[10] ——, "On identifying codes," in *Codes and Association Schemes, Proceedings of the DIMACS Workshop on Codes and Association Schemes*, ser. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, A. Barg and S. Litsyn, Eds., Nov. 1999, vol. 56, pp. 97–109.
[11] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*. New York: Marcel Dekker, 1998.
[12] G. Cohen, I. Honkala, S. Litsyn, and A. Lobstein, *Covering Codes*. Amsterdam, The Netherlands: Elsevier, 1997.

The Convex Geometry of Subchannel Attenuation Coefficients in Linearly Precoded OFDM Systems

Jonathan H. Manton, *Member, IEEE*

***Abstract*—Channels with spectral nulls are sometimes dubbed bad channels because they can cause poor performance in communication systems. This correspondence investigates the validity of this intuition by studying the geometry of an orthogonal frequency-division multiplex (OFDM) system. It is shown that the subchannel attenuation coefficients form a natural coordinate system for describing finite-impulse response (FIR) channels in an OFDM framework. It is also shown that channels with spectral nulls are geometrically significant; they form the faces of the convex set of all subchannel attenuation coefficients. This novel perspective makes it immediately clear why the worst performance of a linearly precoded OFDM system is achieved over a channel having the greatest number of spectral nulls. The practical implications of these results are discussed in the correspondence.**

***Index Terms*—Channel zeros, convex analysis, linear precoder, orthogonal frequency-division multiplex (OFDM), spectral nulls, worst case performance.**

I. Introduction

The European digital audio broadcasting (DAB) standard is based on an orthogonal frequency-division multiplex (OFDM) framework [1], [6], [7], [11]. An OFDM framework is the first line of defense against distortions caused by wireless communication channels because it converts frequency-selective fading channels [5], [13] into a series of independent, frequency flat-fading subchannels. This limits the instantaneous data loss to only those symbols transmitted over subchannels currently experiencing deep fades. Since, in a broadcast environment, the frequencies at which these deep fades occur cannot be known by the transmitter, it is necessary to code each symbol over a number of subchannels. One way of doing this is to use a linear precoder [4], [15], and indeed, it is proved in [10] that a large class of linear precoders spread the spectrum of the source symbols in a well-defined manner.

The role of the linear precoder is to reduce the effects of fading. One way of measuring its effectiveness is now described. At any instant in time, the communication channel can be modeled by a finite impulse response (FIR) channel $\boldsymbol{h}$ [5], [13]. The performance of a linearly precoded OFDM system will, in general, depend on the channel $\boldsymbol{h}$. (A specific measure of performance is given later.) If $\boldsymbol{h}$ is such that a number of subchannels heavily attenuate the transmitted symbols then it may be expected that the performance is poor. (In fact, one aim of this correspondence is to investigate the validity of this intuition.) Similarly, it may be expected that there exists a favorable channel $\boldsymbol{h}$ resulting in exceptionally good performance. Thus, the effectiveness of a precoder can be measured by the ratio of its best-to-worst performances, where best and worst are taken over some feasible set of channels. Alternatively, the robustness of a precoder can be measured by its worst performance.

Therefore, in order to design or compare linear precoders, it is useful to understand the characteristics of channels over which linearly precoded OFDM systems achieve their best and worst performances. This correspondence presents a number of fundamental results contributing to this understanding. In doing so, the natural geometry of OFDM systems is also uncovered.

To facilitate further discussion, consider the following OFDM system [3], [14], [17], [6] which uses frames of length n and a cyclic prefix of length $L-1$, where L is the length of the FIR channel $\boldsymbol{h} \in \mathbb{C}^L$ over which the OFDM system operates. Due to the cyclic prefix, it is necessary for $n \geq L-1$. In the frequency domain, the received symbols $\boldsymbol{Y} \in \mathbb{C}^n$ are related to the transmitted symbols $\boldsymbol{X} \in \mathbb{C}^n$ by the equations

$$\boldsymbol{Y}_k = \boldsymbol{H}_k \boldsymbol{X}_k, \qquad k = 0, \ldots, n-1 \tag{1}$$

where $\boldsymbol{H}_k$ is the attenuation in the kth subchannel, and is given by

$$\boldsymbol{H}_k = \sum_{i=0}^{L-1} \boldsymbol{h}_i e^{-\jmath 2\pi \frac{ik}{n}}. \tag{2}$$

All vectors are indexed from zero. In a linearly precoded OFDM system, the transmitted symbols are a linear function of the p source symbols $\boldsymbol{s} \in \mathbb{C}^p$, that is,

$$\boldsymbol{X} = P\boldsymbol{s} \tag{3}$$

for some precoder matrix $P \in \mathbb{C}^{n \times p}$ with $n \geq p$.

