Exact Filters for Doubly Stochastic AR Models with Conditionally Poisson Observations

Jamie Evans and Vikram Krishnamurthy

***Abstract*—In this paper the authors derive exact filters for the state of a doubly stochastic auto-regressive (AR) process with parameters which vary according to a nonlinear function of a Gauss–Markov process. The observations consist of a discrete-time Poisson process with rate a positive function of the Gauss–Markov process. The dimension of the sufficient statistic increases linearly with the number of observed events.**

***Index Terms*—Doubly stochastic models, exact filters, nonlinear filters, Poisson observations.**

I. Introduction

In this paper we derive optimal (minimum mean squared error) recursive filters for random parameter auto-regressive (AR) models in discrete time. The AR parameter varies as an exponential or polynomial function of a linear Gaussian dynamical system—the *driving* process. The observations form a doubly stochastic, discrete-time Poisson process with rate proportional to the square of the driving process.

The vector of statistics required to fully specify the filtered estimate at a particular time is known as the sufficient statistic for the filter. The new filters we derive here propagate a sufficient statistic with a dimension which increases linearly with the number of observed Poisson events.

Filtering of doubly stochastic Poisson processes has been considered in both continuous [1]–[5] and discrete time [6] for both continuous and discrete-valued rate processes. General filtering results for doubly stochastic point processes in discrete time are presented in [7]. Note that [7] uses a different interpretation of a discrete-time Poisson process where in each time interval either no events occur or one event occurs. Our model can be seen as a generalization of this result which allows any number of events with probabilities governed by the state of the driving process.

A key motivation for studying this problem stems from the authors' recent study of random parameter AR models where the driving process is observed in Gaussian noise [8], [9]. Several new finite-dimensional filters were discovered for these models which is quite surprising, considering the rarity of such filters. This paper provides the analogs of these results for the Poisson observation case.

Doubly stochastic Poisson processes have been widely used in many areas including optical communications and medical diagnosis [1], teletraffic source modeling [10], and computer network analysis [7]. The model we examine here also appears promising for applications to general multisensor applications such as image-enhanced tracking [11]–[13]. In image-enhanced tracking a maneuvering target is modeled by a Markov jump linear system (a random parameter AR process driven by a finite state Markov chain). Image sensors are used to obtain target orientation measurements from which information about the state of the Markov chain can be obtained. These measurements are then used in conjunction with the noisy observations of the state (position, velocity) of the target to estimate the true state. It is usually impractical to implement the optimal filter due to exponential growth in computational requirements with time [14]. However, if the estimate is based solely on the image sensor measurements then the optimal filter turns out to be finite-dimensional [13], [15]. In the absence of state observations, we use the term *image-based* tracking.

In the model under consideration here the finite state Markov chain representing the mode of the maneuvering target is replaced by a continuous-valued random process. This more accurately reflects the underlying continuum of possible modes or orientations of the target, and in fact early work on tracking with orientation measurements did not discretize the mode space [16], [17].

When a finite state Markov process is used to model the mode, the mode is traditionally observed indirectly via a Markov (mode) modulated vector Poisson process. This model is based on properties of image sensors and image processing algorithms. In fact this is also primary motivation for discretizing the orientation space. When the orientation measurements are not discretized, the orientation is assumed to be observed directly in Gaussian noise. While our continuous mode and Poisson observation model does not exactly match existing models, we believe there is much promise for other "image-based" applications, especially considering the explicit form of the optimal filters.

The paper is organized as follows. In Section II we define the signal model followed in Section III by the introduction of a measure change which simplifies derivation of the filters. In Section IV we derive recursions for filtered densities of the linear Gaussian driving process and the random parameter AR model. These recursions are solved explicitly in Section V for the case when the AR parameter is an exponential and a polynomial function of the driving process. Section V contains the main contribution of the paper.

II. Signal Model

All random processes are defined initially on the probability space $(\Omega, \mathcal{F}, P)$. We begin with the scalar linear stochastic difference equation

$$x_{k+1} = A_{k+1} x_k + w_{k+1} \tag{1}$$

where $x_k \in R$ with x_0 a Gaussian random variable with zero mean and nonzero variance Q_0 and $A_k \in R$ is deterministic. The process $\{w_k\}$ is a sequence of independent, zero mean Gaussian random variables and w_k has nonzero variance Q_k. The sequence $\{w_k\}$ is assumed independent of x_0.

