigorous proof based on the eigenvalues of the system matrix.

Theorem 1 is more general than is required to prove the BOP. Indeed it can be used to prove results about binary outputs of CNNs using output functions other than f_L [9].

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