If the received symbols $\boldsymbol{Y}$ are corrupted by additive white Gaussian noise with unit variance then the mean-square error (MSE) of estimating the source symbols $\boldsymbol{s}$ using the maximum-likelihood estimator, under the simplifying assumption that the receiver knows the channel perfectly, is given by

$$\operatorname{tr}\left\{(P^H \Lambda(\boldsymbol{h}) P)^{-1}\right\} \tag{4}$$

where $\operatorname{tr}\{\cdot\}$ is the trace operator, H denotes Hermitian transpose, and $\Lambda(\boldsymbol{h})$ is a diagonal matrix with kth diagonal element, indexed from zero, equal to

$$\Lambda_{kk}(\boldsymbol{h}) = |\boldsymbol{H}_k|^2. \tag{5}$$

The maximum-likelihood estimate of $\boldsymbol{s}$ has the smallest MSE (4) of any unbiased estimator. Thus, the MSE (4) is a sensible indication of the performance of a linearly precoded OFDM system. This correspondence proves that the worst performance of any given precoder P, defined to be

$$\sup_{\substack{\boldsymbol{h} \in \mathbb{C}^L \\ \|\boldsymbol{h}\|^2 = 1}} \operatorname{tr}\left\{(P^H \Lambda(\boldsymbol{h}) P)^{-1}\right\} \tag{6}$$

occurs when the channel $\boldsymbol{h}$ has $L-1$ spectral nulls (defined in Section II). Here, the constraint $\|\boldsymbol{h}\|^2 = \boldsymbol{h}^H \boldsymbol{h} = 1$ is a power constraint; without it, the worst performance occurs when $\boldsymbol{h} = [0, \ldots, 0]^T$. This result is derived by studying the geometry of the subchannel attenuation coefficients $\Lambda(\boldsymbol{h})$. It is proved that the set of all $\Lambda(\boldsymbol{h})$ with $\|\boldsymbol{h}\|^2 = 1$ is convex, and moreover, its extreme points [16] are a subset of the set of all $\Lambda(\boldsymbol{h})$ with the channel $\boldsymbol{h}$ having $L-1$ spectral nulls. It is

Manuscript received June 21, 2001; revised November 7, 2001. This work was performed while the author was a Tan Chin Tuan Exchange Fellow at Nanyang Technological University, Singapore.

The author is with the ARC Special Research Centre for Ultra-Broadband Information Networks, Department of Electrical and Electronic Engineering, The University of Melbourne, VIC 3010, Australia (e-mail: j.manton@ee.mu.oz.au).

Communicated by G. Caire, Associate Editor for Communications.
Publisher Item Identifier S 0018-9448(02)03091-2.

also proved that the MSE (4) is convex in Λ. The simplicity of this geometry suggests that the subchannel attenuation coefficients Λ, and not the channel coefficients $\boldsymbol{h}$, form a natural coordinate system (from a mathematical perspective[1]) in which to study certain aspects of OFDM systems.

The practical significance of these results is that they imply that globally convergent convex optimization techniques [16] can be used to determine the worst MSE (6) and the best MSE ((6) with $\inf$ replacing $\sup$) of any linear precoder. This enables the performance of any two precoders to be compared theoretically as opposed to via simulation. Moreover, it is hoped that the discovered geometry of the subchannel attenuation coefficients $\Lambda(\boldsymbol{h})$ will aid in the design of linear precoders as well as in developing novel channel identification algorithms.

It is believed that the results of this correspondence hold, at least qualitatively, for quite general communication systems. This is because the linear precoder can undo the inverse discrete Fourier transform operation performed by an OFDM system, implying that the only restriction in studying an OFDM system is the compulsory cyclic prefix. However, it is proved in [9] that a cyclic prefix is the most efficient way of ensuring that channels with unstable inverses can be inverted accurately, a consequence of the fact that a cyclic prefix ensures that the Cramer–Rao bound on the MSE of the estimate of the source symbols depends only on the magnitude Λ of the channel spectrum and not on its phase. In other words, it is highly desirable for the performance of a communication system to be a function of Λ only, in which case, the results of this correspondence hold. Viewed from this perspective, this correspondence gives a straightforward and general answer to the question, "Is it true that channel spectral nulls cause poor performance in communication systems, and if so, why?"