The observation process is the doubly stochastic, discrete-time Poisson process $\{n_k\}$, $k \geq 0$ with rate $(C_k x_k)^2$ where $C_k \in R$ is deterministic. We thus have

$$\mathbf{E}\{I(n_k = n) \mid x_k\} = \frac{(C_k x_k)^{2n}}{n!} \exp[-(C_k x_k)^2] \tag{2}$$

where $\mathbf{E}$ denotes expectation under P and I is an indicator function.

The process x_k drives the scalar *doubly stochastic* AR process

$$s_{k+1} = f_{k+1}(x_k) s_k + u_{k+1}, \qquad s_0 = 1 \tag{3}$$

where $s_k \in R$, $f_k : R \to R$ is a real valued function in x and k, and $\{u_k\}$ is a sequence of zero mean random variables independent of x_0 and the processes w_k and n_k.

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Define the sigma fields

$$\mathcal{G}_k = \sigma\{x_0, x_1, \cdots, x_k, n_0, n_1, \cdots, n_k\}$$
$$\mathcal{Y}_k = \sigma\{n_0, n_1, \cdots, n_k\}$$

with corresponding complete filtrations $\{\mathcal{G}_k\}$ and $\{\mathcal{Y}_k\}$.

Aim: To derive a filter for s_k, i.e., to compute the filtered estimate $\hat{s}_k = \mathbf{E}\{s_k \mid \mathcal{Y}_k\}$. We assume s_0 is known.

Remark 1: In the sequel we assume that for all k, $u_k = 0$ a.s. so that (3) is replaced by

$$s_{k+1} = f_{k+1}(x_k)s_k, \qquad s_0 = 1. \tag{4}$$

No generality is lost due to the independence and zero mean conditions on the u_k process. This assumption and the fact that s_0 is known, means that s_k is $\mathcal{G}_{k-1}$-measurable.

III. Measure Change

In this section we introduce a change of measure which simplifies derivation of the filtered densities. Similar methods are discussed for both finite and continuous state-space models in [18].

Suppose on the probability space $(\Omega, \mathcal{F}, \bar{P})$, $\{x_k\}$ is a sequence of independent Gaussian random variables with zero mean and covariance matrix Q_k, and $\{n_k\}$ is a homogeneous Poisson process with unit rate. Further assume that the processes x_k and n_k are independent under $\bar{P}$.

For convenience define

$$\psi_l(x) = (2\pi)^{-1/2} Q_l^{-1/2} \exp\left(-\frac{1}{2}x^2 Q_l^{-1}\right), \qquad x \in R$$
$$\phi(y, n) = \frac{y^n}{n!}\exp(-y), \qquad y \in R, \qquad n = 0, 1, \cdots.$$

Write

$$\begin{aligned}\lambda_0 &= \frac{\phi((C_0 x_0)^2, n_0)}{\phi(1, n_0)}\\ \lambda_l &= \frac{\phi((C_l x_l)^2, n_l)}{\phi(1, n_l)} \frac{\psi_l(x_l - A_l x_{l-1})}{\psi_l(x_l)}, \qquad l \geq 1.\end{aligned} \tag{5}$$

For $k \geq 0$ set

$$\Lambda_k = \prod_{l=0}^{k} \lambda_l \tag{6}$$

and define a new probability measure P on the premeasure space $(\Omega, \mathcal{G}_k)$ by setting the $\mathcal{G}_k$ restriction of the Radon–Nikodym derivative of P with respect to $\bar{P}$ to Λ_k

$$\left.\frac{dP}{d\bar{P}}\right|_{\mathcal{G}_k} = \Lambda_k.$$

The following result relating conditional expectations under P and $\bar{P}$ will be used repeatedly.

Lemma 2: If $\{\phi_k\}$ is a $\mathcal{G}$-adapted integrable sequence of random variables and $\mathcal{H}$ is a subsigma field of $\mathcal{G}_k$, then

$$\mathbf{E}\{\phi_k \mid \mathcal{H}\} = \frac{\bar{\mathbf{E}}\{\Lambda_k \phi_k \mid \mathcal{H}\}}{\bar{\mathbf{E}}\{\Lambda_k \mid \mathcal{H}\}} \tag{7}$$

where $\bar{\mathbf{E}}$ denotes expectation under measure $\bar{P}$.

