

Dynamic Allocation of Subcarriers and Transmit Powers in an OFDMA Cellular Network

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Abstract

This paper considers the problem of minimizing outage probabilities in the downlink of a multiuser, multicell OFDMA cellular network with frequency selective fading, imperfect channel state information and frequency hopping. The task is to determine the allocation of powers and subcarriers for users to ensure that the user outage probabilities are as low as possible. We formulate a min-max outage probability problem and solve it under the constraint that the transmit power spectrum at each base station is flat. In particular, we obtain a subchannel allocation algorithm that has complexity $O(L \log L)$ in L , the number of users in the cell. We also consider suboptimal but implementable approaches with and without the flat transmit power spectrum constraint. We conclude that the flat transmit spectrum approach has merit, and warrants further study.

Index Terms

Cellular network, resource allocation, power control, orthogonal frequency division multiple access (OFDMA), subcarrier allocation, power spectrum, fading channels, outage capacity, fast frequency hopping, interference averaging.

I. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) is an important technique for communicating over frequency selective channels. By dividing the available bandwidth into orthogonal, non-interfering

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subcarriers and adopting a parallel transmission strategy, it offers better immunity to the multipath fading effects of the wireless channel than single carrier transmission systems. OFDM is widely deployed in commercial systems such as xDSL modems [1], [2] and low mobility wireless LANs [3]. It is also part of WiMax [4], and a strong candidate for future wireless cellular systems.

Although OFDM typically multiplexes low rate data substreams from a single user onto all the subcarriers, a cellular network can use orthogonal frequency division multiple access (OFDMA), in which the data streams from different users are multiplexed onto subsets of the subcarriers. This paper considers the downlink resource allocation problem in an OFDMA cellular system.

We consider the problem of allocating subcarriers and powers to users within each cell, subject to meeting a data rate requirements. In the classical approach to power control [5], [6], channels are first allocated to mobiles based on their rate requirements, and then power control adjusts the power levels to account for the locations of the mobiles in the network. On the downlink, this implies that the transmit power spectral density at a base station varies across the system bandwidth.

We propose a novel approach to power control in which the transmit power spectrum is always *flat*. If more power must be allocated to a mobile, such as when it moves close to the cell boundary, then there are two independent ways to achieve this: by increasing the cell power level as a whole, or by increasing the number of subcarriers allocated to the mobile. We will show that this new approach significantly reduces the maximum outage probability in the system.

We work with the premise that the base stations have knowledge of the statistics of channel conditions, but not the instantaneous channel gains. In this setting, a commonly used metric of performance is the outage probability, the fraction of time for which the rate is not achieved. The primary objective in this paper is to minimize the maximum outage probability across all mobiles in the network. We formulate this problem in Section III-A as a joint optimization over transmit powers and subcarrier allocations.

One difficulty in solving this problem is that the outage probability for a user is a complicated function that depends not only on the transmit power and fading characteristics of the signal for the mobile itself, but also on those for mobiles in other cells, interfering signals that have been sent by

other base stations. Since the mobiles are attempting to meet their individual rate targets, the transmit powers of all base stations are coupled in a complicated way. In Section III, we solve the problem assuming that there exists a Genie who can instantly return for us the outage probability of any user, as a function of the power levels and subcarrier allocations in the network.

In Section V, we provide a more practical power and subcarrier allocation algorithm, the *Power First Algorithm*, that does not require a Genie. This algorithm is suboptimal with respect to the min-max objective function of Section III-A, but simulation results show that it performs only slightly worse than the optimal solution. In the *Power First Algorithm*, powers are *first* allocated to the base stations according to a novel distributed power control algorithm, and subcarriers are *then* allocated to the users in a manner that approximately minimizes the maximum outage probability across the cell, given the powers. This algorithm provides a flat transmit power spectrum at each base station.

A nice feature of the Power First Algorithm is that the power allocation and subcarrier allocation are separable problems: Even if the powers are not optimal with respect to the network as a whole, the subcarrier allocation algorithm solves a local min-max optimization problem within the cell. Indeed, the power allocation can be run on a slow time-scale while the subcarrier allocations simultaneously run at a faster time-scale, although here we analyze the case when the subcarrier allocation is run after the power allocation completes. This separability is due to the flat transmit power spectrum at each base station.

In Section VI, we contrast the Power First Algorithm with the *Subchannel First Algorithm*, which takes the classical approach to power control: The bandwidth allocation is proportional to the rate requirement of the link, and per-link power spectral densities are determined to meet the rate requirement of each link. This approach does not provide flat transmit power spectra to the base stations, and we compare its performance with that of the Power First Algorithm in Section VIII.

A. Structure of the paper

In Section II, we introduce a model for OFDMA that includes the assumption of frequency hopping over the subcarriers [7, Section 4.4.2]. Thus, we distinguish between physical subcarriers and the

logical *subchannels* that are allocated to each mobile. Subchannel allocations within a cell involve the specification of subcarrier hopping patterns that maintain orthogonality between users in the same cell, but which provide time diversity with respect to the fading parameters, and with respect to interference from users in other cells. Together, these assumptions imply that the outage probability is a function of the *numbers* of logical subchannels allocated to a mobile, not the particular choice of physical subcarriers (which would be the case if there were no frequency hopping). This avoids a combinatorial explosion in the optimization problems that we consider in this paper.

In Section III-A we pose the *Joint Power and Subchannel Allocation Problem* which will be solved by the *Genie-aided Joint Algorithm* in Section III-C. Before tackling this problem, we first consider the local problem of minimizing the maximum outage probability in each cell. The resulting *Genie-aided subchannel allocation algorithm*, is then a subroutine in the *Genie-aided Joint Algorithm*. The latter algorithm provides a useful benchmark, but it cannot be implemented in practice: It requires knowledge of expectations over random parameters which appear difficult to estimate, and it requires considerable cooperation between the base stations.

In Section V-B we fix the powers, and formulate an approximate version of the subchannel allocation problem, but one that does not require a Genie. We provide the *Practical Subchannel Allocation Algorithm* to solve this problem, and show that it has complexity $L \log L$ in the number, L , of mobiles in the cell (see Theorem 8). Motivated by this, we make the modeling assumption that the power control and subchannel allocation tasks can be performed sequentially, with power levels first selected by the base stations, and then held fixed while subchannels are allocated. In Section V-A, we provide a simple, measurement-based, decentralized power control scheme that minimizes the sum of powers subject to a fade margin. This scheme enforces the constraint that the transmit power spectrum be flat. Given the final powers found by this scheme, we then apply the Practical Subchannel Allocation Algorithm. In Section VIII we compare the performance of the resulting *Power First Algorithm* with the *Subchannel First Algorithm*, and we find that the performance of the Power First Algorithm is significantly better with respect to the min-max outage probability objective from Section III-A.

B. Related work and assumptions

Resource allocation in OFDM systems has received considerable attention in the literature. Much work assumes channel state information (CSI) is available at the transmitter. In single cell scenarios, the maximum total rate is obtained by water pouring [8]. Note that the subcarrier allocation problem is combinatorial, and becomes very difficult to solve once a fairness criterion is considered: suboptimal approaches are considered in [9], [10].

In multi-cell networks, the resource allocation problem is further complicated by inter-cell interference. Iterative water pouring techniques can be considered [11]. However, the iterative water-pouring approach is not, in general, guaranteed to achieve the maximum total rate, nor does it provide fairness among the users.

General formulations of optimization problems for allocating bandwidths and powers for networks of interfering links are provided in [12]. A common theme is the NP hardness of all problems posed, both as the number of subcarriers grows large, for fixed links, and as the number of links grows large, for fixed numbers of subcarriers. Part of the complexity comes from the combinatorial nature of the problem, in that there are many subcarriers, and each has a different channel gain (although there are typically strong correlations between neighbouring subcarriers). The other difficulty is a lack of convexity when interference is taken into account [13]. Although the paper [12] is focused on time-invariant problem formulations, with applications to Digital Subscriber Lines (DSL), these difficulties apply equally to OFDM wireless networks which, even worse, are typically time-varying. This motivates the search for problem formulations that avoid these difficulties.

There are many heuristic approaches to resource allocation in DSL networks that involve the allocation of spectrum to the different links. Collectively, this topic is known as “spectrum balancing” and suboptimal approaches include game-theoretic methods, including iterative water filling [14], [15]; high SNR approximations, and the use of Geometric programming methods [16], [17]; methods of successive convex approximation [18], and dual decomposition methods [19]. Other recent papers on this topic include [20], [21].

Interference is just as significant in wireless networks, but wireless links are typically time varying. This requires the problems to be solved in real-time, which only adds to the computational difficulties. To base a spectrum allocation algorithm on channel state information requires channel measurement, feedback, computation, and convergence all to take place before the channels change. This may be possible in low mobility scenarios, but seems more difficult for high mobility scenarios. Much current research in wireless is devoted to overcoming these difficulties.

Recent work on fair allocation in wireless mesh networks, decomposes the problem into subcarrier and power allocations, and time scheduling [22]. Assuming the users have their own CSI, [23] computes the optimal allocation strategy under a collision model of packet interference. An approach to joint spectrum allocation, power control, routing, and congestion control for wireless networks is provided in [24].

The problem of minimizing power levels subject to rate constraints on the individual links, in a multi-cell context, was addressed in [25]. The subcarriers are allocated to users in a heuristic fashion, and then iterative power control takes place. In the bandwidth-constrained power minimization problem [26], an upper bound is imposed on the number of subcarriers to be allocated to each user to minimize the mutual interference between users.

All of the work referred to above assumes CSI is available at the transmitters, which may not be realistic in mobile, cellular scenarios, especially when the channel conditions vary quickly with time. In this case, the resource allocation needs to be performed based on statistical knowledge of the channel conditions. Such resource allocation problems have been studied in [27] and [28]. While [27] considers a single user rate maximization problem subject to an outage probability target, [28] investigates the problem of characterizing the outage probability region for a single cell system. In contrast, the present paper considers an outage probability based resource allocation problem for a multiple user, multiple cell system.

The present paper considers real-time data transmission, in which coding over time is limited to one hop of the frequency hopping cycle, but each mobile gets frequency diversity from the multiple

subcarriers it is allocated during the hop. Outage capacity and outage probability are then the appropriate metrics to consider. It is in this context, with frequency hopping, that we propose the idea of constraining the transmit power spectrum to be flat. In other settings, a non-flat spectrum may be preferable.

Recent work [29] shows that a flat transmit power spectrum is *not* optimal if spectrum can be allocated as a function of the position of the mobile in the cell. By coordinating the spectrum allocation, the interference is no longer white, and everyone benefits. This “fractional re-use” [29], [30] can be very beneficial, but it does not integrate well with frequency hopping and interference averaging. The difficulty with fractional power re-use is that it is only applicable if the channel changes slowly, so that joint optimization over all variables is possible. The virtue of interference averaging is that rapid variations can be averaged out. In a mobile radio scenario, it may be possible to combine the merits of both approaches. This is a topic for future research. In the present paper, we consider a bandwidth over which all mobiles are hopping, and study the merits of a flat transmit power spectrum constraint for this scenario.

II. SYSTEM MODEL

Consider the downlink of an OFDM cellular network which consists of a set of N base stations, denoted by $\mathcal{N} = \{1, 2, \dots, N\}$. Each base station $n \in \mathcal{N}$ has a set C_n of users. Let the number of subcarriers in the system be N_c .

Assume that the fading on the subcarriers is too fast to track at the base station, and that the base station only has statistical knowledge of the fading. In this setting, a natural measure of performance is the outage probability. A user will be in outage if the total mutual information between sent and received signals summed over the allocated subchannels falls short of the threshold needed to support the target data rate of the user.

The outage probability of a particular user will depend not only on the allocation of the subcarriers for that user, and powers on those allocated subcarriers, but also on the interference experienced on the allocated subcarriers, which will in turn depend on the power allocation in other cells. Due to the difficulties associated with characterizing the user outage probability as a function of all these

parameters, we

- use frequency hopping based on a Latin square design [7, Section 4.4.2] and,
- constrain each base station to use a uniform transmit power spectral density (uniform PSD) across the frequency band.

With the use of frequency hopping, the users will now be allocated logical *subchannels* (which are the hopping patterns across the subcarriers as specified by the Latin square design) instead of physical subcarriers.

The use of a uniform PSD at each base station, together with the use of a Latin square design for frequency hopping, achieves the effect of making all subchannels in any given link statistically identical. This makes it sufficient to model the number of subchannels for each user, instead of individual allocations of physical subcarriers to users. Since we will focus on outage capacity, we will assume coding occurs over a single hop, so the diversity that a user obtains equals the number of allocated subchannels.

Each base station is assumed to have access to all available subcarriers, i.e., the frequency reuse factor is 1 (however, this assumption can easily be relaxed). Consequently, each base station will have N_c available subchannels. Let $\mathbf{\Pi} = \{1, 2, \dots, N_c\}$. The apportionment of subchannels to users within a cell n can be modelled by a subchannel allocation vector $\boldsymbol{\eta}_n \in \mathbf{\Pi}^{L_n}$, where L_n denotes the number of links (users) in cell n . Since each user must be allocated at least one subchannel, we require $N_c \geq L_n$ for all $n \in \mathcal{N}$.

