Differential and Algebraic Geometry in Signal Processing

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Outline

- What is Signal Processing?
- Part I: Algebraic Geometry in Signal Processing
- Part II: Differential Geometry in Signal Processing
- Part III: A Key Problem

Signal Processing

- Concerned with the processing of signals, i.e. a signal comes in, we do something to it, a signal (or estimate) goes out.
- Usually concerned with electrical or digital signals.
- Radar, mobile telephone, digital television...
- Hardware side, software side, algorithm design, fundamental theory.
- Intersects with statistics, linear algebra, linear operator theory...

Part I: Algebraic Geometry

- Wireless Communications
- Channel Identification
- Algebraic Geometry
- Results

Wireless Communications

- We have a message $x \in \mathbb{C}^n$ we wish to transmit.
- The transmitter converts it into an analogue waveform and transmits it.
- The receiver detects the analogue waveform, filters it to reduce noise, digitally samples it.
- Ideally, what we receive is $y = x + n$ where n models random noise.
- Due to multi-path though, in reality, what we receive is $y = h * x + n$ where $h \in \mathbb{C}^L$ is the "channel impulse response".
- Without any extra information, it is impossible to recover x from y.

Channel Identification (1)

- Given $y = h * x + n$, how can we find h (and subsequently x)?
- Statistical (blind) methods: assume elements of x are independent.
- Finite alphabet methods: assume elements of x belong to a finite set.
- Constant modulus methods: assume elements of x have unit norm.
- Oversampling methods: can convert SISO channel into SIMO.
- Training: Set certain elements of x to known values.

Channel Identification (2)

- Writing $y = h * x$ element-wise shows

$$y_1 = h_0 x_1 + h_1 x_0$$
$$y_2 = h_0 x_2 + h_1 x_1$$
$$y_3 = h_0 x_3 + h_1 x_2$$

- If some elements of x are known, we can add to the above equations the extra equations $x_0 = 0$, $x_5 = 0$, $x_{10} = 0$ for example.
- Whether or not we can identify the channel becomes a question of whether or not the above system of *polynomial equations* has a unique solution (generically).

Algebraic Geometry

- Algebraic geometry deals with polynomial equations. A fundamental object in algebraic geometry is a variety or algebraic set:

$$V = \{x \in \mathbb{C}^n | f_1(x) = \cdots = f_k(x) = 0\}$$

  where the f_i are polynomial maps.

- Of interest are polynomial maps between varieties.

- *Whenever polynomial equations arise in signal processing, we should be turning to algebraic geometry.*

Results

- If $L - 1$ zeros are inserted between blocks in a transmitted sequence then after receiving two blocks, a length L channel can be identified (up to an unknown scaling factor).
  - After receiving one block, there are the same number of equations as unknowns (after fixing an element of h); finite number of solutions generically.
  - After receiving the second block, generically there will be only one solution in common.
- If instead of transmitting x we transmit Px where P is a tall matrix, then as soon as P reaches a certain size, the channel can be identified generically.

Part II: Differential Geometry

- Weighted Low Rank Approximation
- Reformulation
- Manifolds, and Optimisation on Manifolds
- Where do Manifolds Appear in Signal Processing?
- Results
- Summary

Weighted Low Rank Approximation

- Given X, r compute $\arg\min_{R,\ \mathrm{rank}\, R \leq r} \|X - R\|_Q$ where $\|Z\|_Q = \mathrm{vec}\{Z\}^T Q\, \mathrm{vec}\{Z\}$.
- If norm is Frobenius norm, can use SVD.
- Otherwise, standard approach is to write $R = AB$ to enforce the rank constraint (A has r columns) and solve numerically $\arg\min_{A,B} \|X - AB\|_Q$.
- This is an over-parametrisation though; $(AG)(G^{-1}B) = AB$.
- In effect, B simply determines the null space of $R = AB$.

Reformulation

- $\min_{N,\ N^T N = I} \min_{R,\ RN=0} \|X - R\|_Q$ where N has the right number of columns to enforce the rank constraint $\operatorname{rank} R \leq r$.
- In fact, the outer minimisation is really over the Grassmann manifold and the inner minimisation is over the set of matrices R whose null space contains a particular subspace.
- The inner minimisation has a closed form solution.
- The parametrisation is of the minimal possible dimension.
- The SVD solution if the Frobenius norm is used falls out.

Manifolds, and Optimisation on Manifolds

- Whereas a variety is defined by the vanishing of polynomial equations, a set defined by the vanishing of smooth equations which satisfy an additional rank constraint on their Jacobian is a manifold.
- More generally, a smooth manifold is a space which locally looks like $\mathbb{R}^n$ and for which the concept of a smooth function can be defined.
- A variety with its singular points removed is a manifold.
- We are interested in computing $\arg\min_{p \in M} f(p)$ where $f : M \to \mathbb{R}$ is a smooth function.

Manifolds in Signal Processing

- Manifolds arise in signal processing in three ways:
  1. As a smooth constraint; it is known that the parameter x is constrained by $F(x) = 0$.
  2. By quotienting out ambiguity; there is not enough information to determine the parameter exactly but it can be determined up to an equivalence relation. Quotienting $\mathbb{R}^n$ or $\mathbb{R}^{n\times p}$ out by certain equivalence relations results in a manifold.
  3. Naturally; subspaces play a large role in signal processing, and the set of all subspaces of a certain dimension can be made into a manifold — the Grassmann manifold.

Results

- Ready-to-use algorithms for optimisation on the Grassmann and Stiefel manifolds.
- A general theory (with convergence proofs) for optimisation on manifolds.
- Explained why the traditional (Riemannian) approach is not always suitable. (Our theory includes the Riemannian approach as a special case.)
- Considered the more general problem of extending algorithms from Euclidean space to manifolds.
- We wish to investigate filtering and tracking on manifolds.

Summary

- If subspaces, ambiguity, or smooth constraints are involved, often the natural space to use is a manifold.
- Once a problem is formulated naturally, it is easier to come up with (better numerical) solutions to it.

Part III: A Key Problem

- Computers can only do additions, subtractions, multiplications, divisions and "if" statements. (Algebraic geometry!)
- The general signal processing problem is to take a vector x as input and return a vector y as output, where y is related to x according to some rule.
- The key question is, given an upper bound on the allowed complexity of the computation, what is the best "conditional polynomial approximation" to this rule?
- A fundamental example is the non-linear projection problem.

Non-Linear Projection

- Let $X \subset \mathbb{R}^n$ be a reasonable set (e.g. manifold or algebraic variety).
- Given $y \in \mathbb{R}^n$, compute $\arg\min_{x \in X} \|x - y\|$.
- We pose the problem: find polynomials $g_1, \cdots, g_k$ of bounded degree such that $\widehat{x} = \arg\min_{x \in \{g_1(y), \cdots, g_k(y)\}} \|x - y\|$ closely approximates the true rule $\widehat{x} = \arg\min_{x \in X} \|x - y\|$.
- For example, assume that $y = f(x) + n$ where n is noise. If it is the case that we can easily recover x from y if $n = 0$ (often f adds a lot of redundancy), then the main computational challenge is to compute the projection of y onto the image of f. Alternatively, we seek to approximate $\arg\min_x \|f(x) - y\|$ as above.

Conclusion

- Traditionally, signal processing problems were often solved by various linearisation techniques.
- Certain non-linear problems can be tackled directly using differential or algebraic geometry though.
- Not only do these areas of mathematics lead to better signal processing algorithms in some cases, but sometimes signal processing problems motivate new, theoretical questions to be asked in mathematics.