Proof: See [18, Lemma 3.3]. □

We then have the following result which says that under P the dynamic relations given in (1) and (2) hold. The proof is standard (see [18] for example).

Lemma 3: Define $w_l = x_l - A_l x_{l-1}, l \geq 1$. Then under measure P, $\{w_l\}$ is a sequence of independent Gaussian random variables with zero mean and covariance matrix Q_l, and $\{n_l\}$ is a doubly stochastic Poisson process with rate $(C_l x_l)^2$.

In this sense P represents the *real world* measure; however, we will work with $\bar{P}$ since the independence properties under $\bar{P}$ simplifies manipulations involving conditional expectations.

IV. Recursions for Filtered Densities

In this section we derive recursive expressions for unnormalized conditional densities which will be used in the sequel to calculate the finite-dimensional filters. While we will use the measure change from Section III, these recursions can also be derived using standard Bayesian techniques. We prefer the measure change approach as it is somewhat simpler once the machinery of Section III is in place.

Let $\alpha_k(x)$ and $\rho_k(x)$ denote the densities implicitly defined by

$$\bar{\mathbf{E}}\{\Lambda_k g(x_k) \mid \mathcal{Y}_k\} = \int_R \alpha_k(x) g(x)\, dx \tag{8}$$
$$\bar{\mathbf{E}}\{\Lambda_k s_k g(x_k) \mid \mathcal{Y}_k\} = \int_R \rho_k(x) g(x)\, dx \tag{9}$$

for any measurable function $g : R \to R$.

We then have the following theorem which gives recursive expressions for the above densities. The proof is simplified due to the independence properties of the $\{x_k\}$ and $\{n_k\}$ sequences under $\bar{P}$.

The densities defined in (8) and (9) obey the following recursions for $k \geq 1$:

$$\alpha_k(x) = \frac{\phi((C_k x)^2, n_k)}{\phi(1, n_k)} \int_R \psi_k(x - A_k z)\alpha_{k-1}(z)\, dz \tag{10}$$
$$\rho_k(x) = \frac{\phi((C_k x)^2, n_k)}{\phi(1, n_k)} \int_R \psi_k(x - A_k z) f_k(z)\rho_{k-1}(z)\, dz \tag{11}$$

with initial values for the recursions given by

$$\begin{aligned}\alpha_0(x) &= \frac{\phi((C_0 x)^2, n_0)}{\phi(1, n_0)}\psi_0(x)\\ \rho_0(x) &= \alpha_0(x).\end{aligned} \tag{12}$$

Proof: We prove the recursion for $\rho_k(x)$. The proof of (10) is similar and hence omitted.

Once more let $g : R \to R$ be a measurable test function. Using (4)–(6) and the independence properties of the $\{x_k\}$ and $\{n_k\}$ sequences under $\bar{P}$, we arrive at the following chain of equalities:

$$\begin{aligned}&\bar{\mathbf{E}}\{\Lambda_k s_k g(x_k) \mid \mathcal{Y}_k\}\\ &\quad= \bar{\mathbf{E}}\bigg\{\Lambda_{k-1}\frac{\phi((C_k x_k)^2, n_k)}{\phi(1, n_k)}\frac{\psi_k(x_k - A_k x_{k-1})}{\psi_k(x_k)}\\ &\qquad\qquad \times f_k(x_{k-1})s_{k-1}g(x_k) \mid \mathcal{Y}_k\bigg\}\\ &\quad= \frac{1}{\phi_k(1, n_k)}\bar{\mathbf{E}}\bigg\{\Lambda_{k-1}\int_R \phi((C_k x)^2, n_k)\psi_k(x - A_k x_{k-1})\\ &\qquad\qquad \times f_k(x_{k-1})s_{k-1}g(x)\, dx \mid \mathcal{Y}_k\bigg\}\\ &\quad= \frac{1}{\phi(1, n_k)}\int_R\int_R \phi((C_k x)^2)\psi_k(x - A_k z)\\ &\qquad \times f_k(z)\rho_{k-1}(z)g(x)\, dx\, dz\\ &\quad= \int_R \left[\frac{\phi((C_k x_k)^2, n_k)}{\phi(1, n_k)}\int_R \psi_k(x - A_k z)\right.\\ &\qquad\qquad \left.\times f_k(z)\rho_{k-1}(z)\, dz\right] g(x)\, dx.\end{aligned} \tag{13}$$

Since g is an arbitrary test function, equating the right-hand side of (13) with (9) immediately yields (11).

