Optimal Multistage Linear Multiuser Receivers

Louis G. F. Trichard, *Member, IEEE*, Jamie S. Evans, *Member, IEEE*, and Iain B. Collings, *Senior Member, IEEE*

***Abstract*—In this paper, we analyze a linear multiuser receiver for code-division multiple-access systems that is based on a matrix polynomial expansion. We focus on the receiver where the polynomial coefficients are chosen to minimize the mean squared error at the output and observe that the resultant coefficients are also signal-to-interference ratio maximizing. We present a simple derivation for the (known) large system coefficients and signal-to-interference ratio of this optimal multistage receiver and make a significant step toward a direct derivation of Honig and Xiao's recursive expression for this large system signal-to-interference ratio. Finally, we extend these results to take into account arbitrary power distributions.**

***Index Terms*—Large system analysis, multistage receivers, multiuser detection, random spreading, reduced-rank filtering.**

I. Introduction

THERE has recently been an increase in the popularity of linear multistage multiuser (MSMU) receivers for code-division multiple-access (CDMA) communications. In some cases, these receivers are also known as partial parallel interference cancellation (PPIC) receivers. The main attractive feature is their simple structure and ability to achieve close-to linear minimum mean squared error (LMMSE) performance, but do so with less computational complexity [1]–[11].

This paper focuses on an *optimal linear* MSMU receiver based on a weighted polynomial expansion. By "optimal," we mean the linear MSMU receiver which minimizes the MSE between the data symbol and the data estimate for user k at the output of a particular stage m with respect to the m polynomial coefficients. We shall focus on synchronous CDMA systems with K users, and spreading gain N. In this case, the weighted polynomial expansion is given in terms of the N-dimensional cross-correlation matrix of the form $\mathbf{SS}'$, where the columns of $\mathbf{S}$ are the spreading sequences of each user[1]. Linear MSMU receivers based on this form may approximate the LMMSE receiver depending on the choice of polynomial coefficients (for examples, see [2], [9]). It can be shown by the Cayley-Hamilton Theorem (along similar lines as in [2]) that when, $m = K - 1$ stages, then the polynomial coefficients may be chosen appropriately so that LMMSE performance is achieved. There is a greater interest though in the optimal linear MSMU receiver which can achieve near-LMMSE performance for $m \ll K - 1$ which translates to significant computational savings, especially when the number of users in the system is large.

The optimal linear MSMU receiver is intimately related to the reduced-rank Wiener filter for this problem. In fact, both receivers produce the minimum MSE estimate in a subspace of dimension lower than $\min(N, K)$. For further details on this equivalence, see [12]. It is shown in [13] that this receiver also maximizes the signal-to-interference plus noise ratio (SIR) for a particular stage m and user k.

In this paper, we perform a large system analysis of the optimal linear MSMU receiver. We define a large system by taking N and K to infinity while keeping their ratio held fixed [14]–[17]. Recently, large system analysis has been applied to other linear CDMA receivers such as the LMMSE, decorrelating, and matched filter receivers where expressions such as the SIR were determined [16], [17]. Closely related large system analysis of multistage and related reduced-rank receivers has been carried out in [9], [11]. It is important to observe that numerous papers have compared the large system results with corresponding results for finite sized systems (see, for example, [17]–[19]). These papers have found the large system performance predictions to be very useful for gaining insights into the operation of systems of realistic size and especially for optimising design parameters.

The following are the main contributions of this paper.

1) We give a simple expression for the large system optimal MSMU receiver coefficients and corresponding large system SIR, using a straightforward derivation.
2) We take a significant step toward a direct proof of an open problem in [11, p. 1934]. This problem revolves around a direct derivation of the recursive (in the number of stages) expression for the large system SIR.
3) In the case of unequal power users, we give the large system SIR expression for unequal user powers as a continued fraction expansion.

This paper is organized as follows. In Section II, the CDMA system model is described. We describe and optimize the linear MSMU receiver structure, (this optimal linear MSMU receiver

Manuscript received April 3, 2003; revised February 19, 2004; accepted April 11, 2004. The editor coordinating the review of this paper and approving it for publication is K. B. Lee. This work was supported in part by the Australian Research Council. The material in this paper was presented in part at the IEEE International Conference on Communications, New York, April 2002; the IEEE International Symposium on Information Theory, Lausanne, Switzerland, June 2002; and the IEEE International Symposium on Information Theory, Yokohama, Japan, June 2003.

L. G. F. Trichard was with the Kohno Laboratory, Division of Physics, Electrical and Computer Engineering, Graduate School of Engineering, Yokohama National University, Kanagawa 240–8501, Japan. He is now with is with R.G.C. Jenkins & Co., London SW1H 0RJ, U.K. (e-mail: ltrichard@jenkins-ip.com).

J. S. Evans is with the ARC Special Research Centre for Ultra-Broadband Information Networks (CUBIN), Department of Electrical and Electronic Engineering, University of Melbourne, Victoria 3010, Australia (e-mail: jse@ee.unimelb.edu.au).

I. B. Collings is with the Telecommunications Laboratory, School of Electrical and Information Engineering, University of Sydney, NSW 2006, Australia (e-mail: i.colllings@ee.usyd.edu.au).

Digital Object Identifier 10.1109/TWC.2005.847004

[1]Here, $(\cdot)'$ is the matrix transpose.

is called the optimal MSMU receiver), in Section III, and derive the quadratic form of the large system SIR of the optimal MSMU receiver. In Section IV, we prove the large system SIR has a continued fraction expansion. In Section V, we derive the large system SIR of the optimal MSMU receiver for an arbitrary power distribution. Finally, in Section VI, we illustrate some performance results.

II. Signal Model

In this paper, we consider a synchronous direct sequence CDMA channel with K users and a processing gain of N. We assume that users employ binary antipodal modulation and consider a real baseband model [3]. The N-dimensional chip matched filter vector for each symbol interval is given by

$$\mathbf{r} = \sum_{k=1}^{K} \sqrt{P_k} b_k \mathbf{s}_k + \mathbf{n} \tag{1}$$

where b_k is the data bit of user k taking on values of ± 1, P_k is the received power for user k, $\mathbf{s}_k$ is the N-dimensional spreading sequence of user k, and $\mathbf{n}$ is additive white Gaussian noise (AWGN) with zero mean and covariance $\sigma^2 \mathbf{I}$. The $N \times K$ spreading sequence matrix is $\mathbf{S} = [\mathbf{s}_1 \dots \mathbf{s}_K]$ and we define the $N \times (K-1)$ spreading sequence matrix excluding the spreading sequence of user k as $\mathbf{S}_k = [\mathbf{s}_1, \dots, \mathbf{s}_{k-1}, \mathbf{s}_{k+1}, \dots, \mathbf{s}_K]$. We assume a random spreading model where the elements of $\mathbf{S}$ are independent and identically distributed random variables taking values of $\pm 1/\sqrt{N}$ with equal probability. We assume that the spreading sequences are known at the receiver and that we have perfect estimation of the received user powers and the noise variance σ^2. We also denote the diagonal matrix of received user powers, by the $K \times K$ matrix $\mathbf{P} = \operatorname{diag}(P_1, \dots, P_K)$.

At the receiver, a linear filter for user k produces the soft estimate $\hat{b}_k = \mathbf{c}_k' \mathbf{r}$ (with $'$ denoting the matrix transpose) for the filter coefficients $\mathbf{c}_k \in \Re^N$. The SIR for this estimate is

$$\mathrm{SIR}_k^{(N)} = \frac{P_k (\mathbf{c}_k' \mathbf{s}_k)^2}{\mathbf{c}_k' (\mathbf{S}_k \mathbf{P}_k \mathbf{S}_k' + \sigma^2 \mathbf{I}) \mathbf{c}_k}, \tag{2}$$

where $\mathbf{P}_k = \operatorname{diag}(P_1, \dots, P_{k-1}, P_{k+1}, \dots, P_K)$ is a $(K-1 \times K-1)$ diagonal matrix of received user powers excluding the power of user k. The superscript $^{(N)}$ indicates the dependence on the processing gain. In Sections III–IV, we shall concentrate on an equal power regime, where $P_k = P, 1 \le k \le K$. In Section V, we generalize our results to the case where the users have unequal powers.[2]

III. Linear Multistage Receiver Structures

In this paper, we consider the class of MSMU receivers which, when appropriately designed, converge to the LMMSE receiver as the number of stages increase. We shall first review the K dimensional MSMU receiver of [2] (see also [1] and [9]), and then present the N dimensional MSMU receiver that we use in our analysis.