Base station n allocates $\eta_{n,m} \in \mathbb{N} \setminus \{0\}$ subchannels to its user m , and naturally $\eta_{n,m}$ can never exceed N_c , the total number of subcarriers. The feasibility constraint on the allocation vector is:

$$\sum_{m \in C_n} \eta_{n,m} \leq N_c.$$

To maintain consistency with the assumption of a flat transmit power spectrum across all subcarriers, it will be used with equality here:

$$\sum_{m \in C_n} \eta_{n,m} = N_c. \tag{1}$$

Let $\boldsymbol{\eta}$ denote the N -tuple of such allocation vectors for the network.

Link $[n, m]$ is allocated $\eta_{n,m}$ subchannels indexed by a set $H_{n,m}$. Due to frequency hopping, the physical subcarrier allocated to subchannel $i \in H_{n,m}$ changes every hop. Thus, we model the gain on subchannel i by a positive random variable $G_{n,m}^{(i)}$ with a continuous distribution function.

Denote the transmit power of base station n by q_n , and let $\mathbf{q} = (q_n)_{n \in \mathcal{N}}$ be the vector of total powers for the network. Clearly, the SIR achievable on subchannel $i \in H_{n,m}$ is random, since it depends on the random channel gain on the allocated subcarrier, and also on the random gains of all the interfering cells on this subcarrier. Although the hopping pattern corresponding to subchannel $i \in H_{n,m}$ is associated with base station n , there will always be interference from other cells, since each cell spreads its power uniformly over the subcarriers. Let $G_{k,m}^{(i)}$ denote the instantaneous path gain on subchannel $i \in H_{n,m}$ from base station k to mobile m , valid for $k \neq n$. This is the path gain on the particular physical subcarrier that subchannel $i \in H_{n,m}$ has chosen.

Denote the receiver noise power at user $m \in C_n$ by $\sigma_m^2 > 0$. Then, the random signal to interference and noise ratio (*instantaneous SIR*) on subchannel i of link $[n, m]$ is

$$\gamma_{n,m}^{(i)}(\mathbf{q}) = \frac{G_{n,m}^{(i)} q_n}{\sigma_m^2 + \sum_{k \in \mathcal{N}, k \neq n} G_{k,m}^{(i)} q_k}. \quad (2)$$

This formula is a consequence of the flat transmit power spectra of the base stations.

We assume that the rate achievable on this subchannel, in bits per channel use, is $f(\gamma_{n,m}^{(i)}(\mathbf{q}))$, a deterministic function of the SIR. We assume that $f(\gamma)$ is a continuous, increasing function of γ , with $f(0) = 0$. A specific example is the function $f(\gamma) = \log_2(1 + \gamma)$, which applies if the link is optimal with respect to Shannon capacity.

Let the total system bandwidth be W Hz. Since there are N_c subcarriers, the OFDM symbol duration is N_c/W seconds. The total rate *available* to user $m \in C_n$ is then given by $\frac{W}{N_c} \sum_{i \in H_{n,m}} f(\gamma_{n,m}^{(i)})$ bits/sec. In this paper, we normalize the total system bandwidth to unity and work with spectral efficiency, so in this sense the available rate for user $m \in C_n$ (in the given hop) is given by

$$R_{n,m}(\mathbf{q}, \eta_{n,m}) = \sum_{i \in H_{n,m}} R_{n,m}^{(i)}(\mathbf{q}) \text{ bits/sec/Hz}. \quad (3)$$

where

$$R_{n,m}^{(i)}(\mathbf{q}) = \frac{1}{N_c} f(\gamma_{n,m}^{(i)}(\mathbf{q})) \quad \text{bit/sec/Hz.} \quad (4)$$

Let $R_{n,m}^{\text{tar}} > 0$ be the normalized target rate for user $m \in C_n$ in bits/sec/Hz. Then the outage probability of user m , with $\eta_{n,m}$ subchannels, when the power allocation for the network is \mathbf{q} , is given by

$$\theta_{n,m}(\mathbf{q}, \eta_{n,m}) = \mathbb{P} \left(\sum_{i \in H_{n,m}} R_{n,m}^{(i)}(\mathbf{q}) < R_{n,m}^{\text{tar}} \right). \quad (5)$$

We define the cell outage probability as the maximum outage probability of the users in the cell. When the power allocation is \mathbf{q} and the subchannel allocation for the cell is $\boldsymbol{\eta}_n$, we define the outage probability of cell n to be:

$$\Omega_n(\mathbf{q}, \boldsymbol{\eta}_n) = \max_{m \in C_n} \theta_{n,m}(\mathbf{q}, \eta_{n,m}). \quad (6)$$

III. MINIMIZING THE MAXIMUM OUTAGE PROBABILITY

The objective of this paper is to derive good algorithms for allocating powers and subcarriers to users to balance the outage probabilities across the entire network whilst not consuming too much transmit power. There is background noise in the model, so outage probabilities can always be reduced by increasing transmit powers, subject to diminishing returns. Thus we begin with a formulation in which the total transmit power in the entire network of cells is constrained. The algorithm we derive to solve this problem is centralized, but we consider distributed formulations in later sections of the paper.

A. Joint Power and Subchannel Allocation Problem

Suppose that a given total power level, q^{total} , must be shared amongst the base stations in the network. Let \mathcal{Q} denote the set of feasible power vectors:

$$\mathcal{Q} = \{ \mathbf{q} \in \mathbb{R}_+^N : \sum_{n \in \mathcal{N}} q_n = q^{\text{total}} \}, \quad (7)$$

where \mathbb{R}_+ is the strictly positive reals. A joint power and subchannel allocation problem is the following:

$$\min_{\mathbf{q}, \boldsymbol{\eta}} \max_{n \in \mathcal{N}} \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n) \quad (8a)$$

such that

$$\mathbf{q} \in \mathcal{Q}, \quad (8b)$$

$$\sum_{m \in \mathcal{C}_n} \eta_{n,m} = N_c, \quad \forall n \in \mathcal{N}. \quad (8c)$$

This problem formulation has the nice property that it has a solution for *any* set of rate requirements. An alternative formulation is to minimize the total power subject to individual outage probability targets for the users. The latter problem is also interesting, but has the disadvantage that it may be infeasible, and there is no known way to determine, a priori, whether a given problem instance is feasible or not. In the present paper, we find the min-max formulation above to be very useful in comparing the performance of different practical algorithms that we propose in Section IV.

In this section, we will obtain an algorithm to solve the Joint Power and Subchannel Allocation Problem (8), assuming the existence of a Genie which can evaluate the outage probability function $\theta_{n,m}(\mathbf{q}, \eta_{n,m})$ in (5). In practice, there is no known formula for evaluating (5). Moreover, the outage values cannot be physically measured without briefly trying each possible power and subchannel allocation. Thus this algorithm is not in itself a solution for real-time implementation. Our approach will be to use it as an off-line technique to solve the Joint Power and Subchannel Allocation Problem (8), to provide a benchmark against which to compare the performance of practical algorithms. To do this, we will replace the Genie with Monte-Carlo estimates of the outage probabilities. Note that the algorithm becomes of practical interest for real-time implementation as soon as one can replace (5) with a practical technique to measure or estimate the outage probabilities. In Sections V and VI, we will provide practical, distributed, suboptimal algorithms that do not require a Genie.

B. The Genie-aided subchannel allocation algorithm

We begin with the sub-problem of allocating subchannels to users, under a fixed allocation of transmit powers to the base stations in the network. The additional problem of selecting these transmit powers is addressed in Section III-C.

Since the base stations use a uniform transmit PSD, if the transmit power allocation for the network

is fixed, varying the subchannel allocation for the users within a given cell will not affect the subchannel allocation for the users in any other cell. Thus, the subchannel allocation for users in each cell can be done independently, without knock-on effects between cells.

Our aim is to obtain a subchannel allocation for users in each cell that minimizes the maximum outage probability among the users in the cell. The corresponding optimization problem for a typical cell n is:

$$\min_{\boldsymbol{\eta}_n} \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n) \quad (9a)$$

such that

$$\sum_{m \in C_n} \eta_{n,m} = N_c. \quad (9b)$$

Since \mathbf{q} is fixed, the problems in (9), one for each cell, are independent of each other. Furthermore, since the number of subchannels is a discrete quantity, it may not be possible to obtain a subchannel allocation that exactly equalizes the outage probabilities among the same cell users.

Since \mathbf{q} is fixed, we will drop the dependence of $R_{n,m}^{(i)}$ and $\theta_{n,m}$ on \mathbf{q} in the notation in this section.

Before presenting the algorithm to solve (9), we begin with some structural results. The function Ω_n defined in (6) provides the maximum outage probability in cell n . Analogously, define the minimum outage probability in cell n :

$$\omega_n(\mathbf{q}, \boldsymbol{\eta}_n) = \min_{m \in C_n} \theta_{n,m}(\boldsymbol{\eta}_n). \quad (10)$$

Denote the optimal value of the problem (9) by Ω_n^* , and an optimal subchannel allocation in cell n by $\boldsymbol{\eta}_n^*$. Observe that the function $\theta_{n,m}(\boldsymbol{\eta}_n)$ in (5), treating $\eta_{n,m}$ as the argument, is monotonically decreasing in $\eta_{n,m}$. The following lemma and corollary follow from this fact.

Lemma 1: For any subchannel allocation $\boldsymbol{\eta}_n$ satisfying (9b) we have that

$$\omega_n(\mathbf{q}, \boldsymbol{\eta}_n) \leq \Omega_n^* \leq \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n).$$

Proof: If $\omega_n(\mathbf{q}, \boldsymbol{\eta}_n) > \Omega_n^*$ then by the monotonicity of $\theta_{n,m}(\boldsymbol{\eta}_n)$ we have that $\eta_{n,m} \leq \eta_{n,m}^*$ for all $m \in C_n$, with strict inequality for at least one $m \in C_n$. But this contradicts (9b). The second inequality in the statement of the lemma follows from the fact that Ω_n^* is the minimum in (9a). ■

Corollary 2: If $\omega_n(\mathbf{q}, \boldsymbol{\eta}_n) = \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n)$ for a subchannel allocation $\boldsymbol{\eta}_n$ satisfying (9b) then $\boldsymbol{\eta}_n$ is a solution to problem (9).

We now use Lemma 1 to construct an optimal solution to problem (9), the *Genie-aided subchannel allocation algorithm*:

- **Initialization:** Let $\boldsymbol{\eta}_n$ be an arbitrary feasible channel allocation. Set $k \leftarrow 0$. Define $\boldsymbol{\eta}_n^{(0)}$ by

$$\eta_{n,m}^{(0)} \leftarrow \min\{x : \theta_{n,m}(\mathbf{q}, x) \leq \omega_n(\mathbf{q}, \boldsymbol{\eta}_n)\}.$$

- **While** $\sum_{m \in C_n} \eta_{n,m}^{(k)} > N_c$
 - $m \leftarrow \operatorname{argmin}_{m \in C_n} \theta_{n,m}(\eta_{n,m}^{(k)} - 1)$
 - Construct $\boldsymbol{\eta}_n^{(k+1)}$ from $\boldsymbol{\eta}_n^{(k)}$ by setting

$$\eta_{n,m}^{(k+1)} \leftarrow \eta_{n,m}^{(k)} - 1$$

- $k \leftarrow k + 1$

- **endwhile**

Although $\boldsymbol{\eta}_n^{(0)}$ need not in general be feasible, the algorithm terminates with a feasible allocation $\boldsymbol{\eta}_n^{(K)}$ after $K = \sum_{m \in C_n} \eta_{n,m}^{(0)} - N_c \geq 0$ steps. Note that $\theta_{n,m}(\eta_{n,m}^{(k)}) \leq \Omega_n^*$ for all $m \in C_n$ for $k = 0$, and for all k up to $k = K$ by induction on the following lemma, whence $\boldsymbol{\eta}_n^{(K)}$ solves (9).

Lemma 3: For any $\boldsymbol{\eta}_n = (\eta_{n,m})_{m \in C_n}$, if

- $\theta_{n,m}(\eta_{n,m}) \leq \Omega_n^*$, $\forall m \in C_n$, and
- $\sum_{m \in C_n} \eta_{n,m} \geq N_c + 1$,

then

$$\min_{m \in C_n} \theta_{n,m}(\eta_{n,m} - 1) \leq \Omega_n^*. \quad (11)$$

Proof: Let $\nu_m = \min\{i \in \mathbb{Z} : \theta_{n,m}(i) \leq \Omega_n^*\}$. Then $\sum_{m \in C_n} \nu_m \leq N_c$, since at least one feasible vector achieves Ω_n^* . The two hypotheses above imply that $\eta_{n,m} \geq \nu_m$ for all $m \in C_n$, with strict inequality for some $m' \in C_n$. Since $\eta_{n,m}$ is an integer, monotonicity of $\theta(\mathbf{q}, \cdot)$ implies that

$$\theta_{n,m'}(\eta_{n,m'} - 1) \leq \theta_{n,m'}(\nu_{m'}) \leq \Omega_n^*.$$

■

The Genie-aided subchannel allocation algorithm will be an important subroutine in the algorithm, to be presented in Section III-C below, for finding a solution to the Joint Power and Subchannel Allocation Problem (8).