Now at $k=0$ we have

$$\begin{aligned}\bar{\mathbf{E}}\{\Lambda_0 g(x_0) \mid \mathcal{Y}_0\} &= \bar{\mathbf{E}}\left\{\frac{\phi((C_0x_0)^2, n_0)}{\phi(1,n_0)} g(x_0) \mid \mathcal{Y}_0\right\} \\ &= \frac{1}{\phi(1,n_0)} \int_R \phi((C_0x)^2, n_0)\psi_0(x) g(x)\, dx\end{aligned}$$

which on equating with (8) gives (12). Similarly because $s_0 = 1$, it can be shown that $\rho_0(x) = \alpha_0(x)$. □

Remark 5: It is important to note that $\rho_k(x)$ is not the unnormalized conditional density of s_k given $\mathcal{Y}_k$ but is used for determining the conditional mean estimate of s_k given $\mathcal{Y}_k$ as follows.

Corollary 6:

$$\hat{s}_k = \mathbf{E}\{s_k \mid \mathcal{Y}_k\} = \frac{\bar{\mathbf{E}}\{\Lambda_k s_k \mid \mathcal{Y}_k\}}{\bar{\mathbf{E}}\{\Lambda_k \mid \mathcal{Y}_k\}} = \frac{\left(\int_R \rho_k(x)\, dx\right)}{\left(\int_R \alpha_k(x)\, dx\right)}. \tag{14}$$

Proof: The proof is immediate from Lemma 2 and the definitions of $\alpha_k(x)$ and $\rho_k(x)$. □

Remark 7: The results we present can be readily extended to derive finite-dimensional filters for the second moment of s_k. In particular, if

$$\gamma_k(x)\, dx = \bar{\mathbf{E}}\left\{\Lambda_k s_k^2 I(x_k \in dx) \,\middle|\, \mathcal{Y}_k\right\}$$

then we have the recursion

$$\begin{aligned}\gamma_k(x) = {}& \frac{\phi((C_kx)^2, n_k)}{\phi(1,n_k)} \int_R \psi_k(x - A_k z) f_k^2(z) \gamma_{k-1}(z)\, dz \\ &+ S_k \alpha_k(x)\end{aligned}$$

where $S_k = \mathbf{E}\{u_k^2\}$ is the covariance of the zero mean noise process which we can no longer assume to be zero without losing generality.

V. Finite-Dimensional Filters

We begin this section by giving an explicit solution to the recursion of (10) for the unnormalized filtered density of x_k. This leads immediately to a filter for estimating the rate of the discrete-time Poisson process. This filter was derived in [6] where we refer the reader for further details.

The main contribution of this paper appears in Theorems 10 and 12 where explicit solutions of (11) are given for the cases when f_k is an exponential (Theorem 10) and when f_k is a polynomial (Theorem 12). These solutions are used to obtain optimal state estimates for the doubly stochastic AR process of (3).

Theorem 8—Solution for $\alpha_k(x)$: The unnormalized filtered density of x_k is given by

$$\alpha_k(x) = x^{M_k}\left(\sum_{j=0}^{L_k} P_k(j) x^j\right) \exp\left(-\frac{1}{2}x^2\Omega_k^{-1}\right) \tag{15}$$

where the sufficient statistic

$$(M_k, L_k, P_k(0), \cdots, P_k(L_k), \Omega_k)$$

is recursively computed for $k \geq 1$ as

$$\begin{aligned} M_k &= 2n_k, \qquad M_0 = 2n_0 \\ L_k &= L_{k-1} + M_{k-1}, \quad L_0 = 0 \\ P_k(j) &= U_k \sum_{i=i^*}^{L_{k-1}} P_{k-1}(i)\xi(j + M_{k-1}, i), \\ P_0(0) &= (2\pi Q_0)^{-1/2} C_0^{2n_0} e^{-1} \\ \Omega_k &= \left(2C_k^2 + Q_k^{-1} - R_k A_k^2 Q_k^{-2}\right)^{-1} \\ \Omega_0 &= \left(2C_0^2 + Q_0^{-1}\right)^{-1}\end{aligned}$$

where $i^* = \max(0, j - M_{k-1})$ and

$$\begin{aligned}\xi(r,s) &= \binom{r}{s}(R_kA_kQ_k - 1)^i \eta\left(r - s, R_k^{1/2}\right) \\ R_k &= \left(\Omega_{k-1}^{-1} + A_k^2 Q_k^{-1}\right)^{-1} \\ U_k &= R_k^{1/2} Q_k^{-1/2} C_k^{2n_k} e^{-1}\end{aligned}$$

with $\eta(r,\sigma)$ as defined in (21).