A. Constrained LMMSE Receiver

Consider the LMMSE receiver matrix filter for K users which is found by minimizing

$$E[(\mathbf{b} - \mathbf{C}'\mathbf{r})'(\mathbf{b} - \mathbf{C}'\mathbf{r})] \tag{3}$$

with respect to the $N \times K$ matrix of filter coefficients $\mathbf{C}$. In this expression, $\mathbf{b} = [b_1, \dots, b_K]'$ is the vector of K user data symbols and $\mathbf{r}$ is the received vector given in (1). The resulting matrix filter is

$$\mathbf{C} = \mathbf{S}\,(\mathbf{S}'\,\mathbf{S} + \sigma^2\,\mathbf{P}^{-1})^{-1}\mathbf{P}^{-1/2} \tag{4}$$

and the output of this LMMSE receiver are the K data estimates $\hat{\mathbf{b}} = [\hat{b}_1, \dots, \hat{b}_K]'$, given by

$$\hat{\mathbf{b}} = \mathbf{C}'\mathbf{r}. \tag{5}$$

We now restrict attention to the equal power case so that $\mathbf{P} = P\mathbf{I}$. With this restriction, we observe that the matrix inverse in (4) may be written in the form of a weighted matrix polynomial expansion in terms of the K dimensional cross-correlation matrix $\mathbf{S}'\mathbf{S}$ (via the Cayley Hamilton Theorem as in [2]). We have

$$\mathbf{C} = P^{-1/2}\,\mathbf{S}\left(\mathbf{S}'\,\mathbf{S} + \frac{\sigma^2}{P}\mathbf{I}\right)^{-1} \tag{6}$$

$$= \mathbf{S}\sum_{i=0}^{K-1} \omega_i (\mathbf{S}'\,\mathbf{S})^i \tag{7}$$

where the coefficients ω_i are functions of P, σ^2, and the eigenvalues of $\mathbf{S}'\mathbf{S}$.

Motivated by the representation in (7), we consider the m-stage MSMU receiver

$$\mathbf{C}_m = \mathbf{S}\sum_{i=0}^{m} \omega_i\,(\mathbf{S}'\mathbf{S})^i \tag{8}$$

where ω_i are polynomial coefficients and the kth column of $\mathbf{C}_m = [\mathbf{c}_{m,1}, \dots, \mathbf{c}_{m,K}]$ is the MSMU filter coefficient vector for the kth user. In [2], this m-stage receiver is optimized by minimizing the MSE

$$E[(\mathbf{b} - \mathbf{C}_m'\mathbf{r})'(\mathbf{b} - \mathbf{C}_m'\mathbf{r})] \tag{9}$$

with respect to the polynomial coefficients $\omega_0, \dots, \omega_m$. This weighting ensures that this receiver attains full-rank LMMSE performance when $m = K-1$. It is for this particular type of constrained optimization that we use the term "optimal linear MSMU receiver." This MSMU receiver is also known as the constrained LMMSE receiver [2] as it is the LMMSE receiver constrained by the structure of (8).

[2]*Note:* For the remainder of this paper we generally omit the word "linear" when discussing linear MSMU receiver structures.

Now, consider the receiver for user k after m stages, denoted by $\mathbf{c}_{m,k}$, which is simply the kth column of $\mathbf{C}_m$ in (8). More conveniently, we rearrange (8) to give

$$\mathbf{C}_m = \sum_{i=0}^{m} \omega_i \, (\mathbf{S}\,\mathbf{S}')^i \, \mathbf{S}$$

so that

$$\mathbf{c}_{m,k} = \sum_{i=0}^{m} \omega_i \, (\mathbf{S}\,\mathbf{S}')^i \, \mathbf{s}_k.$$

We now generalize the receiver by allowing the coefficients ω_i to depend on the user of interest, and, thus

$$\mathbf{c}_{m,k} = \sum_{i=0}^{m} \omega_{i,k} \, (\mathbf{S}\,\mathbf{S}')^i \, \mathbf{s}_k = \mathbf{\Omega}_m \, \boldsymbol{\omega}_{m,k} \tag{10}$$

where $\boldsymbol{\omega}_{m,k} = [\omega_{0,k}, \ldots, \omega_{m,k}]'$ is the vector of polynomial coefficients for stage m and user k, and $\mathbf{\Omega}_m = [\mathbf{s}_k, (\mathbf{S}\mathbf{S}')\mathbf{s}_k, \ldots, (\mathbf{S}\mathbf{S}')^m \mathbf{s}_k]$. Thus, we see that the receiver for user k is constrained to lie in the column space of the matrix $\mathbf{\Omega}_m$.

An equivalent form to (10) that greatly simplifies our performance analysis can be obtained by observing that the column spaces of $\mathbf{\Omega}_m$ and

$$\mathbf{\Omega}_{m,k} = [\mathbf{s}_k, (\mathbf{S}_k\,\mathbf{S}_k')\mathbf{s}_k, \ldots, (\mathbf{S}_k\,\mathbf{S}_k')^m \mathbf{s}_k]$$

are identical. This allows us to write

$$\mathbf{c}_{m,k} = \sum_{i=0}^{m} \beta_{i,k} (\mathbf{S}_k\,\mathbf{S}_k')^i \mathbf{s}_k = \mathbf{\Omega}_{m,k} \boldsymbol{\beta}_{m,k} \tag{11}$$

where $\boldsymbol{\beta}_{m,k} = [\beta_{0,k}, \ldots, \beta_{m,k}]'$ is a new vector of polynomial coefficients. We observe that

$$\boldsymbol{\beta}_{m,k} = \mathbf{M}_{m,k} \boldsymbol{\omega}_{m,k} \tag{12}$$

where $\mathbf{M}_{m,k}$ is a linear mapping that relates the coefficients of $\boldsymbol{\beta}_{m,k}$ to $\boldsymbol{\omega}_{m,k}$. We previously showed in [13, Corollary 1], that $\mathbf{M}_{m,k}$ is given by

$$\mathbf{M}_{m,k} = \begin{bmatrix} 1 & \hat{\gamma}_0 & \cdots & \hat{\gamma}_{m-1} \\ 0 & 1 & \ddots & \hat{\gamma}_{m-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{13}$$

where $\hat{\gamma}_i = \mathbf{s}_k'(\mathbf{S}\mathbf{S}')^i \mathbf{s}_k$.

B. Receiver Optimization

Consider the linear receiver $\mathbf{c}_{m,k} = \mathbf{\Omega}_{m,k} \boldsymbol{\beta}_{m,k}$ as given in (11), and note that there are two natural ways to optimize the coefficients $\boldsymbol{\beta}_{m,k}$. The first is to choose $\boldsymbol{\beta}_{m,k}$ to minimize the MSE $E[(b_k - \mathbf{c}_{m,k}'\mathbf{r})^2]$. Alternatively, we could attempt to maximize the SIR

$$\mathrm{SIR}_{k,m}^{(N)} = \frac{P(\mathbf{c}_{m,k}'\mathbf{s}_k)^2}{\mathbf{c}_{m,k}'(P\mathbf{S}_k\mathbf{S}_k' + \sigma^2\mathbf{I})\mathbf{c}_{m,k}}.$$

In [13], we showed that both problems are solved by choosing

$$\boldsymbol{\beta}_{m,k} \propto (\mathbf{\Omega}_{m,k}'\mathbf{Z}_k\mathbf{\Omega}_{m,k})^{-1}\mathbf{\Omega}_{m,k}'\mathbf{s}_k, \tag{14}$$

where $\mathbf{Z}_k = \mathbf{S}_k\mathbf{S}_k' + (\sigma^2/P)\mathbf{I}$. The constant of proportionality is irrelevant for maximizing SIR, but for interest sake, we note that for minimizing the MSE, it is equal to

$$\eta_{m,k} = \frac{P^{-1/2}}{1 + \mathbf{s}_k'\mathbf{\Omega}_{m,k}(\mathbf{\Omega}_{m,k}'\mathbf{Z}_k\mathbf{\Omega}_{m,k})^{-1}\mathbf{\Omega}_{m,k}'\mathbf{s}_k}.$$

From this point on, we will take the constant of proportionality to be unity as it is not important in our analysis. The SIR of this receiver is given by

$$\mathrm{SIR}_{k,m}^{(N)} = \mathbf{s}_k'\mathbf{\Omega}_{m,k}(\mathbf{\Omega}_{m,k}'\mathbf{Z}_k\mathbf{\Omega}_{m,k})^{-1}\mathbf{\Omega}_{m,k}'\mathbf{s}_k. \tag{15}$$

C. Large System SIR

In this section, we analyze the SIR of the optimal MSMU receiver as the spreading gain (N) and number of users (K) get large with $\alpha = K/N$ held fixed. Related large system results for linear multistage receivers are also presented in [9] and [11].

In deriving the large system SIR and the large system filter coefficients the key terms of interest are random variables of the form $\mathbf{s}_k'(\mathbf{S}_k\mathbf{S}_k')^i\mathbf{s}_k$ for $(0 \le i \le 2m+1)$, consequently, the following Lemma is useful.