II. The Geometry of the Subchannel Attenuation Coefficients

This section studies the geometry [12] of the set

$$C = \left\{ \Lambda \in \mathbb{R}^{n\times n} \colon \Lambda = \Lambda(\boldsymbol{h}),\, \boldsymbol{h} \in \mathbb{C}^L,\, \|\boldsymbol{h}\|^2 = 1 \right\} \tag{7}$$

where $\Lambda(\boldsymbol{h})$ is defined in (5). It is proved that C is convex, and moreover, its extreme points belong to the set of all $\Lambda(\boldsymbol{h})$ with the channel $\boldsymbol{h}$ having $L-1$ spectral nulls (defined later in Definition 3).

The power spectrum $\Gamma(\omega)$ of a channel $\boldsymbol{h}$ is

$$\Gamma(\omega) = \left| \sum_{k=0}^{L-1} \boldsymbol{h}_k e^{-\jmath\omega k} \right|^2. \tag{8}$$

It is expedient to study first the geometry of the set

$$S = \left\{ \Gamma(\omega) \colon \Gamma(\omega) = \left| \sum_{k=0}^{L-1} \boldsymbol{h}_k e^{-\jmath\omega k} \right|^2,\, \boldsymbol{h} \in \mathbb{C}^L,\, \|\boldsymbol{h}\|^2 = 1 \right\}. \tag{9}$$

Studying S is equivalent to studying C as $n \to \infty$ because $\Lambda_{kk}(\boldsymbol{h}) = \Gamma(2\pi k/n)$. (Later, in (16) and (17), a stronger connection between S and C is made.)

Lemma 1: A function $\Gamma(\omega)$ is the power spectrum of an FIR channel of length at most L if and only if it is an element of the real vector space $\operatorname{span}\{1, \cos\omega, \cos 2\omega, \ldots, \cos(L-1)\omega, \sin\omega, \ldots, \sin(L-1)\omega\}$ and is everywhere nonnegative ($\forall\, \omega,\, \Gamma(\omega) \geq 0$).

Proof: This is a standard result [2]; one direction follows immediately by expanding (8) and the other is a consequence of the spectral representation theorem. □

Theorem 2: The set S, defined in (9), is a compact convex set whose convex dimension is $2L-2$.

[1]There is nothing novel about subchannel attenuation coefficients themselves since they are an inherent part of OFDM systems [3], [14].

Proof: Let $\Gamma_1, \Gamma_2 \in S$ and define $\Gamma = \alpha\Gamma_1 + (1-\alpha)\Gamma_2$ for some $0 < \alpha < 1$. Lemma 1 implies that Γ is the power spectrum of an FIR channel of length at most L. Parseval's theorem, which states that

$$\frac{1}{2\pi} \int_0^{2\pi} \Gamma(\omega)\, d\omega = \|\boldsymbol{h}\|^2 \tag{10}$$

implies that Γ has unit power. Thus, $\Gamma \in S$, proving that S is convex. Its dimension also follows from Lemma 1 and (10); the affine hull of elements of S is the set of trigonometric polynomials of degree $L-1$ with constant term equal to 1, which has dimension $2(L-1)$. Compactness follows from the readily verifiable fact that S, when viewed as a subset of the real vector space in Lemma 1, is closed and bounded. □

A face F of a convex set S is a convex subset of S satisfying the property that if $\Gamma_1, \Gamma_2 \in S$ and $\alpha\Gamma_1 + (1-\alpha)\Gamma_2 \in F$ for some $0 < \alpha < 1$, then both Γ_1 and Γ_2 are in F. The geometry of S is studied below by determining the smallest face about any point in S.

A key result of this section is that power spectra in S which have spectral nulls are geometrically significant. The concept of a spectral null is now made precise. It is based on the factorized form

$$\Gamma(\omega) = c \left| \prod_{i=1}^{L-1} \left(1 - r_i e^{\jmath\omega_i} e^{-\jmath\omega}\right) \right|^2 \tag{11}$$

of (8), where $c \geq 0$ and, for all i, $r_i \geq 0$ and $0 \leq \omega_i < 2\pi$.

Definition 3: The power spectrum $\Gamma(\omega)$ is said to have spectral nulls or zeros at $\omega_1, \omega_2, \ldots, \omega_k$ if it can be written in the form (11) with $r_1 = \cdots = r_k = 1$. This is denoted

$$\operatorname{zeros} \Gamma \supset (\omega_1, \ldots, \omega_k). \tag{12}$$

The number of zeros of Γ is defined to be the largest value of k for which (12) can be made to hold.