C. Solving the Joint Power and Subchannel Allocation Problem (8):

Just as it was useful to have notation for the maximum and minimum outage probabilities within a cell, it is useful to have notation for the maximum and minimum cell outage probabilities across the network. Let

$$\Omega(\mathbf{q}, \boldsymbol{\eta}) = \max_{n \in \mathcal{N}} \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n)$$

and

$$\omega(\mathbf{q}, \boldsymbol{\eta}) = \min_{n \in \mathcal{N}} \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n).$$

Note that $\Omega(\mathbf{q}, \boldsymbol{\eta})$ is the value of the objective function to be minimized in (8). Further, for any subset of cells, $X \subseteq \mathcal{N}$, define:

$$\Omega_X(\cdot, \boldsymbol{\eta}) = \max_{n \in X} \Omega_n(\cdot, \boldsymbol{\eta}_n) \tag{12}$$

which, for fixed X and $\boldsymbol{\eta}$, is a function of power levels \mathbf{q} .

We begin with a method for taking an arbitrary feasible power and subchannel allocation $(\mathbf{q}, \boldsymbol{\eta})$ (thus $\mathbf{q} \in \mathcal{Q}$) and improving it. To do this we will appeal to some simple continuity and monotonicity properties of the mapping $\Omega_X(\cdot, \boldsymbol{\eta})$; see Appendix A. These properties are not surprising, given known results for standard power control [6], and they are stated and proven in Appendix A.

The method of improvement will define a function T that maps a feasible power and subchannel allocation pair $(\mathbf{q}, \boldsymbol{\eta})$ (thus $\mathbf{q} \in \mathcal{Q}$) to a new feasible power allocation vector $\bar{\mathbf{q}}$. Thus, to define the mapping T , we provide a single power update step: Given a feasible power vector $\mathbf{q} \in \mathcal{Q}$ and a subchannel allocation $\boldsymbol{\eta}$, a new feasible power vector, $\bar{\mathbf{q}} = T(\mathbf{q}, \boldsymbol{\eta})$, is generated by *Power Update*. The key property is that this power update improves the objective function, i.e., $\Omega(\bar{\mathbf{q}}, \boldsymbol{\eta}) \leq \Omega(\mathbf{q}, \boldsymbol{\eta})$, with inequality unless the power is already optimal.

Power Update: Define $\psi = \frac{1}{2} (\Omega(\mathbf{q}, \boldsymbol{\eta}) + \omega(\mathbf{q}, \boldsymbol{\eta}))$. Let $D = \{n \mid \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n) > \psi\}$ and $\bar{D} = \mathcal{N} \setminus D \neq \emptyset$. Here, D (if nonempty) consists of cells for which the outage probabilities should be decreased and \bar{D} consists of cells for which the outage probabilities can be increased to achieve that. If $D = \emptyset$ then there is no change: set $\bar{\mathbf{q}} = \mathbf{q}$. Otherwise, calculate $\bar{\mathbf{q}}$ as follows:

- Scale the powers of cells in \bar{D} by the same factor λ until the maximum outage probability of the cells in \bar{D} equals the maximum outage probability of the cells in D . Let the resultant power allocation be $\hat{\mathbf{q}}$, i.e.,

$$\hat{q}_n = \begin{cases} q_n, & n \in D \\ \lambda q_n, & n \in \bar{D}. \end{cases} \quad (13)$$

Note that $\lambda < 1$ as $D \neq \emptyset$, and so $\sum_{n \in \mathcal{N}} \hat{q}_n < \sum_{n \in \mathcal{N}} q_n$. Satisfying the above condition follows immediately from the continuity, and monotonicity results in Appendix A (Lemmas 9 and 10, respectively). As we scale λ from 1 down to 0, the outage probabilities in D decrease, and the outage probabilities in \bar{D} increase, achieving value 1 at $\lambda = 0$, and all are continuous functions of λ . This implies that there exists a unique λ equalizing the maximum outage probabilities in D and \bar{D} .

- Now scale up the powers of all cells by the same factor μ which is given by

$$\mu = \frac{\sum_{n \in \mathcal{N}} q_n}{\sum_{n \in \mathcal{N}} \hat{q}_n} > 1.$$

Let the resultant power allocation be $\bar{\mathbf{q}}$, i.e.,

$$\bar{q}_n = \begin{cases} \mu q_n, & n \in D \\ \lambda \mu q_n, & n \in \bar{D}. \end{cases}$$

with $\sum_{n \in \mathcal{N}} \bar{q}_n = \sum_{n \in \mathcal{N}} q_n$.

This concludes the definition of the power update, and hence of the mapping T .

It is clear that *Power Update* is centralized in the way the values of λ and μ are determined, and it requires the Genie to return the outage probabilities as a function of powers and subchannel allocations.

Power Update takes a vector of powers \mathbf{q} and, if it is not optimal ($D \neq \emptyset$), generates a new vector of

powers \bar{q} by increasing the powers of the cells in D by a factor of μ and decreasing the powers of the cells in \bar{D} by a factor of

$$\lambda\mu = \frac{\lambda \sum_n q_n}{\sum_{n \in D} q_n + \lambda \sum_{n \in \bar{D}} q_n} < 1, \quad (14)$$

where the inequality uses $D \neq \emptyset$.

We now propose an iterative application of the power update procedure:

Genie-aided Joint Algorithm:

- **Initialization:** Start with an initial power allocation $\mathbf{q}^{(0)}$ which is feasible, i.e., satisfies $\sum_{n \in \mathcal{N}} q_n^{(0)} = q^{\text{total}}$. Find an optimal subchannel allocation $\boldsymbol{\eta}^{(0)}$ to work with $\mathbf{q}^{(0)}$ by solving (9) for each cell using the Genie-aided subchannel allocation algorithm. Set $k = 0$.
- **Repeat:**
 - Set $k \leftarrow k + 1$
 - Using $(\mathbf{q}^{(k-1)}, \boldsymbol{\eta}^{(k-1)})$ as the input to the Power Update, obtain a new power allocation $\mathbf{q}^{(k)}$. In other words, set $\mathbf{q}^{(k)} = T(\mathbf{q}^{(k-1)}, \boldsymbol{\eta}^{(k-1)})$.
 - Find an optimal subchannel allocation $\boldsymbol{\eta}^{(k)}$ to work with $\mathbf{q}^{(k)}$ by solving (9) for each cell using the Genie-aided subchannel allocation algorithm.
- **Until false**

We claim that the above algorithm solves the Joint Power and Subchannel Allocation Problem (8), as stated in Theorem 5, below. But first it is necessary to make a statement about the uniqueness of the solution to the Joint Power and Subchannel Allocation Problem. Indeed, we will prove that there is a unique solution for the power allocation, \mathbf{q}^* , in (8). Typically, $\boldsymbol{\eta}^*$ will also be unique, but there are scenarios in which users in the same cell can swap subchannels without affecting the maximum outage probability in the cell. For example, if two users swap a subchannel, one outage probability will decrease and the other will increase; there exist parameters for which the maximum of the two will remain unchanged. In the following, we will use E to denote the set of optimal subchannel allocation vectors.

Theorem 4: There is a unique solution for the optimal power allocation, \mathbf{q}^* , in the Joint Power and

Subchannel Allocation Problem (8). There may be more than one subchannel allocation, $\boldsymbol{\eta}^*$, in the set E , i.e., for which $(\boldsymbol{q}^*, \boldsymbol{\eta}^*)$ solves (8), but any subchannel allocation $\boldsymbol{\eta}^*$ that equalizes $\Omega_n(\boldsymbol{q}^*, \boldsymbol{\eta}^*)$, $n \in \mathcal{N}$ provides an optimal solution. Given \boldsymbol{q}^* , any solution of (9) in each cell will provide a subchannel allocation $\boldsymbol{\eta}^* \in E$.

Proof: See Appendix C.

Let Ω^* denote the optimal value in the Joint Power and Subchannel Allocation Problem, and let $\Omega^{(k)}$ be the value generated at step k of the Genie-aided Joint Algorithm, i.e., $\Omega^{(k)} = \Omega(\boldsymbol{q}^{(k)}, \boldsymbol{\eta}^{(k)})$. The following theorem specifies the convergence properties of Genie-aided Joint Algorithm.

Theorem 5: 1) $\Omega^{(k)}$ is a decreasing sequence that converges to Ω^* as $k \uparrow \infty$.

2) $\boldsymbol{q}^{(k)} \rightarrow \boldsymbol{q}^*$ as $k \uparrow \infty$.

3) The sequence $(\boldsymbol{q}^{(k)}, \boldsymbol{\eta}^{(k)})$ has accumulation points, and for any accumulation point $(\boldsymbol{q}, \boldsymbol{\eta})$, we have

$$\boldsymbol{q} = \boldsymbol{q}^* \text{ and } \boldsymbol{\eta} \in E.$$

4) There exists an integer, M , such that $\forall k \geq M, \boldsymbol{\eta}^{(k)} \in E$.

Proof: see Appendix D.

In summary, the Genie-aided Joint Algorithm starts with an arbitrary feasible power allocation and improves it with respect to the objective function of (8) at each iteration. After each power update, it solves (9) in each cell using the Genie-aided subchannel allocation algorithm (Section III-B). The power level in each cell is guaranteed to converge to the unique optimal power allocation, and the maximum outage probabilities in all cells tend to the same value. After a finite number of steps, the subchannel allocation becomes and remains optimal, although it can switch from one optimal allocation to another.

IV. A PRACTICAL APPROACH TO POWER AND SUBCHANNEL ALLOCATION

In the following sections we present more practical algorithms that do not require evaluation of outage probabilities (5) at every power or bandwidth update. In this section, we present an overview of our approach, which involves decoupling the power updates from the subchannel updates, and using fade margins in the power update part of the algorithm.

Our approach to power control in fading channels is to use fade margins: Random, frequency-selective fading parameters are averaged over frequency to obtain average channel gains, and only these average gains are used in the power control algorithm. The power control algorithm uses enhanced rate targets, to provide a margin to protect against the fluctuations from fading. The objective of the power control algorithm is to minimize total average power [6]. Given the fade margin, the power control algorithm does not need to consult an outage probability Genie during power updates. We remark that selecting the fade margin is a one-parameter optimization problem that can be handled numerically, or one can simply measure performance across a range of possible fade margins and choose a desired operating point.

In standard power control, subchannels are allocated *first*, as a function of the data rate requirements of the users, and then per-user power allocation is used to try to achieve the enhanced data rate requirements of the users. We will consider this approach in Section VI below, and we will thereby obtain a transmit power spectrum that is not in general flat.

In Section V, we propose a novel power control algorithm that *does* provide flat transmit power spectra. In this approach, we reverse the usual ordering of subchannel allocation and power control, and select the power levels to be used by the base stations first. Once the power levels are fixed, we then provide a practical method of subchannel allocation (Section V-B). Since the transmit power spectra at all base stations are flat, subchannels can be re-allocated amongst the users in the cell without any knock-on effects to other cells, just as in Section III-B. The subchannels can be allocated in each cell to approximately minimize the maximum outage probability in the cell, as we describe in Section V-B. The whole algorithm is summarized in Section V-D.

We will investigate the performance of these algorithms as the fade margin is varied from small to large. Given a fade margin, the performance of both proposed algorithms can be measured numerically and compared with each other, and with the Genie-aided Joint Algorithm. The performance metrics are the min-max outage probability measure, (8), and the total average power measure, (7). Note that the Genie-aided Joint Algorithm is parameterized by total power consumption, (7), so it is easy to make

this comparison.

V. POWER FIRST ALGORITHM

This section provides a novel power and subchannel allocation algorithm, which adheres to the framework of having a flat transmit power spectrum at each base station. As the name suggests, the power levels in each cell are chosen *first*, and the subchannels are chosen based on these power levels. In fact, the subchannel allocation algorithm can be run independently of the power allocation algorithm: The only pre-requisite of the subchannel allocation algorithm is that the power levels used by the base stations are fixed on the time-scale of the algorithm, and the transmit spectra are flat. These prerequisites are met by the power control algorithm that we describe in Section V-A. Taken together, the two algorithms can be viewed as a joint method for power and subchannel allocation.

A. Power allocation

In this subsection, we avoid the Genie by proposing a power control algorithm whose objective is that of minimizing the total average transmit power, as in classical power control formulations [6]. We will address the problem of selecting suitable fade margins in Section VII, but in this section, the rate targets are assumed to be the enhanced rate targets chosen after fade margins have been applied, and for the remainder of this subsection, outage probabilities are not considered.