Lemma 1: If we take $N \to \infty$ with $\alpha = K/N$ fixed, the random variable $\mathbf{s}_k'(\mathbf{S}_k\mathbf{S}_k')^i\mathbf{s}_k$ converges in probability to the deterministic moment

$$\psi_i(\alpha) = \int \lambda^i dG(\lambda),$$

where $G(\lambda)$ is the limiting empirical distribution function of the eigenvalues of $\mathbf{S}_k\mathbf{S}_k'$ (see [11], [14], [16], and [17]). The ith moment of the limiting empirical distribution function $\psi_i(\alpha)$ can be calculated recursively as

$$\psi_i(\alpha) = \frac{1}{i+1}[(2i-1)(1+\alpha)\psi_{i-1}(\alpha) - (i-2)(1-\alpha)^2\psi_{i-2}(\alpha)], \tag{16}$$

where $\psi_0(\alpha) = 1$ and $\psi_1(\alpha) = \alpha$.

Proof: See [10] and [11]. ■

The following theorem gives the large system SIR of the optimal MSMU receiver (which minimizes the MSE *and* maximizes the SIR) for a given stage m.

Theorem 1: Let $N, K \to \infty$, with $\alpha = K/N$ held fixed. Then, the SIR of the optimal MSMU receiver, (15), converges in probability to a deterministic scalar $\mathrm{SIR}_m^{(\infty)}$ given by

$$\mathrm{SIR}_m^{(\infty)} = \mathbf{\Upsilon}_m'\mathbf{\Phi}_m^{-1}\mathbf{\Upsilon}_m \tag{17}$$

and the elements of the coefficient filter vector $\boldsymbol{\beta}_{m,k}$ of (14) converge in probability to deterministic scalars given by the elements of the vector

$$\boldsymbol{\beta}_m^{(\infty)} = \mathbf{\Phi}_m^{-1}\mathbf{\Upsilon}_m \tag{18}$$

where the elements of the vector $\boldsymbol{\Upsilon}_m$ *are* $(\boldsymbol{\Upsilon}_m)_i = \psi_i(\alpha)$, *the elements of the matrix* $\boldsymbol{\Phi}_m$ *are* $(\boldsymbol{\Phi}_m)_{i,j} = \psi_{i+j+1}(\alpha) + (\sigma^2/P)\psi_{i+j}(\alpha)$ *for* $0 \le i,\, j \le m$, *and where* $\psi_i(\alpha)$ *is given in Lemma 1.*

Proof: Consider the elements of the vector $\boldsymbol{\Omega}'_{m,k}\mathbf{s}_k$, which can be written

$$(\boldsymbol{\Omega}'_{m,k}\mathbf{s}_k)_i = \mathbf{s}'_k(\mathbf{S}_k\mathbf{S}'_k)^i\mathbf{s}_k$$

for $0 \le i \le m$. From Lemma 1 it follows that

$$(\boldsymbol{\Omega}'_{m,k}\mathbf{s}_k)_i \xrightarrow{p} \psi_i(\alpha) = (\boldsymbol{\Upsilon}_m)_i.$$

In addition, the elements of the matrix $\boldsymbol{\Omega}'_{m,k}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}$ can be written

$$(\boldsymbol{\Omega}'_{m,k}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k})_{i,j} = \mathbf{s}'_k(\mathbf{S}_k\mathbf{S}'_k)^{i+j+1}\mathbf{s}_k + \frac{\sigma^2}{P}\mathbf{s}'_k(\mathbf{S}_k\mathbf{S}'_k)^{i+j}\mathbf{s}_k$$

for $0 \le i$ and $j \le m$. Again, from Lemma 1, we have

$$(\boldsymbol{\Omega}'_{m,k}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k})_{i,j} \xrightarrow{p} \psi_{i+j+1}(\alpha) + \frac{\sigma^2}{P}\psi_{i+j}(\alpha) = (\boldsymbol{\Phi}_m)_{i,j}.$$

Now, considering (14) and (15) and taking limits, we have

$$\begin{aligned}&\mathrm{SIR}_{k,m}^{(N)} \xrightarrow{p} \mathrm{SIR}_m^{(\infty)} = \boldsymbol{\Upsilon}'_m\boldsymbol{\Phi}_m^{-1}\boldsymbol{\Upsilon}_m\\ &\text{and } \boldsymbol{\beta}_{m,k} \xrightarrow{p} \boldsymbol{\beta}_m^{(\infty)} = \boldsymbol{\Phi}_m^{-1}\boldsymbol{\Upsilon}_m.\end{aligned}$$

■

NOTE: Observe that the large system values of the SIR and the filter coefficients given in (17) and (18) depend only on the system load (α), the number of stages (m), and the signal-to-noise ratio (SNR) (P/σ^2). The large system expressions do not depend on the particular realization of the spreading sequences which allow the large system filter coefficients to be calculated offline, even in systems employing long spreading sequences. □

One might also be interested in the large system expression for the coefficient vector $\boldsymbol{\omega}_{m,k}$ in (10) when it is chosen to minimize MSE or to maximize SIR. The development is similar to that just undertaken except that the "moments" of interest are of the form $\mathbf{s}'_k(\mathbf{SS}')^i\mathbf{s}_k$ and these are not as easy to handle as terms of the form $\mathbf{s}'_k(\mathbf{S}_k\mathbf{S}'_k)^i\mathbf{s}_k$. However, in [13], we showed that in the large system limit $\mathbf{s}'_k(\mathbf{SS}')^i\mathbf{s}_k$ converges in probability to

$$\gamma_i(\alpha) = \psi_i(\alpha) + \sum_{j=0}^{i-1}\gamma_{i-j-1}(\alpha)\psi_j(\alpha)$$

where $\gamma_0 = 1$ and $\psi_i(\alpha)$ is given in Lemma 1. This result allows us to directly determine an expression for the optimal coefficient vector in a large system. An alternative approach is to use the transformation of (12) to write $\boldsymbol{\omega}_{m,k} = (\mathbf{M}_{m,k})^{-1}\boldsymbol{\beta}_{m,k}$ and take limits. Observe that the elements of $\mathbf{M}_{m,k}$ above the main diagonal, denoted by $(\mathbf{M}_{m,k})_{i,j}$ (with $i < j$), converge in probability to, $(\mathbf{M}_m^{(\infty)})_{i,j} = \gamma_{j-i-1}(\alpha)$. We then have $\boldsymbol{\omega}_m^{(\infty)} = (\mathbf{M}_m^{(\infty)})^{-1}\boldsymbol{\beta}_m^{(\infty)}$ providing a connection between the (large system) optimal coefficients of the two receiver implementations in (10) and (11).

IV. Continued Fraction SIR Expression

In this section, the SIR of the optimal MSMU receiver is given as a continued fraction. Performing large system analysis simplifies the continued fraction expression to a simple recursion in terms of the system loading and the SNR. This recursion expresses the SIR at each stage directly as a function of the SIR at the previous stage and provides a simple and elegant method for calculating the key performance measure of the optimal multistage receiver. It was first derived in [11]—we provide an alternative derivation of the result in this section.

The optimal MSMU receiver's filter vector subspace V_k^m given by the columns of $\boldsymbol{\Omega}_{m,k}$ can be replaced with an equivalent orthonormal filter vector subspace. A set of orthonormal vectors may be found using a method derived in [11]. The orthonormal set of vectors $V_k^{m,\perp} = (\check{\mathbf{c}}_0, \check{\mathbf{c}}_1, \ldots, \check{\mathbf{c}}_m)$ can be generated by

$$\begin{aligned}
\mathbf{u}_0 &= \mathbf{s}_k & \check{\mathbf{c}}_0 &= \mathbf{s}_k\\
\mathbf{u}_1 &= (P\mathbf{S}_k\mathbf{S}'_k + \sigma^2\mathbf{I})\check{\mathbf{c}}_0 & \check{\mathbf{c}}_1 &= \eta_1(\mathbf{I} - \check{\mathbf{c}}_0\check{\mathbf{c}}'_0)\mathbf{u}_1\\
&\vdots & &\vdots\\
\mathbf{u}_m &= (P\mathbf{S}_k\mathbf{S}'_k + \sigma^2\mathbf{I})\check{\mathbf{c}}_{m-1} & \check{\mathbf{c}}_m &= \eta_m\left(\mathbf{I} - \sum_{j=0}^{i-1}\check{\mathbf{c}}_j\check{\mathbf{c}}'_j\right)\mathbf{u}_m
\end{aligned} \tag{19}$$

where η_i are normalization constants.

In [11, Th. 2, p. 1933], the Multistage Reduced Rank Wiener filter receiver forms a filter vector subspace which was defined by $V_k^{m,\perp}$ in $\Re^N$, and was shown to span the set of vectors $V_{\mathbf{R}_k}^m = (\mathbf{s}_k, (\mathbf{R}_k)\mathbf{s}_k, \ldots, (\mathbf{R}_k)^m\mathbf{s}_k)$, where $\mathbf{R}_k = P\mathbf{S}_k\mathbf{S}'_k + \sigma^2\,\mathbf{I}$.

The equivalence of the set of vectors V^m and $V_{\mathbf{R}_k}^m$ is shown in the following lemma, the proof of which is straightforward and is omitted.