Remark: Definition 3 accounts for multiple zeros because it does not require the ω_i to be distinct.

Lemma 4: For any $\omega_1, \ldots, \omega_{L-1}$, define the sets

$$F_k(\omega_1, \ldots, \omega_k) = \{\Gamma \in S \colon \operatorname{zeros} \Gamma \supset (\omega_1, \ldots, \omega_k)\}, \quad k = 0, \ldots, L-1 \tag{13}$$

where $F_0 = S$. Then the F_k are faces, $F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_{L-1}$, and the dimension of F_k is $2(L-1-k)$. In particular, F_{L-1} consists of a single point.

Proof: The proof is a straightforward generalization of the proof of Theorem 2 based on the factorization

$$\Gamma(\omega) = c \left| \prod_{i=1}^{k} \left(1 - e^{\jmath\omega_i} e^{-\jmath\omega}\right) \right|^2 \left| \prod_{i=1}^{L-k-1} \left(1 - r_i e^{\jmath\omega_i'} e^{-\jmath\omega}\right) \right|^2 \tag{14}$$

of any $\Gamma \in F_k$. (Expand the second term of (14) to obtain a trigonometric polynomial as in Lemma 1. Then follow the steps in the proof of Theorem 2.) □

Remark: The reason why F_k has dimension $2(L-1-k)$ is that each zero removes two degrees of freedom, namely, an r_i and an ω_i, from the choice of power spectrum (11).

Theorem 5: Let $\Gamma \in S$ be a point in S, defined in (9). Let $\omega_1, \ldots, \omega_k$ be all the zeros of Γ (see Definition 3). Then $F_k(\omega_1, \ldots, \omega_k)$, defined in (13), is the smallest face of S containing Γ.

Proof: It must be shown that Γ belongs to the relative interior of F_k [16, Prop. 1.19]. That is to say, it must be shown that for any affine combination $\overline{\Gamma} = \lambda_1\Gamma_1 + \cdots + \lambda_m\Gamma_m$ with $\Gamma_i \in F_k$ and $\lambda_1 + \cdots + \lambda_m = 1$, there exists an $\epsilon > 0$ such that $\Gamma + \epsilon\left(\overline{\Gamma} - \Gamma\right) \in F_k$. Lemma 1 implies that it suffices to show $\exists\epsilon > 0$ such that

$$\forall\,\omega, \quad \Gamma(\omega) + \epsilon\left(\overline{\Gamma}(\omega) - \Gamma(\omega)\right) \geq 0. \tag{15}$$

For ω away from the zeros $\omega_1, \ldots, \omega_k$, it is straightforward to find such an ϵ. (Let $K \subset [0, 2\pi]$ be any compact set excluding the zeros and choose

$$\epsilon = [\min_{\omega \in K} \Gamma(\omega)]/[\max_{\omega \in K} |\Gamma(\omega) - \overline{\Gamma}(\omega)|]$$

if the denominator is nonzero and $\epsilon = 1$ otherwise.) Around a zero, it follows[2] from Lemma 1 and (11) that

$$\Gamma(\omega) = \beta(\omega - \omega_i)^{2d_i} + O((\omega - \omega_i)^{2d_i+1})$$

for some $\beta > 0$, where d_i is the multiplicity of ω_i (that is, d_i equals the number of ω_j equaling ω_i for $j = 1, \ldots, k$). Similarly,

$$\overline{\Gamma}(\omega) = \overline{\beta}(\omega - \omega_i)^{2\overline{d}_i} + O((\omega - \omega_i)^{2\overline{d}_i+1})$$

for some $\overline{\beta} > 0$ and $\overline{d}_i \geq d_i$. Thus, around each zero, it is possible to find an ϵ sufficiently small to ensure (15) holds. $\square$

An important corollary is that the extreme points of S, defined to be the faces consisting of a single point, correspond to power spectrums having $L-1$ spectral nulls.

Corollary 6: A point $\Gamma \in S$ is an extreme point of S if and only if Γ has $L-1$ spectral nulls.