The novel feature of this algorithm is the flat transmit power spectrum at each base station. Compared to the standard power control algorithm [6], the base stations lose the ability to independently vary the per-user power levels, but they can still vary the total power spectral density. Moreover, they *can* control the share of power allocated to each mobile by varying the amount of bandwidth that is allocated to each user. It is important to note that bandwidth allocation we refer to in this subsection is only *virtual*: Only the power levels will be used by the system. Since the bandwidth allocations derived by the algorithm are not actually to be used, we allow them to take continuous values, rather than integer values. In Section V-B, we will propose a subchannel allocation algorithm that does takes account of the discrete nature of the subchannels.

For a given subchannel allocation, let the weight vector $\mathbf{w}_n = \boldsymbol{\eta}_n/N_c$ denote the proportions of (virtual) subchannels allocated to each user in cell n , and \mathbf{w} be the corresponding N -tuple of weight vectors. Constraint (1) becomes $\sum_{m \in C_n} w_{n,m} = 1$. Recall from Section II that the total system bandwidth is normalized to unity. Thus, $w_{n,m}$ represents the normalized bandwidth allocated to user $m \in C_n$. Since the total power at base station n is q_n , the power allocated to user m is $p_{n,m} = w_{n,m}q_n$.

Let $\bar{G}_{k,m}$ be the average path gain of link $[k, m]$ to user m in cell n from a base station k (not necessarily $k = n$). Then, the value for the signal to interference and noise ratio at $m \in C_n$ that we will use is

$$\bar{\gamma}_{n,m}(\mathbf{q}) = \frac{\bar{G}_{n,m}q_n}{\sigma_m^2 + \sum_{k \in \mathcal{N}, k \neq n} \bar{G}_{k,m}q_k} \quad (15)$$

which in practice will be a simple ratio of average power measurements. This can be measured by each user m , and transmitted to its controlling base station n , without global co-ordination. We now formulate the power control problem as a power minimization subject to all users achieving rates $R_{n,m}^{\text{margin}} > R_{n,m}^{\text{tar}}$, which exceed their target rates by a fade margin (see Section VII). The formulation is:

$$\min_{\mathbf{q}, \mathbf{w}} \sum_{n \in \mathcal{N}} q_n \quad (16a)$$

such that for all n ,

$$w_{n,m} f(\bar{\gamma}_{n,m}(\mathbf{q})) \geq R_{n,m}^{\text{margin}}, \quad \forall m \in C_n, \quad (16b)$$

$$\sum_{m \in C_n} w_{n,m} = 1, \quad (16c)$$

$$w_{n,m} > 0, \quad \forall m \in C_n, \quad (16d)$$

$$q_n > 0. \quad (16e)$$

Note that, in the formulation above, the weights $w_{n,m}$ are allowed to be continuous.

The problem (16) is studied in [31] and there it is shown that if there is a solution, it is unique, and an algorithm is proposed that finds the unique solution when it exists. We use it to obtain the power allocation for the base stations. The algorithm is summarized as follows.

Decentralized Power Control Algorithm:

Initialization: Start with any initial power vector $\mathbf{q}^{(0)} > 0$. Set $k \leftarrow 0$.

Repeat:

- Compute a pseudo-weight $\hat{w}_{n,m}$ for each user m given the power vector $\mathbf{q}^{(k)}$ (using (16b)):

$$\hat{w}_{n,m} = \frac{R_{n,m}^{\text{margin}}}{f(\bar{\gamma}_{n,m}(\mathbf{q}^{(k)}))}, \quad \forall m \in C_n.$$

Define $\hat{\sigma}_n = \sum_{m \in C_n} \hat{w}_{n,m}$. Note that the vector $\hat{\mathbf{w}}_n$ computed above is infeasible if $\hat{\sigma}_n > 1$.

- Compute a feasible weight vector $\mathbf{w}_n^{(k)}$ by normalizing $\hat{\mathbf{w}}_n$:

$$w_{n,m}^{(k)} = \frac{\hat{w}_{n,m}}{\hat{\sigma}_n}, \quad \forall m \in C_n.$$

- Use the newly computed $\mathbf{w}_n^{(k)}$ to compute the target transmit power $\hat{\rho}_{n,m}$ for each user $m \in C_n$ (using (16b)):

$$\hat{\rho}_{n,m} = \frac{q_n^{(k)}}{\bar{\gamma}_{n,m}(\mathbf{q}^{(k)})} f^{-1} \left(\frac{R_{n,m}^{\text{margin}}}{w_{n,m}^{(k)}} \right), \quad \forall m \in C_n$$

- Compute the transmit power to use for the next iteration:

$$q_n^{(k+1)} = \begin{cases} \min_{m \in C_n} \hat{\rho}_{n,m}, & \text{if } \hat{\sigma}_n > 1 \\ \max_{m \in C_n} \hat{\rho}_{n,m}, & \text{otherwise.} \end{cases}$$

- Increment k .

Until false

Each iteration of this algorithm can be considered as a mapping from $\mathbf{q}^{(k)}$ to $\mathbf{q}^{(k+1)}$. Although this mapping does not satisfy the monotonicity condition required in Yates' framework [6], the convergence of the algorithm can be proved by examining the sequence of power vectors generated. For any sequence $(\mathbf{q}^{(k)})_{k=0}^{\infty}$ generated by the algorithm, a monotonically non-increasing upper bounding sequence, and a monotonically non-decreasing lower bounding sequence, can be constructed with the property that both bounding sequences provably converge to the minimal solution. This implies that the sequence $(\mathbf{q}^{(k)})_{k=0}^{\infty}$ also converges to the minimal solution. See [31] for the details of this argument.

Once the power levels are deemed to have converged, we then invoke the subchannel allocation part of the joint algorithm, described in Section V-B below.

B. Subchannel Allocation under a Fixed Power Allocation

In this subsection, we assume that power levels are fixed, and power spectra are flat, pre-requisites that are met by the *Decentralized power control algorithm* described in Section V-A. If we had access to the Genie, we could apply the Genie-aided subchannel allocation algorithm (Section III-B), to obtain the optimal subchannel allocation given the fixed power levels. Indeed, we will provide numerical results for this approach in Section VIII, but in this section, we propose a more practical subchannel allocation algorithm that does not require a Genie.

This subsection revisits the subchannel allocation problem (9). It will derive an algorithm amenable to practical implementation, which serves as a subroutine in the *Power First* algorithm of the present section. However, it is more general than that, and can be applied to any power allocation that is flat across the frequency band.

Since \mathbf{q} is fixed, we will again drop the dependence of $R_{n,m}^{(i)}$ and $\theta_{n,m}$ on \mathbf{q} in the notation in this section.

Note that $R_{n,m}^{(i)}$ are all identically distributed because of frequency hopping. Let $Z_{n,m} = \sum_{i \in H_{n,m}} R_{n,m}^{(i)}$, $\mathbb{E}[R_{n,m}^{(i)}] = \mu_{n,m}$ and variance of $R_{n,m}^{(i)}$ be $\beta_{n,m}^2$. Since the power allocation is fixed, each mobile can measure the distribution of its own SIR, $\gamma_{n,m}^{(i)}$, over a sufficient period. From this, it can calculate the distribution of $R_{n,m}^{(i)}$ by (4), and hence $\mu_{n,m}$ and $\beta_{n,m}^2$. These two values are then transmitted to the base station. Define

$$\hat{Z}_{n,m}(\eta_{n,m}) = \frac{Z_{n,m} - \eta_{n,m} \mu_{n,m}}{\sqrt{\eta_{n,m}} \beta_{n,m}}. \quad (17)$$

The outage probability of user m is given by

$$\theta_{n,m}(\eta_{n,m}) = \mathbb{P}(\hat{Z}_{n,m}(\eta_{n,m}) < B_{n,m}(\eta_{n,m})) \quad (18)$$

where

$$B_{n,m}(\eta_{n,m}) = \frac{R_{n,m}^{\text{tar}} - \eta_{n,m} \mu_{n,m}}{\sqrt{\eta_{n,m}} \beta_{n,m}}. \quad (19)$$

The simplification in this section comes by assuming that the $\hat{Z}_{n,m}(\eta_{n,m})$ are approximately identically distributed.

By definition, these random variables share the same first two moments. The approximation is particularly reasonable if all the $\eta_{n,m}$ are moderately large and $R_{n,m}^{(i)}$ are independent, for then the central limit theorem applies. Independence represents the case in which the number of users in the cell is large and no user gets allocated a large proportion of the subchannels. As a result, a user's allocation of subchannels can be well separated in frequency, hence the average correlation between subchannels on a given link can be neglected.

We claim that the approximation is reasonable even when the $\eta_{n,m}$ are not large. The numerical results will demonstrate that this approximation is "reasonable" in that it yields outage probabilities very close to those of the benchmark Genie-aided algorithm.

Under the assumption that $\hat{Z}_{n,m}$ are identically distributed, the right hand side of (18) shows that minimizing the maximum of $\theta_{n,m}(\eta_{n,m})$ is equivalent to minimizing the maximum of $B_{n,m}$'s among the users. This yields the following optimization problem:

$$\min_{\boldsymbol{\eta}_n} \max_{m \in C_n} B_{n,m}(\eta_{n,m}) \quad (20a)$$

such that

$$\sum_{m \in C_n} \eta_{n,m} = N_c, \quad (20b)$$

$$\eta_{n,m} \in \mathbb{Z}_+, \quad \forall m \in C_n. \quad (20c)$$

Note that there are N cells, each with its own subchannel allocation problem. Since the transmit powers at the base stations are held fixed, each problem can be solved independently of the others, as was the case in Section III-B.

C. Practical Subchannel Allocation Algorithm

The Practical Subchannel Allocation Problem (20) is a nonlinear integer programming problem, which is combinatorial in nature. It may have multiple solutions, but has at least one since the number of users in cell n is $L_n \leq N_c$. The following *Practical Subchannel Allocation Algorithm* is an $O(L_n \log L_n)$ algorithm that exactly solves (20).

The first step is to solve the continuous relaxation of (20):

$$\min_{\mathbf{x}_n} \max_{m \in C_n} B_{n,m}(x_{n,m}) \quad (21a)$$

such that

$$\sum_{m \in C_n} x_{n,m} = N_c, \quad (21b)$$

$$x_{n,m} \in \mathbb{R}_+, \quad \forall m \in C_n. \quad (21c)$$

Note that $x_{n,m}$ are not required to be integers and $\mathbf{x}_n = (x_{n,m})_{m \in C_n}$. Note also that the inverse of $B_{n,m}(\cdot)$ is

$$x_{n,m}(B_{n,m}) = \left(\frac{\sqrt{(B_{n,m}\beta_{n,m})^2 + 4\mu_{n,m}R_{n,m}^{\text{tar}} - B_{n,m}\beta_{n,m}}}{2\mu_{n,m}} \right)^2$$

where the “+” has been taken in the quadratic implied by (19) since $\sqrt{x_{n,m}} \geq 0$.

Lemma 6: The problem (21) has a unique solution \mathbf{x}_n^* , and $B_{n,m}(x_{n,m}^*) = \tilde{B}_n^*$ for all $m \in C_n$.

Moreover, $\sum_{m \in C_n} x_{n,m}(B_n)$ is strictly decreasing in B_n .

Proof: The final claim holds since $x_{n,m}(\cdot)$ is strictly decreasing. Thus $\sum_{m \in C_n} x_{n,m}(B_n) = N_c$ has a unique solution $B_n = \tilde{B}_n^*$ since $x_{n,m}(-\infty) = \infty$ and $x_{n,m}(\infty) = 0$. Thus the unique solution to (21) is $x_{n,m}^* = x_{n,m}(\tilde{B}_n^*)$ for all $m \in C_n$, due to the monotonicity of $B_{n,m}(\cdot)$. ■

Lemma 6 shows that (21) can be solved by binary search on B_n .

Let B_n^* be the minimum of the original problem (20). Then, $\tilde{B}_n^* \leq B_n^*$. Now define a vector $\bar{\eta}_n = (\bar{\eta}_{n,m})_{m \in C_n}$ with

$$\bar{\eta}_{n,m} = \lceil x_{n,m}(\tilde{B}_n^*) \rceil, \quad \forall m \in C_n. \quad (22)$$

Since $\bar{\eta}_{n,m} \geq x_{n,m}(\tilde{B}_n^*)$ and $B_{n,m}(\cdot)$ is monotonically decreasing, $B_{n,m}(\bar{\eta}_{n,m}) \leq \tilde{B}_n^* \leq B_n^*$, $\forall m \in C_n$.

Furthermore, $\sum_{m \in C_n} \bar{\eta}_{n,m} \geq N_c$.

If $\sum_{m \in C_n} \bar{\eta}_{n,m} = N_c$ then the problem (21) has an integral solution, which solves (20). Alternatively, $\sum_{m \in C_n} \bar{\eta}_{n,m} \geq N_c + 1$. Note that $B_{n,m}(\bar{\eta}_{n,m}) \leq B_n^*$, for all $m \in C_n$, whence the excess in the number of allocated subchannels can be reduced, analogous to Lemma 3.