Further, the filtered estimate of the Poisson rate is

$$\mathbf{E}\{(C_kx_k)^2 \mid \mathcal{Y}_k\} = C_k^2 \frac{\sum_{j=0}^{L_k} P_k(j)\eta\left(j + M_k + 2, \Omega_k^{1/2}\right)}{\sum_{j=0}^{L_k} P_k(j)\eta\left(j + M_k, \Omega_k^{1/2}\right)}.$$

Proof: See [6]. □

Remark 9: The dimension of the sufficient statistic at time k is the random variable $4 + 2\sum_{j=0}^{k-1} n_j$. Strictly speaking this filter is not finite-dimensional since there is a positive probability of the dimension being arbitrarily large. The total computational cost for filtering a block of k observations is $O(k(\sum_{j=0}^{k-1} n_j)^2)$. Note that Ω_k is independent of the data and can be calculated off-line.

A suboptimal filter with a fixed dimension sufficient statistic was proposed in [6] using an Edgeworth series expansion to approximate $\alpha_k(x)$ by a fixed-order polynomial times a Gaussian. This reduces the total computational cost to $O(k)$.

Also note that as pointed out in [6], Theorem 8 can be extended to the case where x_k is vector valued.

To obtain conditional mean estimates of quantities such as x_k^2 and s_k we need the following normalization constant:

$$\begin{aligned}\Delta_k &= \int_R \alpha_k(x)\, dx \\ &= \sum_{j=0}^{L_k} P_k(j) \int_R x^{j+M_k} \exp\left(-\frac{1}{2}x^2\Omega_k^{-1}\right) dx \\ &= (2\pi\Omega_k)^{1/2} \sum_{j=0}^{L_k} P_k(j) \mathbf{E}\{x^{j+M_k}\}\end{aligned}$$

where x is a normal random variable with mean zero and variance Ω_k on $(\Omega, \mathcal{F}, P)$. Continuing on

$$\Delta_k = (2\pi\Omega_k)^{1/2} \sum_{j=0}^{L_k} P_k(j)\eta\left(j + M_k, \Omega_k^{1/2}\right) \tag{16}$$

where $\eta(r,\sigma)$ is defined in (21).

We now turn to filters for the state of the random parameter AR model starting with the case where f_k is an exponential.

Theorem 10—Solution for $\rho_k(x)$: Exponential Case: If $f_k : R \to R$ with

$$f_k(x) = e_k \exp(a_k x^2 + b_k x) \tag{17}$$

then the solution to the recursion (11) is given by

$$\rho_k(x) = x^{M_k}\left(\sum_{j=0}^{L_k} S_k(j) x^j\right) \exp(H_k x^2 + D_k x) \tag{18}$$

where the sufficient statistic

$$(M_k, L_k, S_k(0), \cdots, S_k(L_k), H_k, D_k)$$

is recursively computed for $k \geq 1$ as

$$
\begin{aligned}
M_k &= 2n_k, \qquad M_0 = 2n_0\\
L_k &= L_{k-1} + M_{k-1}, \quad L_0 = 0\\
S_k(j) &= V_k \sum_{i=i^*}^{L_{k-1}} \sum_{l=j}^{j+M_{k-1}} S_{k-1}(i)\chi(i+M_{k-1}, l, j)\\
S_0(0) &= (2\pi Q_0)^{-1/2} C_0^{2n_0} e^{-1}\\
H_k &= -\frac{1}{2}\left(2C_k^2 + Q_k^{-1} - R_k A_k^2 Q_k^{-2}\right)\\
H_0 &= -\frac{1}{2}\left(2C_0^2 + Q_0^{-1}\right)\\
D_k &= R_k A_k Q_k^{-1}(b_k + D_{k-1}), \qquad D_0 = 0
\end{aligned}
$$

where $i^* = \max(0, j - M_{k-1})$ and

$$
\begin{aligned}
\chi(r,s,t) &= \binom{r}{s}\binom{s}{t} R_k^s (A_k Q_k - 1)^t\\
&\quad \times (b_k + D_{k-1})^{s-t} \eta\left(r - s, R_k^{1/2}\right)\\
R_k &= \left(A_k^2 Q_k^{-1} - 2a_k - 2H_{k-1}\right)^{-1}\\
V_k &= \left(R_k^{1/2} Q_k^{-1/2} C_k^{2n_k} e^{-1}\right) e_k \exp\left(\frac{1}{2} R_k (b_k + D_{k-1})^2\right)
\end{aligned}
$$

with $\eta(r,\sigma)$ as defined in (21).