Lemma 2: Let $V^m = (\mathbf{s}_k, (\mathbf{SS}')\mathbf{s}_k, \ldots, (\mathbf{SS}')^m\mathbf{s}_k)$ *and* $V_{\mathbf{R}_k}^m = (\mathbf{s}_k, (\mathbf{R}_k)\mathbf{s}_k, \ldots, (\mathbf{R}_k)^m\mathbf{s}_k)$ *be two sets of vectors in the vector space* $\Re^N$, *then, span*$(V^m) \equiv$ *span*$(V_{\mathbf{R}_k}^m)$.

With this in mind, the orthonormal set of vectors defined in (19) is now used to give the optimal MSMU receiver for user k and stage m as

$$\mathbf{c}_{m,k}^{\perp} = \sum_{i=0}^{m}\xi_{i,k}\check{\mathbf{c}}_i = \boldsymbol{\Omega}_{m,k}^{\perp}\boldsymbol{\xi}_{m,k} \tag{20}$$

where $\boldsymbol{\xi}_{m,k} = [\xi_{0,k}, \ldots, \xi_{m,k}]'$ is the vector of optimal filter coefficients [cf. $\omega_{i,k}$ in (10) and $\beta_{i,k}$ in (11)], and $\boldsymbol{\Omega}_{m,k}^{\perp} = [\check{\mathbf{c}}_0, \ldots, \check{\mathbf{c}}_m]$.

The following theorem presents the first main contribution of this section, namely, the SIR of the optimal MSMU receiver in terms of a continued fraction expansion.[3]

[3]The continued fraction notation $a/(b + c/(d + \cdots)) \equiv a/b+\;\; c/d + \cdots$ is used to ease readability where possible.

Theorem 2: The SIR of the optimal MSMU receiver is given by

$$
\begin{aligned}
\mathrm{SIR}_{m,k}^{(N)} &= P\mathbf{s}_k'\mathbf{\Omega}_{m,k}^{\perp'}(\mathbf{\Omega}_{m,k}^{\perp'}\mathbf{R}_k\mathbf{\Omega}_{m,k}^{\perp})^{-1}\mathbf{\Omega}_{m,k}^{\perp}\mathbf{s}_k \\
&= \frac{P}{g_0-}\;\frac{f_0^2}{g_1-},\ldots,\frac{f_{m-2}^2}{g_{m-1}-}\;\frac{f_{m-1}^2}{g_m}
\end{aligned}
$$

where $g_i = \check{\mathbf{c}}_i'\mathbf{R}_k\check{\mathbf{c}}_i$ *and* $f_i = \check{\mathbf{c}}_{i+1}'\mathbf{R}_k\check{\mathbf{c}}_i$.

Proof: Following similar lines as before, it can be shown that the SIR of the optimal MSMU receiver for user k and stage m is

$$
\begin{aligned}
\mathrm{SIR}_{m,k}^{(N)} &= P\mathbf{s}_k'\mathbf{\Omega}_{m,k}^{\perp'}(\mathbf{\Omega}_{m,k}^{\perp'}\mathbf{R}_k\mathbf{\Omega}_{m,k}^{\perp})^{-1}\mathbf{\Omega}_{m,k}^{\perp}\mathbf{s}_k \\
&= P\mathbf{e}_1'(\mathbf{\Omega}_{m,k}^{\perp'}\mathbf{R}_k\mathbf{\Omega}_{m,k}^{\perp})^{-1}\mathbf{e}_1 \\
&= P[(\mathbf{\Omega}_{m,k}^{\perp'}\mathbf{R}_k\mathbf{\Omega}_{m,k}^{\perp})^{-1}]_{11} \qquad (21)
\end{aligned}
$$

where $\mathbf{e}_1 = [1, 0, \ldots, 0]'$ and $[.]_{11}$ is the element in the first row and first column.

Due to the orthonormal set of filter vectors it can be shown that $\mathbf{\Omega}_{m,k}^{\perp'}\mathbf{R}_k\mathbf{\Omega}_{m,k}^{\perp}$ is, in fact, tridiagonal, and we need to solve the following set of linear equations:

$$
\begin{bmatrix}
\check{\mathbf{c}}_0'\mathbf{R}_k\check{\mathbf{c}}_0 & \check{\mathbf{c}}_0'\mathbf{R}_k\check{\mathbf{c}}_1 & 0\cdots & 0 \\
\check{\mathbf{c}}_1'\mathbf{R}_k\check{\mathbf{c}}_0 & \ddots & \ddots & 0 \\
0 & \ddots & \check{\mathbf{c}}_{m-1}'\mathbf{R}_k\check{\mathbf{c}}_{m-1} & \check{\mathbf{c}}_{m-1}'\mathbf{R}_k\check{\mathbf{c}}_m \\
0 & \cdots 0 & \check{\mathbf{c}}_m'\mathbf{R}_k\check{\mathbf{c}}_{m-1} & \check{\mathbf{c}}_m'\mathbf{R}_k\check{\mathbf{c}}_m
\end{bmatrix}
\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{bmatrix}
= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (22)
$$

where $\mathrm{SIR}_{m,k}^{(N)} = Px_0$.

Now, let $g_i = \check{\mathbf{c}}_i'\mathbf{R}_k\check{\mathbf{c}}_i$ and $f_i = \check{\mathbf{c}}_{i+1}'\mathbf{R}_k\check{\mathbf{c}}_i$. Then, solving for x_0 [20, p. 213] gives

$$
\begin{aligned}
x_0 &= \frac{1}{g_0+}\;\frac{f_0x_1}{x_0},\;\frac{f_0x_1}{x_0} \\
&= \frac{-f_0^2}{g_1+}\;\frac{f_1x_2}{x_1},\ldots,\frac{f_{m-1}x_m}{x_{m-1}} = \frac{-f_{m-1}^2}{g_m} \qquad (23)
\end{aligned}
$$

back-substituting the equations in (23) gives $\mathrm{SIR}_{m,k}^{(N)}$ as a continued fraction

$$
\mathrm{SIR}_{m,k}^{(N)} = \frac{P}{g_0-}\;\frac{f_0^2}{g_1-}\;\frac{f_1^2}{g_2-},\ldots,\frac{f_{m-2}^2}{g_{m-1}-}\;\frac{f_{m-1}^2}{g_m}. \qquad (24)
$$

■

In order to examine the asymptotic performance, we now make use of some large system results derived in [11, Th. 1]. First, from Theorem 2, we can see that the linear filter for the ith stage of the optimal MSMU receiver for user k is $\check{\mathbf{c}}_i$, defined in (19). The filter for stage i produces the soft data estimate

$$
d_i = \check{\mathbf{c}}_i'\mathbf{r}. \qquad (25)
$$

Now, considering g_i and f_i [defined in (23)], and taking the limit as $N, K \to \infty$ with $\alpha = K/N$ held fixed, we have the following two expressions (derived in [11]):

$$
g_i = \check{\mathbf{c}}_i'\mathbf{R}_k\check{\mathbf{c}}_i = E[|d_i|^2] \xrightarrow{p} P + \alpha P + \sigma^2 \qquad (26)
$$

for $i \geq 1$ and $E[|d_0|^2] \xrightarrow{p} \alpha P + \sigma^2$, and

$$
(f_i)^2 = (\check{\mathbf{c}}_{i+1}'\mathbf{R}_k\check{\mathbf{c}}_i)^2 \xrightarrow{p} \alpha P^2. \qquad (27)
$$

The following theorem gives the large system SIR of Theorem 1 expressed as a continued fraction expansion in terms of only $\alpha = K/N$, the SNR and the large system SIR of the previous stage.

Theorem 3: Let $N, K \to \infty$, *with* $0 < \alpha = K/N < \infty$ *held fixed. Then, the SIR of the optimal* m*-stage MSMU receiver converges in probability to a deterministic scalar given by*

$$
\mathrm{SIR}_m^{(\infty)} = \mathbf{\Upsilon}_m'\mathbf{\Phi}_m^{-1}\mathbf{\Upsilon}_m = \frac{P}{\sigma^2+}\;\frac{\alpha P}{1 + \mathrm{SIR}_{m-1}^{(\infty)}} \qquad (28)
$$

where $\mathrm{SIR}_0^{(\infty)} = P/(P\alpha + \sigma^2)$, $\mathbf{\Upsilon}_m'$ *and* $\mathbf{\Phi}_m$ *are given in Theorem 1.*

Proof: Taking the limit of $\mathrm{SIR}_{m,k}^{(N)}$ (from Theorem 2) as $N, K \to \infty$ with $\alpha = K/N$ held fixed, and using (26) and (27), the SIR of the optimal MSMU receiver for stage m converges in probability to

$$
\begin{aligned}
&\mathrm{SIR}_{m,k}^{(N)} \xrightarrow{p} \mathrm{SIR}_m^{(\infty)} \\
&\quad = \frac{P}{P\alpha + \sigma^2-}\;\frac{\alpha P^2}{P\alpha + P + \sigma^2-},\ldots,\frac{\alpha P^2}{P\alpha + P + \sigma^2} \qquad (29)
\end{aligned}
$$

where the term $\alpha P^2/(P\alpha + P + \sigma^2-)$ is repeated $m-1$ times.