Lemma 7: For any $\boldsymbol{\eta}_n = (\eta_{n,m})_{m \in C_n}$, if

- $B_{n,m}(\eta_{n,m}) \leq B_n^*$, $\forall m \in C_n$, and
- $\sum_{m \in C_n} \eta_{n,m} \geq N_c + 1$, then,

$$\min_{m \in C_n} B_{n,m}(\eta_{n,m} - 1) \leq B_n^*. \quad (23)$$

Proof: Let $\boldsymbol{\eta}_n^* = (\eta_{n,m}^*)_{m \in C_n}$ be a subchannel allocation that solves (20) (not necessarily unique).

Since $\sum_{m \in C_n} \eta_{n,m} \geq N_c + 1$ and $\sum_{m \in C_n} \eta_{n,m}^* = N_c$, there exists a user $m' \in C_n$ such that $(\eta_{n,m'} - 1) \geq \eta_{n,m'}^*$. Since $B_{n,m}(\cdot)$ is monotonically decreasing, $B_{n,m'}(\eta_{n,m'} - 1) \leq B_{n,m'}(\eta_{n,m'}^*) \leq B_n^*$. ■

An integral solution to Problem (20) can be constructed by successively applying Lemma 7 zero or more times, starting with the allocations in $\bar{\boldsymbol{\eta}}_n$, and terminating after at most L_n steps in such a solution. This provides us with an algorithm, but also the proof of the following theorem:

Theorem 8: Problem (20) for cell n can be solved in time $O(L_n \log L_n)$, where L_n is the number of mobiles in cell n .

Proof: First note that \tilde{B}_n^* is the minimum of the relaxed problem (21). Problem (21) can be solved by first determining the value of \tilde{B}_n^* and then using it to find \mathbf{x}_n^* . Initial bounds on \tilde{B}_n^* can be constructed by selecting an $\mathbf{x}_n > 0$ which satisfies $\sum_{m \in C_n} x_{n,m} = N_c$, and then setting $B_n^u = \max_{m \in C_n} B_{n,m}(x_{n,m})$ and $B_n^l = \min_{m \in C_n} B_{n,m}(x_{n,m})$, giving $\sum_{m \in C_n} x_{n,m}(B_n^l) \geq N_c \geq \sum_{m \in C_n} x_{n,m}(B_n^u)$. The bisection search converges exponentially fast, at a rate independent of L_n . The computation of $\sum_{m \in C_n} x_{n,m}(B_n)$ at each step of the bisection search is linear in L_n .

The algorithm to compute the solution to (20) is constructed as follows. We start with $\boldsymbol{\eta}_n^{(0)} = \bar{\boldsymbol{\eta}}_n$ and note that this allocation involves $r_n = \sum_{m \in C_n} \bar{\eta}_{n,m} - N_c$ excess subchannels, which need to be removed. We do this iteratively. Suppose $\boldsymbol{\eta}_n^{(k)}$ has $r_n - k > 0$ excess subchannels, and satisfies $B_{n,m}(\eta_{n,m}^{(k)}) \leq B_n^*$ for all $m \in C_n$. Then Lemma 7 applies. In particular, removing a subchannel from user m' that satisfies

$$m' = \arg \min_{m \in C_n} B_{n,m}(\eta_{n,m}^{(k)} - 1) \quad (24)$$

results in a new subchannel allocation $\boldsymbol{\eta}_n^{(k+1)}$ that has $r_n - (k + 1)$ excess subchannels, and satisfies $B_{n,m}(\eta_{n,m}^{(k+1)}) \leq B_n^*$ for all $m \in C_n$. Note that implementation of (24) does not require knowledge of B_n^* . By induction, we obtain after r_n steps that there are no excess subchannels, so we must be at a

solution to the problem (20). The number of iterations is linear in L_n . Each iteration requires (24), which is a minimization over L_n elements. At an initial cost of $L_n \log L_n$, a priority queue can be used, with cost $O(\log L_n)$ per iteration [32]. ■

The proof of Theorem 8 provides an algorithm to solve the practical subchannel allocation problem which we shall call the Practical Subchannel Allocation Algorithm. Note that the algorithm is not equivalent to subtracting one subchannel each from the r_n users with the smallest $B_{n,m}(\bar{\eta}_{n,m} - 1)$, as there are cases where more than one subchannel must be removed from the same user.

The relevance of the Practical Subchannel Allocation Problem (20) is based on an approximation, and the Practical Subchannel Allocation Algorithm solves this problem. We will measure the true outage probability performance of the algorithm in Section VIII, and examine the validity of the approximation via numerical examples.

D. Power first: a joint power and subchannel allocation algorithm

In summary, we have proposed two distinct algorithms that are to be applied one after the other:

Power First Algorithm: A two stage power and subchannel allocation algorithm

- 1) Power allocation: Select increased target rates using a fade margin (Section VII). Choose base station transmit powers using the above decentralized power control algorithm, using averaged channel gains and averaged interference measurements.
- 2) Subchannel Allocation: With the above transmit powers, determine a subchannel allocation using the Practical Subchannel Allocation Algorithm to minimize the maximum outage probability of the users.

Let us now consider some implementation issues.

We assume that the mobile can measure both its own signal's received power, and the total interference plus noise power, in each of its subchannels. This enables the mobile to measure both the numerator, and denominator, of (2), for each of its subchannels, which is required in the subchannel allocation part of the algorithm. By averaging these measurements across the subchannels, and across several hops, the mobile obtains moving average estimates of the numerator and denominator, respectively, of (15),

a quantity required in the power allocation part of the algorithm. During the averaging process, the mobile's signal hops across the entire system frequency band.

During the Power First Algorithm, the mobile obtains (15), as above, and sends it back to the base station; the base station then computes the next power level to use based on feedback from all users in the cell. The subchannel allocation part requires the mean and variance of $R_{n,m}^{(i)}$, as defined in (4), but this is a simple function of (2). Again, the mobile can keep track of these statistics, and send the mean and variance back to the base station, so that the base station can compute a new subchannel allocation. In practice, the time taken to collect the statistics will dominate the time to run the Practical Subchannel Allocation Algorithm. Note that the statistics only need to be collected once, prior to the commencement of the Practical Subchannel Allocation Algorithm.

Finally, note that we have presented these two components (power allocation and subchannel allocation) as occurring in *series*, but this is only for ease of exposition. In practice, the two algorithms could be run on different time-scales: For example, the power updates could occur less frequently than the subchannel updates. The numerical results in the present paper are for the order as presented above, however.

VI. SUBCHANNEL FIRST ALGORITHM

In this section, we present the standard approach, in which subchannels are allocated *first*, as a function of the data rate requirements of the users, and then per-user power allocation is used to try to achieve the data rate requirements of the users [7]. The Subchannel First Algorithm allocates subchannels in direct proportion to the data rate requirements of the users. The remaining task is then to select the transmit power levels on each subchannel.

As in Section V, we will apply a fade margin to obtain enhanced rate targets. However, once the enhanced rate targets are fixed, the power control algorithm we employ is the standard one [6]. In applying the standard power control algorithm, we use average powers, averaged over the fading parameters. The objective of the standard power control algorithm is to minimize the sum of all transmit powers subject to meeting the (enhanced) data rate constraints of the users.

The subchannel first algorithm is summarized as follows.

Subchannel First Algorithm:

- 1) Subchannel allocation: The base station allocates subchannels to each user in direct proportion to the data rate requirements of the users.
- 2) Power allocation: Select increased target rates using a fade margin (Section VII). Apply the standard power control algorithm [6] in each subchannel. Interference measurements are averaged over many hops, and hence over all users in other cells (due to frequency hopping).

Note that the transmit power spectrum at a base station under the Subchannel First Algorithm will not in general be flat. For example, a user with a poor average gain will receive more transmit power than a user with higher gain and the same data rate target. Both users will operate at the same spectral efficiency, so the amount of transmit power allocated per subchannel will be different for the two users.

VII. FADE MARGINS

We now discuss the way fade margins are incorporated into the above heuristic algorithms. We describe three different methods, which we term a) power margin, b) additive rate margin and c) multiplicative rate margin. All three are applicable to both the Power First and the Subchannel First algorithm. Each approach has one free parameter; increasing the parameter decreases the maximum outage probability, but increases the total power consumption.

In the power margin approach, the data rate is not altered, but once the power levels are calculated, each is incremented by a fixed amount (in dB). This is the IS95 approach to fade margins. In the Subchannel First Algorithm, this amounts to scaling the power on each link by a common factor. In the Power First Algorithm, the total power from each base station is scaled by a common factor. Using a power margin decreases the outage probability of all links at the expense of increased transmit power.

The additive rate margin approach adds a fixed amount, ΔR , to the data rate requirement of each user. Similarly, the multiplicative rate margin approach scales the rate target of each user by a fixed constant. Note that if data rate requirements are the same for all users then these two are equivalent. However, they differ when traffic is heterogeneous, as in Section VIII.

If L_n is equal for all cells n , then power margins and rate margins are equivalent in the interference-free, high-SNR case. However, they differ when L_n differs, which is the focus of this paper.

The rate margin approach (either kind) gives more power to links operating at high spectral efficiency than it gives to links operating at low spectral efficiency. This follows because power increases exponentially with rate. Recall that the power first algorithm assigns power to a base station and a virtual bandwidth to each link to meet the data rate requirements, when fading is not considered. In the virtual bandwidth allocation, links with good channel conditions (high average SIR) get less virtual bandwidth and operate at high spectral efficiency. When we consider fading, these links get less frequency diversity, and suffer higher outage probabilities. Rate margins compensate for the lack of diversity: they result in more virtual bandwidth being allocated to such links, forcing other links to operate at higher spectral efficiency. We would expect then that outage probabilities will be fairer, given the diversities are now more similar. Indeed, numerical results confirm that rate margins are better than a power margin, in minimizing the maximum outage probability for a given power consumption.

One scheme can be called *superior* to another if the maximum outage probability is measured to be lower for the same total amount of transmit power. Numerical results indicate that rate margins are clearly superior to power margins for both the Power First Algorithm and the Subchannel First Algorithm. Of the two rate margins, the multiplicative is better than the additive, albeit only marginally so for the Subchannel First Algorithm. For this reason we use a multiplicative rate margin in the numerical results presented in the following section.

VIII. NUMERICAL RESULTS

In this section, we numerically evaluate and compare the performances of all the schemes proposed in this paper. We also address the effectiveness of various heuristics, including the heuristic subchannel allocation scheme proposed in Section V-C.

Primarily, we will evaluate and compare the performances of the Subchannel First Algorithm, the Power First Algorithm, and the Genie-aided Joint Algorithm. We will compare these with the performance of three other algorithms, which we will call “Subchannel only”, “Flat spectrum, rounding”, and

“Power first, Genie reallocation”, to highlight the effects of individual building blocks.

“Subchannel only” is the Practical Subchannel Allocation Algorithm (Section V-C) applied after allocating each base station an equal amount of transmit power: All the complexity is in the subchannel allocation. This is the reverse of the Subchannel First Algorithm, in which all the complexity is in the power control. By comparing “Subchannel only” with both the Subchannel First Algorithm, and the Power First Algorithm, we will demonstrate the relative importance of power control. Note that “Subchannel only” is *not* the subchannel allocation part of the Subchannel First Algorithm, it is the subchannel allocation part of the Power First Algorithm.

The purpose of “Flat spectrum, rounding” is to show that the subchannel allocation used in the Power First Algorithm is actually needed. Recall that the Power First Algorithm has a power allocation part followed by a subchannel allocation part. The power first part provides a virtual bandwidth allocation, and this bandwidth allocation can be rounded to obtain a discrete subchannel allocation. Rounding is performed by thresholding the fractional part of the number of channels allocated, with a threshold chosen such that the rounded numbers sum up to be N_c .

The purpose of “Power first, Genie reallocation” is to demonstrate the effectiveness of the heuristic subchannel allocation scheme proposed in Section V-C. In “Power first, Genie reallocation”, the power allocation part of the Power First Algorithm is run first, but the Practical Subchannel Allocation Algorithm (Section V-C) is replaced by the Genie-aided subchannel allocation algorithm (Section III-B). If the heuristic based on the Central limit theorem were no good, then there would be a significant difference in performance between the Power First Algorithm and “Power first, Genie reallocation”.

The results of all six schemes are depicted in Fig. 2, for the following cellular radio scenario. The cellular network consists of 7 hexagonal cells with each cell having a radius of 500 m. The base stations are located at the center of their respective cells. There are 70 users, uniformly randomly distributed in the network (refer to Fig. 1). Each user is served by the base station with the strongest average channel strength to that user. The rate target for each user is chosen uniformly at random from a set $\{r, 2r, 3r, 4r\}$, where r is a scaling factor to enable us to consider the impact of different traffic loads.

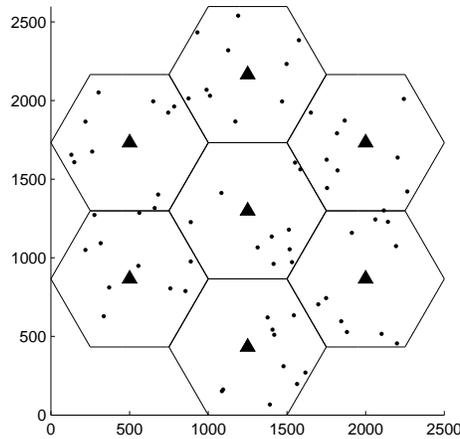


Fig. 1. Network configuration

Log-distance path loss [33] with a path loss exponent of 4 is used (with the reference distance set to 50 m). The log-normal shadowing has a mean of 0 dB and a standard deviation of 8 dB for all distances. The noise spectral density is 10^{-19} W/Hz at each receiver.