Further, the minimum mean squared estimate of s_k is computed from

$$\hat{s}_k = \mathbf{E}\{s_k \mid \mathcal{Y}_k\} = \Gamma_k / \Delta_k$$

where

$$
\begin{aligned}
\Gamma_k &= \left(-\frac{\pi}{H_k}\right)^{1/2} e^{-D_k^2/4H_k} \sum_{j=0}^{L_k} \sum_{i=0}^{j+M_k} \binom{j+M_k}{i}\\
&\quad \times S_k(j)\left(-\frac{D_k}{2H_k}\right)^i \eta(j + M_k - i, (-2H_k)^{-1/2})
\end{aligned}
$$

and Δ_k is defined in (16).

Proof: The result follows inductively upon substitution of (18) into the recursion of (11). For full details the reader is referred to [19]. □

Remark 11: The form of $\rho_k(x)$ is a polynomial times a Gaussian just as for $\alpha_k(x)$. The dimension of the sufficient statistic at time k is the random variable $5 + 2\sum_{j=0}^{k-1} n_j$. Again the total computational cost for filtering a block of k observations is $O(k(\sum_{j=0}^{k-1} n_j)^2)$. Note that H_k and D_k are independent of the data and can thus be calculated off-line.

We next consider the situation where f_k is a polynomial.

Theorem 12—Solution for $\rho_k(x)$: Polynomial Case: If $f_k : R \to R$ with

$$f_k(x) = \sum_{l=0}^{p} e_k(l) x^l \tag{19}$$

then the solution to the recursion (11) is given by

$$\rho_k(x) = x^{M_k}\left(\sum_{j=0}^{N_k} T_k(j) x^j\right) \exp\left(-\frac{1}{2} x^2 \Omega_k^{-1}\right) \tag{20}$$

where the sufficient statistic

$$(M_k, N_k, T_k(0), \cdots, T_k(N_k), \Omega_k)$$

is recursively computed for $k \geq 1$ as

$$
\begin{aligned}
M_k &= 2n_k, \quad M_0 = 2n_0\\
N_k &= N_{k-1} + M_{k-1} + p, \qquad N_0 = 0\\
T_k(j) &= U_k \sum_{l=l^*}^{p} \sum_{i=i^*}^{N_{k-1}} e_k(l) T_{k-1}(i) \xi(l + i + M_{k-1}, j)\\
T_0(0) &= (2\pi Q_0)^{-1/2} C_0^{2n_0} e^{-1}\\
\Omega_k &= \left(2C_k^2 + Q_k^{-1} - R_k A_k^2 Q_k^{-2}\right)^{-1}\\
\Omega_0 &= \left(2C_0^2 + Q_0^{-1}\right)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
l^* &= \max(0, j - M_{k-1} - N_{k-1})\\
i^* &= \max(0, j - M_{k-1} - p)\\
\xi(r,s) &= \binom{r}{s}(R_k A_k Q_k - 1)^s \eta\left(r - s, R_k^{1/2}\right)\\
R_k &= \left(\Omega_{k-1}^{-1} + A_k^2 Q_k^{-1}\right)^{-1}\\
U_k &= R_k^{1/2} Q_k^{-1/2} C_k^{2n_k} e^{-1}
\end{aligned}
$$

with $\eta(r,\sigma)$ as defined in (21).

Further, the minimum mean squared estimate of s_k is computed from

$$\hat{s}_k = \mathbf{E}\{s_k \mid \mathcal{Y}_k\} = \Xi_k / \Delta_k$$

where

$$\Xi_k = (2\pi\Omega_k)^{1/2} \sum_{j=0}^{N_k} T_k(j) \eta\left(j + M_k, \Omega_k^{1/2}\right)$$

and Δ_k is defined in (16).