Now, for $i = 0$ and $i = 1$, clearly

$$
\mathrm{SIR}_0^{(\infty)} = \frac{P}{P\alpha + \sigma^2} \qquad (30)
$$

$$
\mathrm{SIR}_1^{(\infty)} = \frac{P}{P\alpha + \sigma^2-}\;\frac{\alpha P^2}{P\alpha + P + \sigma^2}. \qquad (31)
$$

Factoring out $P\alpha$ in the denominator and letting $x = \sigma^2 + P\alpha$ gives

$$
\begin{aligned}
\mathrm{SIR}_1^{(\infty)} &= \frac{P}{\sigma^2 + \alpha P\left(1 - \frac{P}{P+x}\right)} \\
&= \frac{P}{\sigma^2+}\;\frac{\alpha P}{1 + \mathrm{SIR}_0^{(\infty)}} \qquad (32)
\end{aligned}
$$

where $\mathrm{SIR}_0^{(\infty)} = P/x$.

Now assume for $i = m$ that the following holds:

$$
\mathrm{SIR}_m^{(\infty)} = \frac{P}{\sigma^2+}\;\frac{\alpha P}{1 + \mathrm{SIR}_{m-1}^{(\infty)}}. \qquad (33)
$$

Now, let

$$
\begin{aligned}
x_m = P\alpha + \sigma^2 - \Bigg[&\frac{\alpha P^2}{P\alpha + P + \sigma^2-} \\
&\cdots\;\frac{\alpha P^2}{P\alpha + P + \sigma^2-}\;\frac{\alpha P^2}{P\alpha + P + \sigma^2}\Bigg] \qquad (34)
\end{aligned}
$$

where the term $\alpha P^2/((P\alpha+P+\sigma^2)-)$ in the square brackets is repeated $m-1$ times, and from (29) $\mathrm{SIR}_m^{(\infty)} = P/x_m$. Then, for $i=m+1$, $\mathrm{SIR}_{m+1}^{(\infty)}$ is

$$\mathrm{SIR}_{m+1}^{(\infty)} = \frac{P}{P\alpha+\sigma^2-}\left[\frac{\alpha P^2}{P\alpha+P+\sigma^2-},\ldots,\frac{\alpha P^2}{P\alpha+P+\sigma^2}\right] \tag{35}$$

where the term $\alpha P^2/(P\alpha+P+\sigma^2-)$ is repeated m times. Factoring out αP in the denominator gives

$$\mathrm{SIR}_{m+1}^{(\infty)} = \frac{P}{\sigma^2+\alpha P\left[1-\frac{P}{P\alpha+P+\sigma^2-}\frac{\alpha P^2}{P\alpha+P+\sigma^2-}\cdots\frac{\alpha P^2}{P\alpha+P+\sigma^2}\right]}, \tag{36}$$

where the term $\alpha P^2/(P\alpha+P+\sigma^2-)$ is repeated $m-1$ times. Substituting x_m into (36) gives

$$\begin{aligned}\mathrm{SIR}_{m+1}^{(\infty)} &= \frac{P}{\sigma^2+\alpha P\left(1-\frac{P}{P+x_m}\right)}\\ &= \frac{P}{\sigma^2+\alpha P\left(\frac{1}{1+\frac{P}{x_m}}\right)}\\ &= \frac{P}{\sigma^2+}\ \frac{\alpha P}{1+\mathrm{SIR}_m^{(\infty)}}.\end{aligned} \tag{37}$$

■

NOTE: In Theorem 2, we established a direct connection between the standard quadratic form of the SIR and the continued fraction form. This is an important step, and provides a partial solution to the open problem stated in [11, p. 1934]. The open problem is to prove directly that the quadratic form is equal to the continued fraction form using only properties of the large system moments. In Theorem 3, however, we have relied on results from [11] to complete the proof. □

V. Unequal Power Users

In this section, we extend our developments to account for systems where the users have unequal powers. We are able to derive a continued fraction expansion for the SIR and take the large system limit however we do not obtain a recursive expression for the large system SIR in this unequal power case. This is consistent with [11], where it was shown that, in the unequal power case, the large system SIR does not satisfy the obvious extension of the recursive expression.

The N-dimensional MSMU receiver for m stages and user k is[4]

$$\mathbf{c}_{m,k} = \sum_{i=0}^{m}\beta_{i,k}^{\mathrm{UEP}}(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^i\mathbf{s}_k = \boldsymbol{\Omega}_{m,k}\boldsymbol{\beta}_{m,k} \tag{38}$$

where $\boldsymbol{\beta}_{m,k} = [\beta_{1,k}^{\mathrm{UEP}},\ldots,\beta_{m,k}^{\mathrm{UEP}}]'$, and $\boldsymbol{\Omega}_{m,k} = [\mathbf{s}_k,(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')\mathbf{s}_k,\ldots,(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^m\mathbf{s}_k]$.

In this section on unequal power, users we will focus our attention on the continued fraction expansion of the SIR. As in the previous section, we first change the filter vector subspace to an orthogonal filter vector subspace. The optimal MSMU receiver's filter vector subspace, is now given by the set of vectors $V_k^m = (\mathbf{s}_k,(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')\mathbf{s}_k,\ldots,(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^m\mathbf{s}_k)$, with its equivalent orthogonal filter vector subspace, denoted by $V_k^{m,\perp} = (\mathbf{v}_0,\ldots,\mathbf{v}_m)$. In the unequal user power case, it is simpler to use the Gram Schmidt Orthogonalization (GSO) algorithm to find a set of orthogonal vectors, in place of the approach in (19). Performing the GSO algorithm on V_k^m to get a set of orthogonal vectors gives $\mathbf{v}_0 = \mathbf{s}_k$ and the ith vector is $\mathbf{v}_i = \mathbf{u}_i - \sum_{j=0}^{i-1}(\mathbf{v}_j'\mathbf{u}_i/\|\mathbf{v}_j\|^2)\mathbf{v}_j$, where $\|\mathbf{v}_j\|^2 = \mathbf{v}_j'\mathbf{v}_j$.

Now, for $\boldsymbol{\Omega}_{m,k}^{\perp} = [\mathbf{v}_0,\ldots,\mathbf{v}_m]$ the filter coefficient vector of the optimal MSMU receiver for user k and stage m is

$$\mathbf{c}_{m,k}^{\perp} = \boldsymbol{\Omega}_{m,k}^{\perp}\boldsymbol{\beta}_{m,k}^{\perp} \tag{39}$$

where it can be shown that $\boldsymbol{\beta}_{m,k}^{\perp} = \zeta_{m,k}^{\perp}(\boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}^{\perp})^{-1}\mathbf{e}_1$ minimizes the MSE (and also maximizes the corresponding SIR by direct extension of [13, Th. 1]) for a given stage m and user k, and $\zeta_{m,k}^{\perp} = P_k^{1/2}/(1+P_k\mathbf{e}_1'(\boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}^{\perp})^{-1}\mathbf{e}_1)$, with $\mathbf{e}_1 = \boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{s}_k = [1,0,\ldots,0]'$.

A. Continued Fraction SIR—Unequal Powers

We now give the SIR of the optimal MSMU receiver for unequal power users. First, by substituting (39) into (2) results in the SIR of the form

$$\begin{aligned}\mathrm{SIR}_{m,k}^{\mathrm{UEP}(N)} &= P_k\mathbf{s}_k'\boldsymbol{\Omega}_{m,k}^{\perp'}(\boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}^{\perp})^{-1}\boldsymbol{\Omega}_{m,k}^{\perp}\mathbf{s}_k\\ &= P_k[(\boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}^{\perp})^{-1}]_{00}.\end{aligned} \tag{40}$$

It can be shown for $0\le i,j\le m$ that

$$[\boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}^{\perp}]_{ij} = \begin{cases}\mathbf{v}_i'\mathbf{Z}_k\mathbf{v}_j & \text{for } i\le j+1 \text{ and } j\le i+1,\\ 0, & \text{otherwise.}\end{cases} \tag{41}$$

As well, $\mathbf{v}_i'\mathbf{Z}_k\mathbf{v}_{i-1} = \mathbf{v}_{i-1}'\mathbf{Z}_k\mathbf{v}_i$ is also equal to

$$\mathbf{v}_i'(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')\mathbf{v}_{i-1} = \mathbf{v}_i'\mathbf{u}_i \tag{42}$$

for $1\le i\le m$. Observe that $\boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}^{\perp}$ is tridiagonal. Then, $[(\boldsymbol{\Omega}_{m,k}^{\perp'}\mathbf{Z}_k\boldsymbol{\Omega}_{m,k}^{\perp})^{-1}]_{00}$ is found by solving the following linear equations:

$$\begin{array}{cc}\mathbf{v}_0'\mathbf{Z}_k\mathbf{v}_0x_0+(\mathbf{v}_1'\mathbf{u}_1)x_1 & =1\\ (\mathbf{v}_1'\mathbf{u}_1)x_0+\mathbf{v}_1'\mathbf{Z}_k\mathbf{v}_1x_1+(\mathbf{v}_2'\mathbf{u}_2)x_2 & =0\\ \cdots & \cdots\\ (\mathbf{v}_m\mathbf{u}_m)x_{m-1}+\mathbf{v}_m'\mathbf{Z}_k\mathbf{v}_mx_m & =0.\end{array} \tag{43}$$

The SIR can be written as $\mathrm{SIR}_{m,k}^{\mathrm{UEP}(N)} = P_kx_0$.