The system implements Latin square design based fast frequency hopping. There are $N_c = 113$ subcarriers in the system (when N_c is a prime, there is a simple construction of Latin square design based fast frequency hopping). As the FFT implementation requires the number of subcarriers to be a power of 2, this can correspond to a system with 128 subcarriers, with the remainder to be used as pilot and null subcarriers. Within each link, the subcarrier gains are independent and Rayleigh distributed about the flat fading gain. Monte Carlo simulations were used for calculating the outage probabilities of the users.

Figure 2 plots the maximum user outage probability against the total symbol energy for three different values of r . The symbol energy (plotted on the horizontal axis) is a function of the multiplicative rate margin applied, but a different function for each algorithm: A particular rate margin will not have the same effect on each algorithm, but we can compare the algorithms across the whole range of rate margins, which is the way Fig. 2 was generated. From Fig. 2, we can compare the outage performance of each algorithm at the same level of total transmit power.

The first thing to notice from Fig. 2 is the extremely poor performance of “Subchannel only”, which

confirms the importance of power control. This result is not unexpected, since more heavily loaded cells need more power, and an equal allocation of power between cells is too rigid. Moreover, our metric captures the *maximum* outage probability; “Subchannel only” might not look so bad if we measured average outage probability, for example. The Power First Algorithm does much better, but “Flat spectrum, rounding” shows that the initial virtual subchannel allocation in the Power First Algorithm needs to be improved. We conclude that the power allocation in “Subchannel only”, and the subchannel allocation in “Flat spectrum, rounding”, are both deficient.

Now consider the Power First Algorithm and, in particular, the heuristic Practical Subchannel Allocation Algorithm (Section V-C). In “Power first, Genie reallocation”, the heuristic Practical Subchannel Allocation Algorithm is replaced by the Genie-aided subchannel allocation algorithm (Section III-B). There is very little difference between the two. Further numerical results (not shown) reveal that the Genie-aided subchannel allocation algorithm produces an almost identical allocation to the heuristic, with only half a dozen of the $7N_c = 791$ subchannels allocated differently. Although the number of subchannels is typically small, the good performance of the heuristic inspired by the central limit theorem is presumably because the Rayleigh fading distribution is already quite similar to a Gaussian, and the central limit theorem applies to the sum of any number of Gaussian random variables. These two schemes are outperformed by the Genie-aided Joint Algorithm because the latter is free to readjust powers based on the new channel allocation.

We now focus on the three proposed schemes: the Power First Algorithm, the Subchannel First Algorithm and the Genie-aided Joint Algorithm. Notice that the outage performance of all schemes improves with transmit power. With smaller rate margins, these schemes have similar maximum outage performance. However, the Genie-aided Joint Algorithm and the Power First Algorithm benefit more from increased rate margins than does the Subchannel First Algorithm. Overall, the performance of the Genie-aided Joint Algorithm is the best of these three schemes, while the Subchannel First Algorithm is the worst. One can conclude that Power First Algorithm is significantly better than Subchannel First Algorithm in the scenario that we simulated.

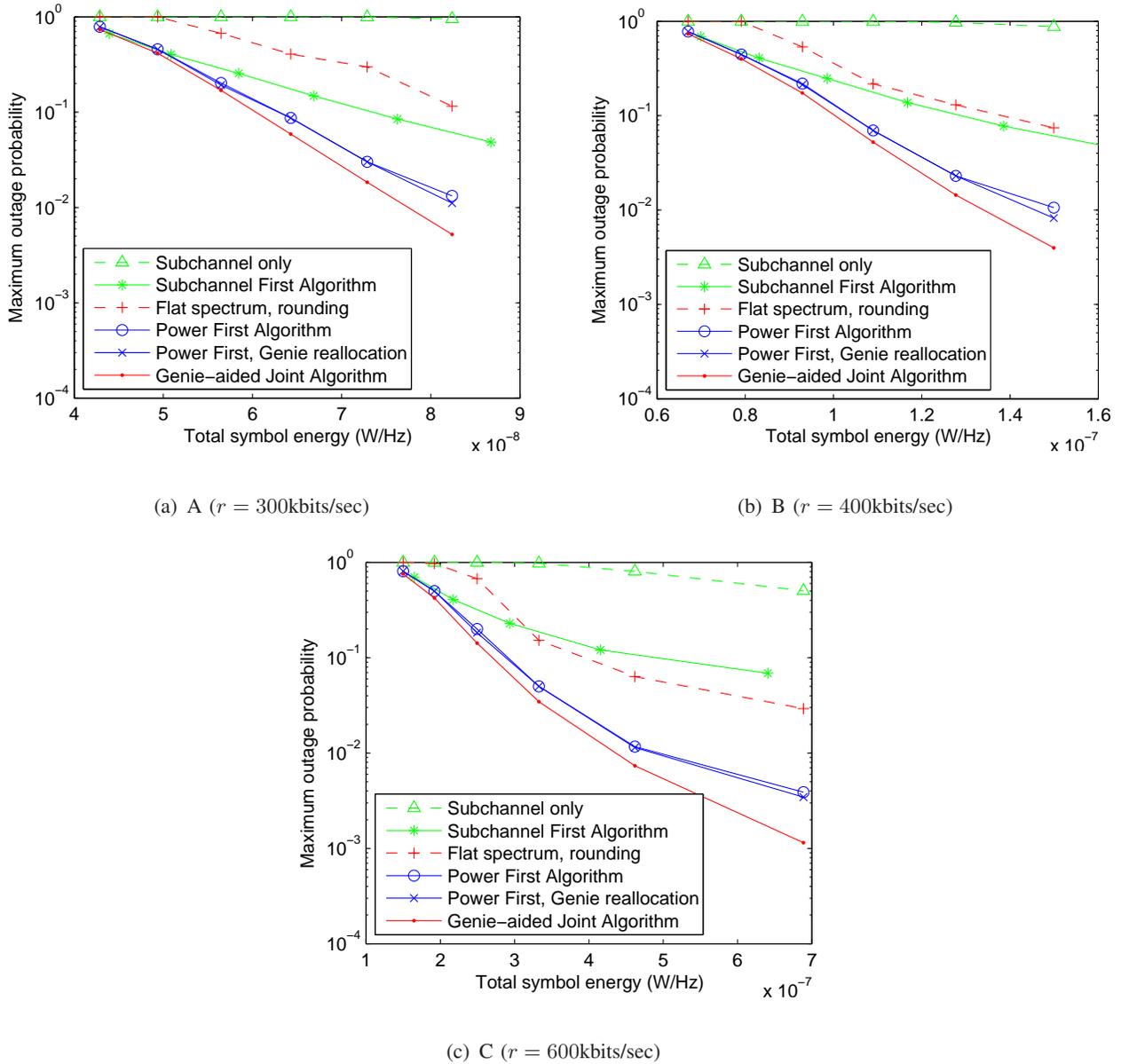


Fig. 2. The outage performance of the schemes with varying rate margins. The total symbol energy increases with the rate margin applied. The r values of 300, 400, 600 kbit/sec were used.

We now compare the distribution of user outage probabilities for the three schemes: the Power First Algorithm, the Subchannel First Algorithm, and the Genie-aided Joint Algorithm. Figure 3 shows the distribution of user outage probabilities for all schemes when $r = 400$ kbits/sec. The rate margins of 30% (Power First Algorithm) and 26% (Subchannel First Algorithm) were chosen to make the total symbol energy equal (to 1.09×10^{-7} W/Hz) for all schemes. The outage probabilities of users are grouped by cells, with the probability values sorted in descending order in each cell. Note that the outage performances of the Power First Algorithm and the Genie-aided Joint Algorithm are much better

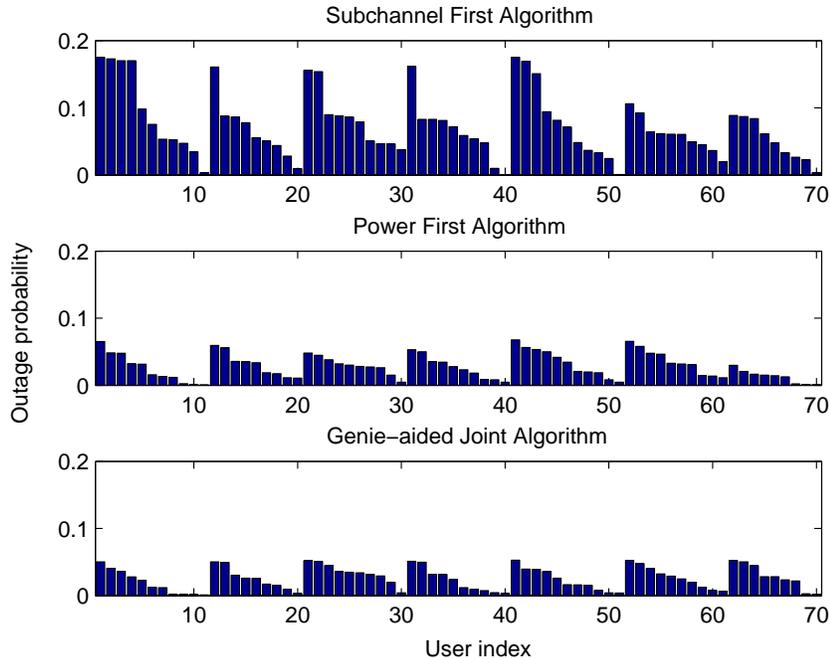


Fig. 3. Outage performance of the users when $r = 400$ kbits/sec. The rate margins for the Subchannel First Algorithm (26%) and the Power First Algorithm (30%) were selected to make all schemes use a same total symbol energy. The outage probabilities of same cell users are sorted in descending order.

than that of the Subchannel First Algorithm. Furthermore, the Genie-aided Joint Algorithm indeed achieves nearly equal cell outage probabilities as it solves the min-max outage probability minimization problem (8). The Power First Algorithm also has similar cell outage probabilities even though it takes a suboptimal (relative to the objective in (8a)), but realizable, two stage approach to resource allocation.

In a dynamic environment, it is important that a resource allocation algorithm be able to track changing conditions. The rapid convergence of the Power First Algorithm is demonstrated in Fig. 4, which shows that the power allocation converges after around six iterations for the case $r = 300$ kbit/sec.

IX. CONCLUSIONS

This paper has considered the problem of minimizing the maximum outage probability across all users in a multiple user, multiple cell OFDM cellular network under frequency selective fading. The problem involves the joint allocation of power and bandwidth to each user.

Much of the literature on resource allocation for OFDM cellular networks focuses on problem

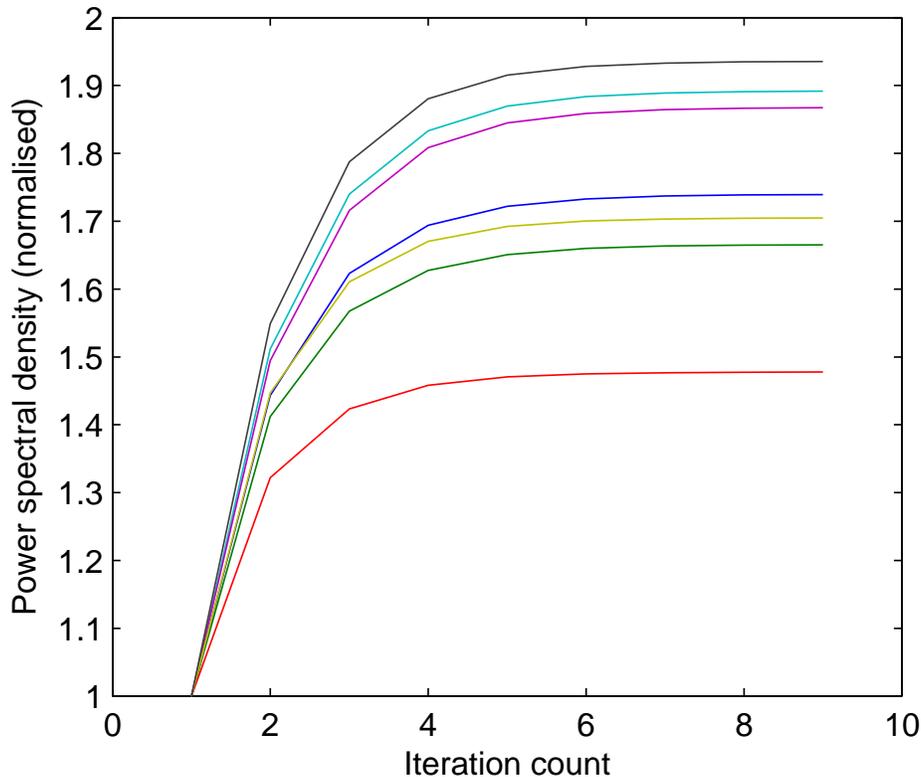


Fig. 4. Time evolution of power spectral density of each cell during the first phase of the Power First Algorithm, normalized by the power spectral density after the first iteration.

formulations which turn out to be NP-hard [12]. Our formulation turns out to be tractable, and this is primarily because we adopt a frequency hopping framework, which benefits from interference averaging, and makes all subchannels essentially equivalent. Nevertheless, the system is interference-limited, and there is a strong coupling between all the users in the network. The tractability of the problem we pose requires proof, which we supply in this paper.