Proof: The result follows inductively upon substitution of (20) into the recursion of (11). For full details the reader is referred to [19]. □

Remark 13: The form of $\rho_k(x)$ is once more a polynomial times a Gaussian. The dimension of the sufficient statistic at time k is the $4 + 2\sum_{j=0}^{k-1} n_j + kp$ which grows linearly with time and with the number of observed events. The total computational cost for filtering a block of k observations is now $O(k(\sum_{j=0}^{k-1} n_j)^2 + k^3)$.

VI. Conclusion

In this paper we derive exact filters for the state of a random parameter AR process when the random parameter is either an exponential or a polynomial function of a linear Gaussian driving process. The observations consist of a discrete-time Poisson process which has its rate modulated by the square of the driving process. The new filters are specified by a sufficient statistic which increases with time. In the exponential case, the growth is linear in the number of observed events while in the polynomial case, the increase is linear in both time and the number of events.

We note that it is also possible to derive a filter when $f_k(x)$ is a polynomial times an exponential. For general $f_k(x)$ one could approximate the function by a polynomial times an exponential and develop suboptimal approaches along the lines of the extended Kalman filter. This is a subject for future research.

Appendix I
Noncentral Moments of Gaussian Random Variables

Assume that Z is a Gaussian random variable on $(\Omega, \mathcal{F}, P)$ with mean m and variance σ^2 and let $\mathbf{E}$ denote expectation with respect to P. We are interested in evaluating the moments

$$m_n = \mathbf{E}\{Z^n\} = (2\pi\sigma^2)^{-1/2} \int_R z^n \exp\left(-\frac{1}{2}(z - m)^2 \sigma^{-2}\right).$$

We will use a binomial expansion to express m_n in terms of the central moments of Z which are then readily expressed in a simple recursive form. We have

$$m_n = \mathbf{E}\{(Z-m+m)^n\} = \mathbf{E}\left\{\sum_{j=0}^{n}(Z-m)^{n-j}m^j\right\} = \sum_{j=0}^{n}\mathbf{E}\{(Z-m)^{n-j}\}m^j = \sum_{j=0}^{n}\eta(n-j,\sigma)m^j$$

where

$$\eta(r,\sigma) = \begin{cases} 1, & r=0 \\ 0, & r>0 \text{ and odd} \\ 1\cdot 3\cdots(r-1)\sigma^r, & r>0 \text{ and even} \end{cases} \quad (21)$$

is the well-known expression for the rth central moment of a normal random variable [20, p. 24]. The moments can also be computed recursively from

$$\eta(r,\sigma) = \begin{cases} 1, & r=0 \\ 0, & r>0 \text{ and odd} \\ (r-1)\sigma^2\eta(r-2,\sigma), & r>0 \text{ and even.} \end{cases} \quad (22)$$

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An Exponentially Stable Adaptive Control for Force and Position Tracking of Robot Manipulators

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***Abstract*—The problem of controlling a robot manipulator while the end effector is in contact with an environment of finite but unknown stiffness is considered in this paper. An exponentially stable control law is derived starting from a passivity-based position control algorithm. The original position trajectory is scaled along the interaction direction so as to achieve force tracking as well as position tracking along the unconstrained directions. A passivity-based adaptive algorithm is designed to avoid the explicit computation of the scaling factor, which depends on the unknown stiffness of the environment, leading to time-varying PID control actions on the force error.**

***Index Terms*—Adaptive control, force control, manipulators, position control.**

I. Introduction

For the execution of robotic tasks requiring interaction with the environment it is necessary to control not only the position of a manipulator but also the force exerted at the contact. A controlled interaction with the environment can be sought by imposing a suitable dynamic behavior or impedance between contact force and manipulator end-effector position [1]. Explicit force feedback measurements are not strictly required in such a case, but a desired force cannot generally be specified. If force regulation or tracking is desired, then explicit control strategies making a proper use of force measurements must be adopted. Hybrid [2] and parallel [3] position/force control belong to this category.

For the design of force/position control algorithms the environment may be assumed either rigid or compliant. In the rigid case, kinematic constraints are imposed on the robot motion, and thus a desired position trajectory and a desired time-varying force can be realized by means of state feedback laws relying on the analytic description of the environment geometry [4]. When the environment is not rigid, its compliance characteristics must be properly taken into account. Asymptotic stability of a force/position regulator without exact knowledge of the environment stiffness has been proved [5] by using a PI-force plus PD-position scheme. It must be pointed out, however, that the transient behavior is uncertain as long as the stiffness of the environment is not known. Adaptive algorithms with

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