Now, from (41) and (42), we can define

$$\mathbf{v}_i'\mathbf{u}_j \equiv \hat{f}_{i,j} = \begin{cases}\hat{\psi}_{i+j}-\sum_{l=0}^{i-1}\frac{\hat{f}_{l,i}\hat{f}_{l,j}}{\hat{f}_{l,l}}, & i,j\ge 0 \text{ and } j\ge i\\ \mathbf{s}_k'\mathbf{s}_k=1, & i=0 \text{ and } j=0\\ 0, & j<i\end{cases} \tag{44}$$

where $\hat{\psi}_i = \mathbf{s}_k'(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^i\mathbf{s}_k$. Similarly, we can define

$$\mathbf{v}_i'(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')\mathbf{v}_i \equiv \hat{g}_i = \begin{cases}\hat{f}_{i,i+1}-\frac{\hat{f}_{i-1,i}\hat{f}_{i,i}}{\hat{f}_{i-1,i-1}}, & \forall i\ge 1\\ \hat{g}_0=\hat{f}_{0,1}, & i=0.\end{cases} \tag{45}$$

[4]The superscript $^{\mathrm{UEP}}$ denotes unequal power users.

where $\hat{f}_{i,j}$ is given in (44). This gives, $\mathbf{v}_i'\mathbf{Z}_k\mathbf{v}_i = \hat{g}_i + \sigma^2\hat{f}_{i,i}$ and $\mathbf{v}_i'(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')\mathbf{v}_{i-1} = \hat{f}_{i,i}$ for $1 \le i \le m$ with $\hat{f}_{0,0} = 1$. Solving (43) with respect to x_0 [20, p. 213] gives

$$\begin{aligned} x_0 &= \frac{1}{\hat{g}_0 + \sigma^2 +} \quad \frac{\hat{f}_{1,1}x_1}{x_0}, \quad \frac{\hat{f}_{1,1}x_1}{x_0} \\ &= \frac{-\hat{f}_{1,1}^2}{\hat{g}_1 + \sigma^2\hat{f}_{1,1} +} \quad \frac{\hat{f}_{2,2}x_2}{x_1}, \ldots, \frac{\hat{f}_{m,m}x_m}{x_{m-1}} \\ &= \frac{-\hat{f}_{m,m}^2}{\hat{g}_m + \sigma^2\hat{f}_{m,m}} \end{aligned} \tag{46}$$

back-substituting the equations in (46) gives $\mathrm{SIR}_{m,k}^{\mathrm{UEP}(N)}$ as a continued fraction expansion

$$\mathrm{SIR}_{m,k}^{\mathrm{UEP}(N)} = \frac{P_k}{\hat{g}_0 + \sigma^2 -} \; \frac{(\hat{f}_{1,1})^2}{\hat{g}_1 + \sigma^2\hat{f}_{1,1} -} \; \cdots \; \frac{(\hat{f}_{m,m})^2}{\hat{g}_m + \sigma^2\hat{f}_{m,m}}. \tag{47}$$

B. Large System SIR—Unequal Power Users

In the large system, we are primarily interested in terms of the form $\mathbf{s}_k'(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^i\mathbf{s}_k$ for $(0 \le i \le 2m+1)$. By taking the limit as $N, K \to \infty$ with $\alpha = K/N$ fixed, this corresponds to finding the ith moment of the limiting empirical distribution of the eigenvalues of $\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k'$, which we denote by $\psi_i(\alpha, F(P))$, where $F(P)$ is the limiting empirical distribution of $\mathbf{P}_k$. A technique for finding these moments was suggested in [11] and [21], however, we will use an alternate method given in [22] where it was shown that $\hat{\psi}_i = \mathbf{s}_k'(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^i\mathbf{s}_k \xrightarrow{p} \psi_i(\alpha, F(P))$ where

$$\begin{aligned} \psi_i(\alpha, F(P)) &\equiv \psi_i(\alpha, F) \\ &= \sum_{j=1}^{i} \alpha^{i-j+1} \sum \frac{i!}{d_1! \cdots d_j! j!} E^{d_1}[P] \cdots E^{d_j}[P^j] \end{aligned} \tag{48}$$

where $i \ge 1$, $E^{d_j}[P^j] = (\int P^j \, dF)^{d_j}$, and $\psi_0(\alpha, F) = 1$.

The inner summation of (48) is over all nonnegative solutions to the equations

$$\begin{aligned} d_1 + d_2 + \cdots + d_j &= i - j + 1 \\ d_1 + 2d_2 + \cdots + jd_j &= i. \end{aligned}$$

Now, the SIR of the optimal MSMU receiver given in (47) converges in probability to

$$\begin{aligned} \mathrm{SIR}_{m,k}^{\mathrm{UEP}(N)} &\xrightarrow{p} \mathrm{SIR}_{m}^{\mathrm{UEP}(\infty)} \\ &= \frac{P_k}{g_0 + \sigma^2 -} \; \frac{(f_{1,1})^2}{g_1 + \sigma^2 f_{1,1} -} \cdots \frac{(f_{m,m})^2}{g_m + \sigma^2 f_{m,m}} \end{aligned} \tag{49}$$

where taking the limit of (44) and (45) gives $\hat{f}_{i,j} \xrightarrow{p} f_{i,j} = \psi_{i+j}(\alpha, F) - \sum_{l=0}^{j-1}(f_{l,i}f_{l,j}/f_{l,l})$ for $i, j \ge 0$, where $f_{0,0} = 1$ and $f_{i,j} = 0$ for $j < i$, and $\hat{g}_i \xrightarrow{p} g_i = f_{i,i+1} - f_{i-1,i}f_{i,i}/f_{i-1,i-1}$ for $1 \le i$, where $g_0 = \psi_1(\alpha, F)$.

Here, convergence in probability holds since $\hat{f}_{i,j}$ and $\hat{g}_i$ are finite sums and products of terms of the form $\mathbf{s}_k'(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^i\mathbf{s}_k$ where in the limit each term converges in probability to $\psi_i(\alpha, F(P))$.

We have now given the large system SIR of the optimal MSMU receiver for both equal and unequal power users. In the following section, we will illustrate the use of these expressions.

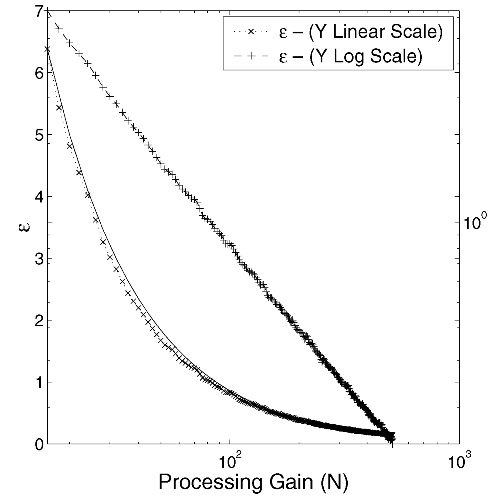

Fig. 1. Average empirical squared error versus processing gain ($m = 4$, $SNR = 12$ dB and $\alpha = 0.75$).

VI. Numerical Results

In this section, we empirically show that the variance of the SIR of the optimal MSMU receiver decreases proportionally to $1/N$, as N increases. We then illustrate the usefulness of the equal power large system SIR by estimating the number of stages needed for the optimal MSMU receiver's bit-error rate (BER) to come within 5% of the LMMSE receiver's BER. Finally, we examine two examples of using the large system SIR expression of the optimal MSMU receiver for unequal power users.