The min-max problem is formulated under the constraint that the transmit power spectrum at each base station must be flat. This is a non-standard constraint, and the solution to this problem provides a novel method of power control that does not fall into the classical framework of Yates [6]. The total power level at each base station is updated at each step of the power control algorithm, but the way that it is allocated to each user in the cell is via a novel method of subchannel allocation. Indeed, we provide the Genie-aided Joint Algorithm: a joint power and bandwidth allocation algorithm to solve the min-max optimization problem. We show it does much better than the standard approach of first

allocating the bandwidth in a fixed way, and then providing a power only update algorithm.

The Genie-aided Joint Algorithm suffers from the fact it is centralized, and it also requires a method to compute outage probabilities, which we do not provide in this paper. In the terminology of the paper, the Genie-aided Joint Algorithm requires a Genie to supply these outage probabilities, when needed by the algorithm. For this reason we propose a number of more practical algorithms, which are decentralized, and which do not require the Genie. Of particular interest is the Power First Algorithm, which is similar to Genie-aided Joint Algorithm in that each base station controls an overall power level, uses a flat transmit power spectrum, and controls the per-user power allocation via the allocation of subchannels to the individual users.

The Power First Algorithm is a two stage power and subchannel allocation algorithm. The power allocations are obtained first by applying a multiplicative rate margin to the user rate targets and using the average received power measurements. The second stage allocates numbers of subchannels to users. This allocation is performed by the Practical Subchannel Allocation Algorithm within each cell in order to minimize the maximum outage probability in the cell. This balancing of outage probabilities is done independently in each cell using the locally measured channel statistics, independent of other cells. In practice, the two stages can run on different time-scales; the Practical Subchannel Allocation Algorithm can be run for any fixed allocation of transmit powers, not just the optimal ones.

A key benefit of using the Power First Algorithm as a method for resource allocation is that it avoids a combinatorial explosion as the number of users and subcarriers grows large. In particular, the first stage (power control) is not combinatorial (it is a continuous relaxation of a combinatorial problem) and it converges exponentially fast, as is typical for power control algorithms of this type. The second stage Practical Subchannel Allocation Algorithm is combinatorial, but converges in time $O(L_n \log L_n)$, where L_n is the number of users in the cell, as shown in Theorem 8. The use of frequency hopping is crucial at this step.

We compared the performance of the Power First Algorithm with the standard approach of first allocating subchannels (independently of the interference coupling between cells) and then doing standard

power control (referred to as the Subchannel First Algorithm in this paper). Numerical investigations showed that the proposed Power First Algorithm is significantly superior with respect to the objective of minimizing the maximum outage probability. One plausible explanation is that a flat transmit power spectrum reduces the interference fluctuations in the system. In the Subchannel First Algorithm, per-user power densities are allocated according to user locations, and as a result interference levels fluctuate more wildly than in the Power First Algorithm. This may be harmful to outage capacity, although we remark that the same conclusion cannot be made with respect to ergodic capacity.

APPENDIX

A. Continuity and monotonicity properties of various outage probability expressions

The following lemmas will be used in the proof of convergence of the Genie-aided joint algorithm in Section III-C.

Lemma 9: For any subchannel allocation $\boldsymbol{\eta}$, the following functions are continuous functions of the first argument on the closure of \mathcal{Q} , for any $n \in \mathcal{N}$, $m \in C_n$ and $X \subseteq \mathcal{N}$:

$$\begin{aligned} \theta_{n,m}(\cdot, \boldsymbol{\eta}) &= \Omega_n(\cdot, \boldsymbol{\eta}) \\ \Omega_X(\cdot, \boldsymbol{\eta}) &= \max_{n \in X} \Omega_n(\cdot, \boldsymbol{\eta}_n) & \Omega(\cdot, \boldsymbol{\eta}) &= \Omega_{\mathcal{N}}(\cdot, \boldsymbol{\eta}) \\ \omega_X(\cdot, \boldsymbol{\eta}) &= \min_{n \in X} \Omega_n(\cdot, \boldsymbol{\eta}_n) & \omega(\cdot, \boldsymbol{\eta}) &= \omega_{\mathcal{N}}(\cdot, \boldsymbol{\eta}) \end{aligned}$$

Proof: The continuity of θ follows from the continuity of $R_{n,m}^{(i)}(\mathbf{q})$ and the assumption that the fading has a continuous distribution. For a detailed argument, see [34].

The remaining statements follow since the maximum and minimum of continuous functions are continuous. ■

Lemma 10: The cell outage probability Ω_n satisfies

- 1) $\Omega_n(\mathbf{q}, \boldsymbol{\eta}_n)$ is monotonically decreasing in q_n .
- 2) $\Omega_n(\mathbf{q}, \boldsymbol{\eta}_n)$ is monotonically increasing in q_k , for all $k \neq n$.
- 3) If $\alpha > \beta$, then

$$\Omega_n(\alpha \mathbf{q}, \boldsymbol{\eta}_n) < \Omega_n(\beta \mathbf{q}, \boldsymbol{\eta}_n). \quad (25)$$

Lemma 10 is a corollary of the following:

Lemma 11: The per-user outage probability has the following monotonicity properties:

- 1) $\theta_{n,m}(\mathbf{q}, \eta_{n,m})$ is monotonically decreasing in q_n .
- 2) $\theta_{n,m}(\mathbf{q}, \eta_{n,m})$ is monotonically increasing in q_k , for all $k \neq n$.
- 3) If $\alpha > \beta$, then

$$\theta_{n,m}(\alpha \mathbf{q}, \eta_{n,m}) < \theta_{n,m}(\beta \mathbf{q}, \eta_{n,m}). \quad (26)$$

Proof: The events of interest strictly increase or decrease, in the sense of set inclusion, with the corresponding changes in power levels. The result then follows from the axioms of probability. For a detailed argument, see [34]. ■

B. Structural results about $T(\mathbf{q}, \boldsymbol{\eta})$, $\Omega(\mathbf{q}, \boldsymbol{\eta})$, and some related functions

This section contains results useful for proving Theorem 5.

An important monotonicity property of T is stated in the following lemma.

Lemma 12: If $\omega(\mathbf{q}, \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta})$ then

$$\Omega(T(\mathbf{q}, \boldsymbol{\eta}), \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta}). \quad (27)$$

Proof: We recall the definition of the set D from *Power Update* in Section III-C. Let $\bar{\mathbf{q}}$ denote $T(\mathbf{q}, \boldsymbol{\eta})$. Since $\omega(\mathbf{q}, \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta})$, whence $D \neq \emptyset$, it follows that T is defined in terms of (13). Let $\hat{\mathbf{q}}$ denote the vector in (13), where λ is the scaling of the powers of the cells in \bar{D} . Note that the existence and uniqueness of λ and the fact that $\lambda < 1$ is established in *Power Update* in Section III-C. Then

$$\Omega(\hat{\mathbf{q}}, \boldsymbol{\eta}) = \max_{n \in \mathcal{N}} \Omega_n(\hat{\mathbf{q}}, \boldsymbol{\eta}_n) < \max_{n \in \mathcal{N}} \Omega_n(\mathbf{q}, \boldsymbol{\eta}_n) = \Omega(\mathbf{q}, \boldsymbol{\eta}) \quad (28)$$

by Lemma 10(2), since the maximum outage occurs at a cell in D , and $\lambda < 1$. On the other hand, Lemma 10(3) implies that $\Omega(\bar{\mathbf{q}}, \boldsymbol{\eta}) < \Omega(\hat{\mathbf{q}}, \boldsymbol{\eta})$. The result follows. ■

The remainder of this section will establish the following lemma, which is used in Appendix D.

Lemma 13: If $\omega(\mathbf{q}, \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta})$ then there exists a neighbourhood A of \mathbf{q} such that

$$\Omega(T(\mathbf{q}', \boldsymbol{\eta}), \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta}) \quad \forall \mathbf{q}' \in A.$$

The proof of this lemma is complicated by the fact that T is not continuous in \mathbf{q} , because of the discontinuity in the set D , as \mathbf{q} varies. In the remainder of this section, the subchannel allocation $\boldsymbol{\eta}$ will be fixed, and will be omitted from the notation where it is not ambiguous. We will make explicit the dependence of D and ψ on \mathbf{q} , with

$$D(\mathbf{q}) = \{n : \Omega_n(\mathbf{q}) > \psi(\mathbf{q})\}. \quad (29)$$

Note that $\psi(\mathbf{q})$ is continuous in \mathbf{q} by Lemma 9.

The set of indices, $D(\mathbf{q})$, in (29), is not continuous in \mathbf{q} . To avoid discontinuities in T that occur when $D(\mathbf{q})$ changes, we will fix the set of indices under consideration, as follows: Let (X, \bar{X}) be a partition of $\{n = 1, 2, \dots, N\}$, let $\Omega_X(\mathbf{q}) = \max_{n \in X} \Omega_n(\mathbf{q})$, and let $\mathcal{Q}_X = \{\mathbf{q} \in \mathbb{R}_+^N : \Omega_{\bar{X}}(\mathbf{q}) < \Omega_X(\mathbf{q})\}$. Let $T_X : \mathcal{Q}_X \rightarrow \mathbb{R}_+^N$ be the mapping defined by the *Power Update* procedure, but with the set D replaced by X . Thus, the powers of the cells in \bar{X} are scaled down by a common factor λ until the maximum outage probability of the cells in X is the same as the maximum outage probability of the cells in \bar{X} . Let $\lambda_X(\mathbf{q})$ denote the value of λ at which this equality occurs. The resulting power allocation is

$$\hat{\mathbf{q}}_n(\mathbf{q}) = \begin{cases} q_n, & n \in X \\ \lambda_X(\mathbf{q})q_n, & n \in \bar{X}. \end{cases} \quad (30)$$

Then all the powers in $\hat{\mathbf{q}}$ are scaled up by the common factor $\mu(\hat{\mathbf{q}}) = \|\mathbf{q}\|_1 / \|\hat{\mathbf{q}}\|_1$, and $T_X(\mathbf{q})$ is defined to be this new value:

$$T_X(\mathbf{q}) = \begin{cases} \mu(\hat{\mathbf{q}}(\mathbf{q}))q_n, & n \in X \\ \mu(\hat{\mathbf{q}}(\mathbf{q}))\lambda_X(\mathbf{q})q_n, & n \in \bar{X}. \end{cases} \quad (31)$$

Note the similarity between the definition of T_X and the mapping T defined in Section III-C of the paper. Indeed, $T(\mathbf{q}) = T_{D(\mathbf{q})}(\mathbf{q})$ for $\mathbf{q} \in \mathcal{Q}$. The following lemma is more general than Lemma 12, but is proven in exactly the same way.

Lemma 14: For any $\mathbf{q} \in \mathcal{Q}_X$ with $\omega(\mathbf{q}) < \Omega(\mathbf{q})$, we have $\Omega(T_X(\mathbf{q})) < \Omega(\mathbf{q})$.

However, in contrast to T , the mapping T_X is continuous:

Lemma 15: Each T_X is continuous on its domain \mathcal{Q}_X .

Proof: We first show that $\lambda_X(\mathbf{q})$ is continuous. Note that the domain of T_X is open. Consider an arbitrary $\mathbf{q} \in \mathcal{Q}_X$. Define $g_1 : \mathcal{Q}_X \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $g_1(\mathbf{q}, \ell)$ is the maximum outage probability of the cells in X when the powers of cells in \bar{X} are scaled by ℓ . Define g_2 as the maximum outage probability in \bar{X} under the same conditions. Let $h = g_1 - g_2$. Note that h is continuous and strictly increasing in its second argument, always negative when $l = 0$, and positive when $l = 1$, which implies that $h(\mathbf{q}, l) = 0$ has the unique solution $l = \lambda_X(\mathbf{q})$ for any $\mathbf{q} \in \mathcal{Q}_X$. For any $\mathbf{q}_0 \in \mathcal{Q}_X$, let ℓ_0 be the unique point for which $h(\mathbf{q}_0, \ell_0) = 0$. Since h is defined in a neighbourhood around (\mathbf{q}_0, ℓ_0) , and is continuous, the version of the implicit function theorem in [35] implies that there is a unique, continuous function $\ell(\cdot)$ defined in a neighbourhood of \mathbf{q}_0 for which $h(\mathbf{q}, \ell(\mathbf{q})) = 0$. Thus $\lambda_X(\mathbf{q}) = \ell(\mathbf{q})$ is continuous in \mathbf{q} at every $\mathbf{q} \in \mathcal{Q}_X$. Hence $\hat{\mathbf{q}}(\mathbf{q})$ is also continuous in \mathbf{q} . It is clear that $\mu(\hat{\mathbf{q}})$ is a continuous function of $\hat{\mathbf{q}}$. From the definition (31), each component of T_X is a composition of continuous functions, and hence T_X is continuous. ■

Lemma 16: Suppose $\omega(\mathbf{q}) < \Omega(\mathbf{q})$. Then for all \mathbf{q}' in a neighbourhood of \mathbf{q} , we have that $D(\mathbf{q}) \subseteq D(\mathbf{q}')$ and $\Omega_{D(\mathbf{q}')}(\mathbf{q}) = \Omega(\mathbf{q})$.

Proof: Under the hypothesis, there is an $\epsilon > 0$ such that

$$D(\mathbf{q}) = \{n : \Omega_n(\mathbf{q}) \geq \psi(\mathbf{q}) + \epsilon\} \neq \emptyset \quad (32)$$

Since Ω_n and ψ are continuous functions of \mathbf{q} , it follows that for \mathbf{q}' sufficiently close to \mathbf{q} , we have that both $\psi(\mathbf{q}') < \psi(\mathbf{q}) + \epsilon/2$ and $\Omega_n(\mathbf{q}') > \Omega_n(\mathbf{q}) - \epsilon/2$. If $n \notin D(\mathbf{q}')$ then $\Omega_n(\mathbf{q}') < \psi(\mathbf{q}') + \epsilon/2 < \psi(\mathbf{q}) + \epsilon$. But (32) then implies that $n \notin D(\mathbf{q})$, establishing the first claim. The second claim follows since the cell with highest outage probability is in $D(\mathbf{q})$, hence in $D(\mathbf{q}')$. ■

We are now in a position to prove the main result of this section: *Proof of Lemma 13:*

Proof: Assume that the statement in the lemma is false. Then there exists a sequence of power vectors $\mathbf{q}^{(k)}$ converging to \mathbf{q} with the properties that $D(\mathbf{q}^{(k)}) = D'$ and

$$\Omega(T(\mathbf{q}^{(k)})) = \Omega(T_{D'}(\mathbf{q}^{(k)})) \geq \Omega(\mathbf{q}), \quad \text{for all } k. \quad (33)$$

Note that D' need not be $D(\mathbf{q})$, but $\mathbf{q} \in \mathcal{Q}_{D'}$. To see this, note that $\omega(\mathbf{q}) < \Omega(\mathbf{q})$ implies $\psi(\mathbf{q}^{(k)})$ is bounded away from $\Omega(\mathbf{q})$, whence $\Omega_{\bar{D}'}(\mathbf{q}^{(k)})$ is also, yet $\Omega_{D'}(\mathbf{q}^{(k)}) \rightarrow \Omega_{D'}(\mathbf{q}) = \Omega(\mathbf{q})$ by Lemma 16.

Thus, $\Omega_{D'}(\mathbf{q}) < \Omega_{D'}(\mathbf{q})$, which gives $\mathbf{q} \in \mathcal{Q}_{D'}$ as claimed. Thus, by Lemma 14,

$$\Omega(T_{D'}(\mathbf{q})) < \Omega(\mathbf{q}) \quad (34)$$

But by Lemma 15, $\Omega(T_{D'}(\mathbf{q}^{(k)})) \rightarrow \Omega(T_{D'}(\mathbf{q}))$, and thus, by (33),

$$\Omega(T_{D'}(\mathbf{q})) \geq \Omega(\mathbf{q}). \quad (35)$$

But this contradicts (34), establishing the lemma. \blacksquare

C. Proof of Theorem 4

The following lemmas are useful in establishing the *uniqueness* of the power allocation in solutions to (8).

Lemma 17: Let $(\mathbf{q}^*, \boldsymbol{\eta}^*)$ be a solution to (8). Then,

1) All cell outages are equal:

$$\Omega_n(\mathbf{q}^*, \boldsymbol{\eta}_n^*) = \Omega^*, \quad \forall n \in \mathcal{N}. \quad (36)$$

2) For all $n \in \mathcal{N}$, $\boldsymbol{\eta}_n^*$ solves (9) for cell n .

Proof: For 1), suppose that $\omega(\mathbf{q}^*, \boldsymbol{\eta}^*) < \Omega(\mathbf{q}^*, \boldsymbol{\eta}^*)$. By Lemma 12 (Appendix B), $\Omega(T(\mathbf{q}^*, \boldsymbol{\eta}^*), \boldsymbol{\eta}^*) < \Omega(\mathbf{q}^*, \boldsymbol{\eta}^*)$, which contradicts the optimality of $(\mathbf{q}^*, \boldsymbol{\eta}^*)$. For 2), suppose that $\boldsymbol{\eta}_n^*$ does not solve (9) in cell n . Let $\boldsymbol{\eta}'_n$ be an optimal subchannel allocation in cell n . Construct a new subchannel allocation, $\boldsymbol{\eta}^\dagger$, from $\boldsymbol{\eta}^*$ by replacing $\boldsymbol{\eta}_n^*$ with $\boldsymbol{\eta}'_n$. Then $\omega(\mathbf{q}^*, \boldsymbol{\eta}^\dagger) < \Omega(\mathbf{q}^*, \boldsymbol{\eta}^\dagger)$ and it then follows from Lemma 12 that $\Omega(T(\mathbf{q}^*, \boldsymbol{\eta}^\dagger)) < \Omega(\mathbf{q}^*, \boldsymbol{\eta}^*)$ which is a contradiction to the fact that $(\mathbf{q}^*, \boldsymbol{\eta}^*)$ is an optimal solution to (8). \blacksquare

To prove the uniqueness of the optimal power allocation, we need to show that if $(\mathbf{q}^\dagger, \boldsymbol{\eta}^\dagger)$ solves problem (8), then properties 2) and 3) below must hold in addition to 1) (which holds by definition).

Lemma 18: Let $(\mathbf{q}^*, \boldsymbol{\eta}^*)$ be a solution to (8), and let Ω^* be the optimal value (the minimum). Suppose that an allocation $(\mathbf{q}^\dagger, \boldsymbol{\eta}^\dagger)$ of powers and subchannels has the following properties:

1) $\sum_{n \in \mathcal{N}} q_n^\dagger = q^{\text{total}}$ and

2) $\boldsymbol{\eta}_n^\dagger$ solves (9) under the power allocation of q_n^\dagger , for all $n \in \mathcal{N}$.

3) There exists an Ω^\dagger such that

$$\Omega_n(\mathbf{q}^\dagger, \boldsymbol{\eta}_n^\dagger) = \Omega^\dagger, \quad \forall n \in \mathcal{N}.$$

Then, $\mathbf{q}^* = \mathbf{q}^\dagger$ and $\Omega^* = \Omega^\dagger$.

Proof of Lemma 18

Proof: Suppose that $\mathbf{q}^* \neq \mathbf{q}^\dagger$. Define

$$\delta = \max_{n \in \mathcal{N}} \frac{q_n^\dagger}{q_n^*} > 1 \quad (37)$$

where the inequality follows from hypothesis 1). Let b be a cell that achieves this maximum. We have that $q_b^\dagger = \delta q_b^*$ and $\delta q_n^* \geq q_n^\dagger$. From Lemma 11(3), in Appendix A, we have

$$\theta_{b,m}(\mathbf{q}^*, \boldsymbol{\eta}_{b,m}^*) > \theta_{b,m}(\delta \mathbf{q}^*, \boldsymbol{\eta}_{b,m}^*), \quad \forall m \in C_b. \quad (38)$$

Since $\delta q_b^* = q_b^\dagger$ and $\delta q_n^* \geq q_n^\dagger, \forall n \neq b$ with at least one with strict inequality, we obtain the following inequality using Lemma 11(2):

$$\theta_{b,m}(\delta \mathbf{q}^*, \boldsymbol{\eta}_{b,m}^*) > \theta_{b,m}(\mathbf{q}^\dagger, \boldsymbol{\eta}_{b,m}^*), \quad \forall m \in C_b. \quad (39)$$

Since $\boldsymbol{\eta}_b^\dagger$ is optimal for cell b under \mathbf{q}^\dagger , we also have

$$\max_{m \in C_b} \theta_{b,m}(\mathbf{q}^\dagger, \boldsymbol{\eta}_{b,m}^*) \geq \max_{m \in C_b} \theta_{b,m}(\mathbf{q}^\dagger, \boldsymbol{\eta}_{b,m}^\dagger). \quad (40)$$

Thus, $\Omega_b(\mathbf{q}^*, \boldsymbol{\eta}_n^*) > \Omega_b(\mathbf{q}^\dagger, \boldsymbol{\eta}_n^\dagger)$. Consequently $\Omega^* > \Omega^\dagger$, which contradicts the optimality of $(\mathbf{q}^*, \boldsymbol{\eta}^*)$.

Thus $\mathbf{q}^* = \mathbf{q}^\dagger$. The result follows since both $\boldsymbol{\eta}^*$ and $\boldsymbol{\eta}^\dagger$ solve (9) for \mathbf{q}^* . \blacksquare

Theorem 4 is an immediate corollary of Lemmas 18 and 17.

D. Proof of Theorem 5

Proof: Lemma 12 (Appendix B) implies

$$\Omega(\mathbf{q}^{(k)}, \boldsymbol{\eta}^{(k-1)}) \leq \Omega(\mathbf{q}^{(k-1)}, \boldsymbol{\eta}^{(k-1)}) = \Omega^{(k-1)},$$

since $T(\mathbf{q}, \boldsymbol{\eta}) = \mathbf{q}$ if $\omega(\mathbf{q}, \boldsymbol{\eta}) = \Omega(\mathbf{q}, \boldsymbol{\eta})$. But since the Genie-aided subchannel allocation algorithm (Section III-B) cannot increase the maximum outage probability in each cell,

$$\Omega^{(k)} = \Omega(\mathbf{q}^{(k)}, \boldsymbol{\eta}^{(k)}) \leq \Omega(\mathbf{q}^{(k)}, \boldsymbol{\eta}^{(k-1)})$$

These imply $\Omega^{(k)}$ is non-increasing in k . Since it is bounded,

$$\Omega^{(k)} \downarrow \Omega \quad \text{for some } \Omega \quad (41)$$

Now notice that since $\mathbf{q}^{(k)} \in \mathcal{Q}$, the sequence $(\mathbf{q}^{(k)}, \boldsymbol{\eta}^{(k)})$ is bounded, and hence accumulation points exist, and the powers at the accumulation points are in the closure of \mathcal{Q} . Now $\Omega^{(k)} \downarrow \Omega(\mathbf{q}, \boldsymbol{\eta})$ for any accumulation point $(\mathbf{q}, \boldsymbol{\eta})$ of the sequence $(\mathbf{q}^{(k)}, \boldsymbol{\eta}^{(k)})$, so $\Omega(\mathbf{q}, \boldsymbol{\eta})$ must be the same for all accumulation points of that sequence. We now show that $\omega(\mathbf{q}, \boldsymbol{\eta}) = \Omega(\mathbf{q}, \boldsymbol{\eta})$ at any accumulation point $(\mathbf{q}, \boldsymbol{\eta})$. Let $(k_i)_{i=1}^{\infty}$ denote a convergent subsequence, with $\boldsymbol{\eta}^{(k_i)} = \boldsymbol{\eta}$ for all i , $\mathbf{q}^{(k_i)} \rightarrow \mathbf{q}$ as $i \uparrow \infty$. Assume instead that $\omega(\mathbf{q}, \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta})$. Then by Lemma 13 in Appendix B,

$$\Omega(T(\mathbf{q}^{(k_i)}, \boldsymbol{\eta}), \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta})$$

for i sufficiently large. But $\mathbf{q}^{(k_{i+1})} = T(\mathbf{q}^{(k_i)}, \boldsymbol{\eta})$, and hence for i sufficiently large,

$$\Omega^{(k_{i+1})} \leq \Omega(\mathbf{q}^{(k_{i+1})}, \boldsymbol{\eta}) < \Omega(\mathbf{q}, \boldsymbol{\eta})$$

which contradicts (41). Thus it must be that $\Omega(\mathbf{q}, \boldsymbol{\eta}) = \omega(\mathbf{q}, \boldsymbol{\eta})$ and by Corollary 2 (Section III-B), η_n is a solution of (9) for all n . The three hypotheses of Lemma 18 (Appendix C) are therefore true at $(\mathbf{q}, \boldsymbol{\eta})$ from which it follows that $\mathbf{q} = \mathbf{q}^*$, and Theorem 4 gives $\boldsymbol{\eta} \in E$. This establishes claims 1)–3) in the statement of Theorem 5. The last claim, 4), follows from the fact that if it were not true, there would exist an infinite subsequence converging to an accumulation point $(\mathbf{q}', \boldsymbol{\eta}')$ with the property that $\boldsymbol{\eta}'$ is not an element of E , but that would contradict claim 3) in Theorem 5. ■

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