A. Average Empirical Squared Error

We have proved in Theorem 1 that $\mathrm{SIR}_{m,k}^{(N)}$ converges in probability to the large system SIR of the optimal MSMU receiver, $\mathrm{SIR}_m^{(\infty)}$, as N, K increases with $\alpha = K/N$ held fixed. We shall now show this convergence in another manner with the empirical mean square error, which is the MSE between $\mathrm{SIR}_{m,k}^{(N)}$ and $\mathrm{SIR}_m^{(\infty)}$, denoted $\epsilon^{(N)} = \overline{(\mathrm{SIR}_m^{(\infty)} - \mathrm{SIR}_{m,k}^{(N)})^2}$. This gives an indication of the relationship between the variance of the SIR and N. An example of this convergence is illustrated in Fig. 1. In this plot, we have plotted $\epsilon^{(N)}$ using 1000 samples of $\mathrm{SIR}_{k,m}^{(N)}$ for each value of N in the range $8 \le N \le 512$ (incrementing in steps of 4). For each sample of $\mathrm{SIR}_{k,m}^{(N)}$, we randomly generated the spreading sequences. This plot uses $m = 4$, $\alpha = 0.75$, $8 \le N \le 512$, and SNR $= 12$ dB. We empirically found that $\epsilon^{(N)} \approx \hat{\epsilon}^{(N)} = 130N^{-1.1}$ providing evidence that the empirical MSE of the SIR decreases as $1/N$. We have also shown similar results for MSMU receivers based on iterative solution methods, for examples, see [10] and [23]. Further, this result for the optimal MSMU receiver complements those results and that presented in [24], where it has been proven for the decorrelator and LMMSE receiver that the fluctuations around the large system SIR are proportional to $1/N$.

B. LMMSE BER Convergence

It has recently been shown that as the system size increases, the BER of the direct LMMSE receiver converges almost surely

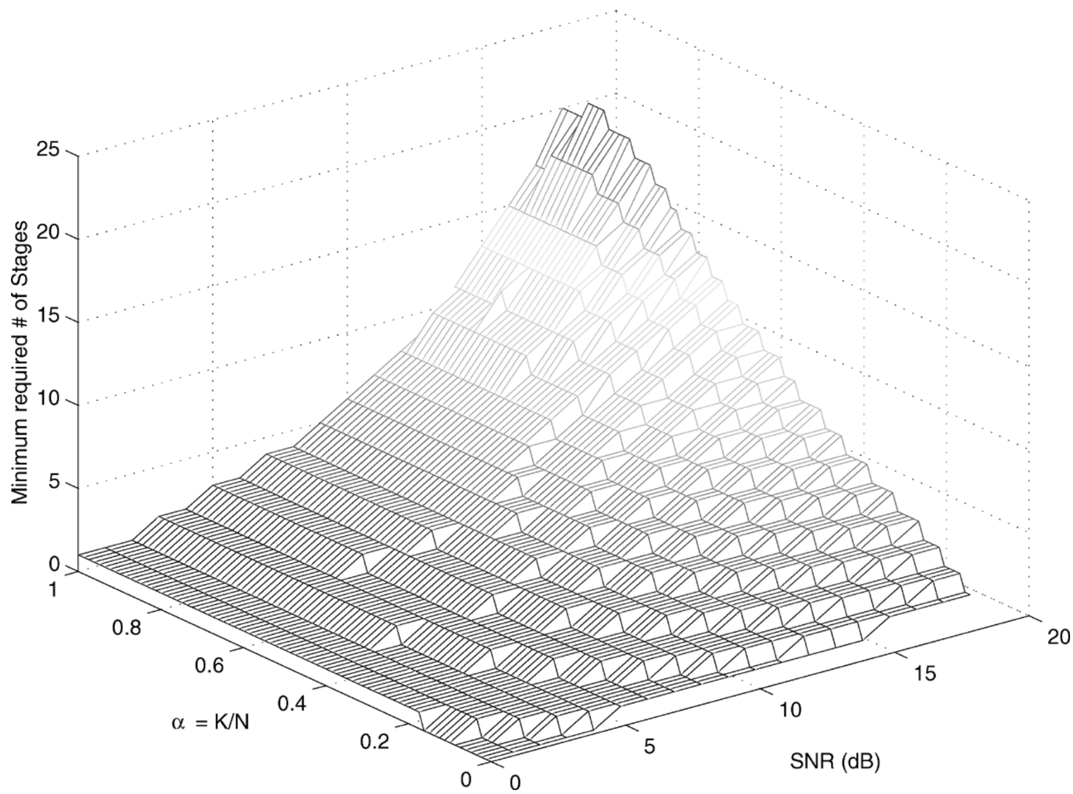

Fig. 2. Minimum required number of stages versus $\alpha = K/N$ versus SNR for equal user powers (within 5% of LMMSE BER).

to $Q(\sqrt{\mathrm{SIR}_{\mathrm{LMMSE}}^{(\infty)}})$ (constrained to antipodal signalling, equal power users in synchronous CDMA), where $\mathrm{SIR}_{\mathrm{LMMSE}}^{(\infty)}$ is the large system SIR of the LMMSE receiver and $Q(\cdot)$ is the Gaussian Q–function, see [17] and [25]. This result has also been found to be applicable for linear multistage receivers [26]. Therefore, we may couple these results with our large system SIR expression to calculate the BER of the optimal MSMU receiver for equal power users, giving $P_{b,m} \approx Q(\sqrt{\mathrm{SIR}_m^{(\infty)}})$.

Fig. 2 shows the number of stages required for the optimal MSMU receiver's BER performance to come within 5% of the BER performance of the LMMSE receiver, that is $P_{b,m} \approx Q(\sqrt{\mathrm{SIR}_m^{\infty}}) \leq 1.05 \times Q(\sqrt{\mathrm{SIR}_{\mathrm{LMMSE}}^{(\infty)}})$. This plot uses, $0.0125 \leq \alpha = K/N \leq 1$ and $0 \leq \mathrm{SNR} \leq 18$. We can see that as the SNR increases and as the number of users increases, more stages are necessary to achieve near-LMMSE performance.

C. Unequal Power Users

This section illustrates the performance of the optimal MSMU receiver using (47) for various user power distributions. Fig. 3 shows results for a system loading of $\alpha = 0.75$, $\mathrm{SNR} = 10$ dB, for up to 15 stages. The corresponding LMMSE and single user matched filter (SUMF) receiver's large system SIR for each group is indicated by the arrows. We verify our large system SIR expression showing that it of course matches that of the Multistage Reduced Rank Wiener (MSWF) filter receiver given by [11] where the large system SIR is indicated by the dashed line and $\circ$ marks. In this case, the optimal MSMU receiver can attain near-LMMSE performance by Stage 8.

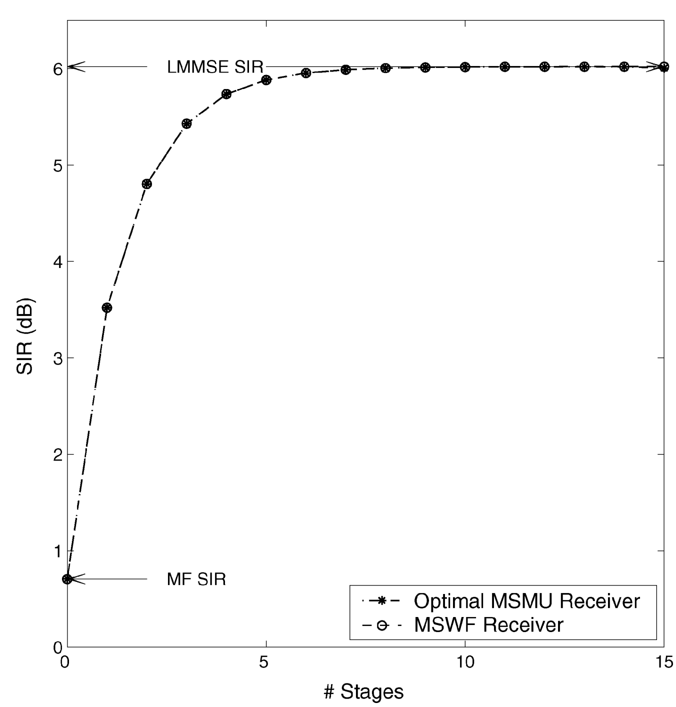

Fig. 3. SIR versus stages for equal user powers ($\mathrm{SNR} = 10$ dB, $\alpha = 0.75$ and $m = 15$).

A multiple data rate scenario such as in third–generation (3G) and future standards, can be modeled using a discrete power distribution where users are grouped by their data rate, for example higher data rates require higher powers. The performance of the optimal MSMU receiver attained in each group is shown

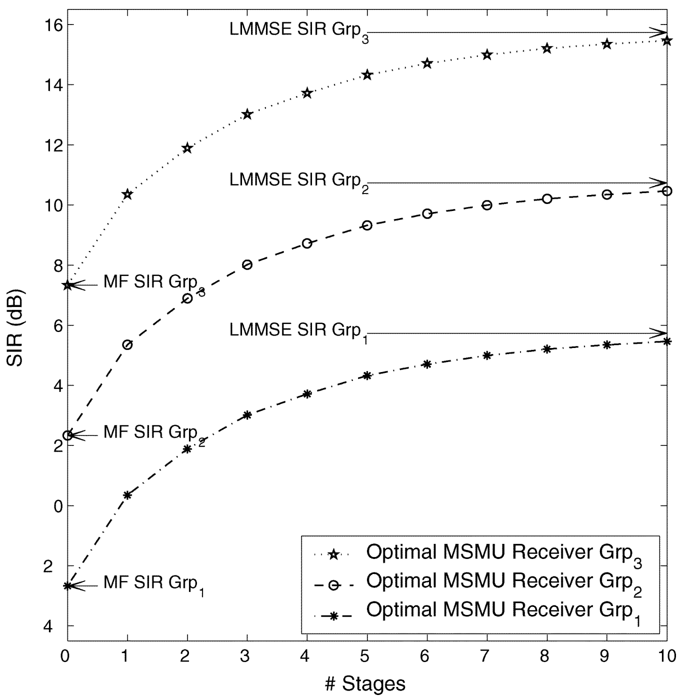

Fig. 4. SIR versus stages for multiple data rages (SNR $= 10$ dB, $\alpha = 0.75$, and $m = 10$).

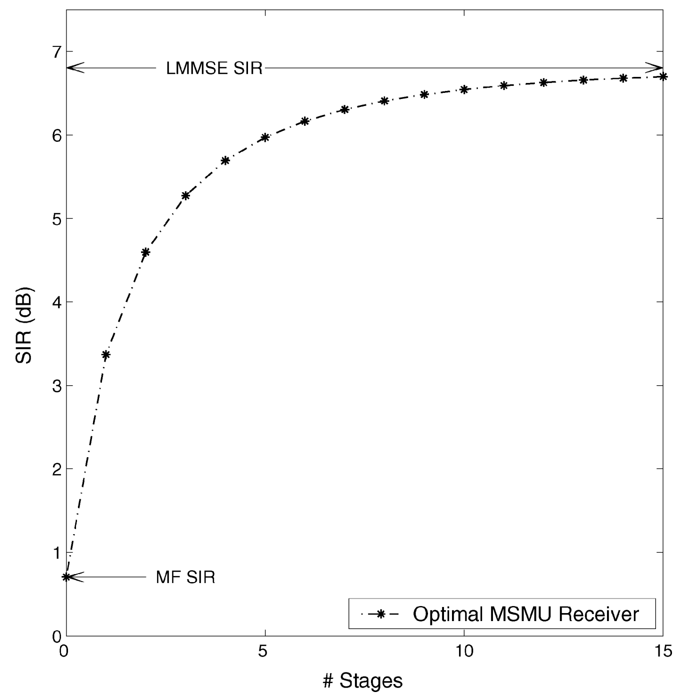

Fig. 5. SIR versus stages for frequency flat Rayleigh fading (SNR $= 10$ dB, $\alpha = 0.75$, and $m = 15$).

in Fig. 4. This plot uses a system loading of $\alpha = 0.75$, the desired user in Group 1 has an SNR $= 10$ dB and $P_{(2)}/P_{(1)} = P_{(3)}/P_{(2)} = 5$ dB. The fraction of users in Groups 1, 2, and 3 are 0.7, 0.2, and 0.1 respectively. It can be seen that at least 10 stages are needed to attain near-LMMSE performance.

The increase in the number of stages compared to the equal rate case can be explained by the higher power users causing more multiple access interference. Another reason has been discussed in [2] where they noted that by applying an eigenvalue-eigenvector decomposition to $\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k'$, then the eigenvalues of (38), $(\sum_{i=0}^{m}\omega_{i,k}(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k')^i\mathbf{s}_k)$ form a polynomial which approximates the eigenvalues of $(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k' + \sigma^2\mathbf{I})^{-1}$. If λ_j is the jth eigenvalue of $\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k'$, then the jth eigenvalue of $(\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k' + \sigma^2\mathbf{I})^{-1}$ is $h(\lambda_j) = 1/(\lambda_j + \sigma^2)$. As well, the jth eigenvalue of the matrix polynomial for stage m is $h_m(\lambda_j) = \sum_{i=0}^{m}\omega_{i,j}\lambda_j^i$ which approximates $h(\lambda_j)$ when $\omega_{i,j}$ is chosen appropriately. However, it was shown in [2] that there is a limited ability for $h_m(\lambda_j)$ to approximate $h(\lambda_j)$ for very small (as well as very large values) of $\lambda_j + \sigma^2$ without increasing m. In an unequal user power scenario, then the possibility that λ_j is very small (or large) results in the need for more stages in order for $h_m(\lambda_j)$ to approximate $h(\lambda_j)$. This is reflected in our large system SIR results.

We now show the effect of the frequency nonselective Rayleigh-fading channel. The performance of the optimal MSMU receiver for this channel is shown in Fig. 5. This plot uses $\alpha = 0.75$ and SNR $= 10$ dB, and it was assumed that the average received power was $E[P] = 1$. As the received amplitudes, $\sqrt{P_k}$, are Rayleigh distributed this results in the user power distribution being the simple exponential distribution. Once again, the eigenvalue spread, (ratio between maximum and minimum eigenvalues of $\mathbf{S}_k\mathbf{P}_k\mathbf{S}_k'$), has increased and more stages are necessary to attain near-LMMSE performance compared with the equal power scenario in Fig. 3.

We have shown some simple examples of the various user power distributions that can be considered using the unequal large system SIR expression. The analysis of the large system SIR for the optimal MSMU receiver in multipath-fading channels and incorporating channel estimation uncertainty is an ongoing topic of research.

VII. Conclusion

In this paper, we analyzed the MSMU receiver based on a weighted matrix polynomial expansion. We presented an alternative derivation of the large system SIR of the optimal MSMU receiver, and made significant steps toward a direct proof of the recursive large system SIR expression. Finally, we derived the SIR and the large system SIR for the optimal MSMU receiver in an unequal user power regime.

Acknowledgment

The authors would like to thank Prof. M. Honig for our discussions on the topic of continued fraction representations. His insights have been extremely valuable in preparing this manuscript.

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Louis G. F. Trichard (S’98–M’03) was born in Cape Town, South Africa, in 1973. He received the B.S. degree in computer science and mathematics and the B.E. degree in electrical engineering (information systems engineering) from the University of Sydney, Sydney, Australia, in 1996 and 1998, respectively. He continued his studies in the Telecommunications Laboratory in the School of Electrical and Information Engineering, University of Sydney, where he submitted his Ph.D. thesis in 2001 and received the Ph.D. degree in 2002 in electrical engineering.

In December 2001, he moved to Tokyo, Japan, where he held a JSPS Postdoctoral Research Fellowship at the Kohno Laboratory, Division of Physics, Electrical and Computer Engineering, Graduate School of Engineering, Yokohama National University, Kanagawa, Japan. In 2003, he moved to London, U.K., where, in 2005, he attained the Certificate in Intellectual Property Law from Queen Mary College Westfield, University of London, and is now currently qualifying to become a patent attorney at R.G.C. Jenkins & Co. His current research interests include multiuser detection, wireless communications, and ultrawideband communications.

Dr. Trichard was awarded the Telstra Australia Prize in electrical engineering in 1998.

Jamie S. Evans (S’93–M’98) was born in Newcastle, Australia, in 1970. He received the B.S. degree in physics and the B.E. degree in computer engineering from the University of Newcastle, Callaghan, Australia, in 1992 and 1993, respectively, and received the University Medal upon graduation. He received the M.S. and the Ph.D. degrees from the University of Melbourne, Parkville, Victoria, Australia, in 1996 and 1998, respectively, both in electrical engineering, and was awarded the Chancellor’s Prize for Excellence for his Ph.D. thesis.

From March 1998 to June 1999, he was a Visiting Researcher in the Department of Electrical Engineering and Computer Science, University of California, Berkeley. He returned to Australia to take up a position as Lecturer at the University of Sydney, Sydney, Australia, where he stayed until July 2001. Since that time, he has been with the Department of Electrical and Electronic Engineering, University of Melbourne, where he is currently an Associate Professor. His research interests are in communications theory, information theory, and statistical signal processing with current focus on wireless communications networks.

Dr. Evans currently serves on the Editorial Board of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS.

Iain B. Collings (S’92–M’95–SM’02) was born in Melbourne, Australia, in 1970. He received the B.E. degree in electrical and electronic engineering from the University of Melbourne, in 1992, and the Ph.D. degree in systems engineering from the Australian National University, Canberra, in 1995.

In 1995, he was a Research Fellow with the Australian Cooperative Research Center for Sensor Signal and Information Processing, Adelaide, Australia, where he worked in the area of radar signal processing. From 1996 to 1999, he was a Lecturer at the University of Melbourne. Since 1999, he has been with the School of Electrical and Information Engineering, University of Sydney, Sydney, Australia, where he is currently an Associate Professor. His current research interests include synchronization, channel estimation, equalization, and multicarrier modulation, for time-varying and frequency-selective channels.

Dr. Collings is currently an Editor for the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS and is a Guest Editor for the *EURASIP Journal on Applied Signal Processing*. He has also served on the Technical Program Committees for the IEEE GLOBECOM Conference, Taipei, Taiwan, 2002, the IEEE Vehicular Technology Conference, Orlando, FL, 2003, and the IEEE International Symposium on Spread Spectrum Techniques and Applications, Sydney, Australia, 2004.