Proof: Lemma 4 and Theorem 5 together imply that the only faces of S consisting of a single point are the faces $F_{L-1}(\omega_1, \ldots, \omega_{L-1})$ for arbitrary $\omega_1, \ldots, \omega_{L-1}$. $\square$

The geometry of C is deduced from the geometry of S by considering the projection operator $\mathcal{P}_n$ defined as

$$\mathcal{P}_n\Gamma(\omega) = \begin{cases} \Gamma(\omega), & \text{if } \omega = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2\pi(n-1)}{n} \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

The geometry of $\mathcal{P}_nS$ is identical to that of C because the map ψ_n, defined by

$$\psi_n(\Gamma(\omega)) = \operatorname{diag}\{\Gamma(0), \Gamma(2\pi/n), \ldots, \Gamma(2\pi(n-1)/n)\} \tag{17}$$

is a vector space isomorphism from the range space of $\mathcal{P}_n$ to the space of all $n \times n$ diagonal matrices.

Theorem 7: The set C in (7) is convex and compact.

Proof: Since S in Theorem 2 is convex and compact, and $\mathcal{P}_n$ in (16) is a projection, $\mathcal{P}_nS$ is convex and compact. $\square$

It is a consequence of the sampling theorem that $\mathcal{P}_n$ is one-to-one if and only if $n \geq 2L-1$. Thus, the geometry of C is identical to that of S if and only if $n \geq 2L-1$. However, it turns out that under the milder restriction $n \geq L-1$, the extreme points of $\mathcal{P}_nS$ are a subset of the extreme points of S.

Lemma 8: Define $C, S, \mathcal{P}_n$, and ψ_n as above. Let F_Γ and F_Λ denote the smallest faces about the points $\Gamma \in S$ and $\Lambda \in C$, respectively. Then $\Lambda = \psi_n(\mathcal{P}_n\Gamma)$ implies $\psi_n(\mathcal{P}_nF_\Gamma) \subset F_\Lambda$.

Proof: This follows from standard properties of projections [16]. $\square$

Theorem 9: For any $n \geq L-1$ and $\boldsymbol{h} \in \mathbb{C}^L$, a necessary condition for $\Lambda(\boldsymbol{h})$, defined in (5), to be an extreme point of C, defined in (7), is for the power spectrum $\Gamma(\omega)$ of $\boldsymbol{h}$, defined in (8), to have $L-1$ spectral nulls (Definition 3).

Proof: Assume to the contrary that there exists an $\boldsymbol{h}$ such that $\Lambda(\boldsymbol{h})$ is an extreme point of C but that $\Gamma(\omega)$ does not have $L-1$ zeros. Then, by Lemma 8, $\mathcal{P}_nF_\Gamma$ consists of a single point. From Lemma 4 and Theorem 5, there exists an F_{L-2} such that $F_{L-2} \subset F_\Gamma$. It will be shown that $\mathcal{P}_nF_{L-2}$ contains an element other than Λ, proving the theorem by contradiction. Since Γ has less than $L-1$ zeros, there exists an i such that Λ_{ii} is nonzero. Then $\Gamma_1(\omega) = (1 - e^{j2\pi i/n}e^{-j\omega})\Gamma(\omega)$, after scaling to ensure unit power, also lies in F_{L-2} but is such that the ith diagonal element of $\psi_n(\mathcal{P}_n\Gamma_1)$ is zero. Thus, $\psi_n(\mathcal{P}_n\Gamma_1) \neq \Lambda$, as required. $\square$

Remark: The proof of Theorem 9 shows that, provided $n \geq L-1$, a necessary and sufficient condition for Λ to be an extreme point of C is for its pre-image $\{\Gamma \in S: \psi_n(\mathcal{P}_n\Gamma) = \Lambda\}$ to be an extreme point of S.

III. The Convexity of the MSE

Determining the worst performance (6) of a linearly precoded OFDM system appears to be difficult because the MSE (4), as a function of the channel $\boldsymbol{h}$, has no nice properties. This section shows how this difficulty is overcome by treating the MSE (4) as a function of Λ instead. Indeed, it is proved that (4) is convex in Λ.

Theorem 10: Let C be any convex subset of the set of all positive semidefinite diagonal $n \times n$ matrices. Then the MSE (4), when viewed as a function of Λ over the domain C, is convex.

Proof: For any $\Lambda_1, \Lambda_2 \in C$, define

$$f(\alpha) = \operatorname{tr}\left\{(P^H(\alpha\Lambda_1 + (1-\alpha)\Lambda_2)P)^{-1}\right\}.$$

It must be proved that

$$\forall\, 0 < \alpha < 1, \qquad f(\alpha) \leq \alpha f(1) + (1-\alpha)f(0). \tag{18}$$

If either $f(0)$ or $f(1)$ is infinite, (18) holds by convention [16, Sec. 2.1]. Assuming for the moment that the inverse exists, define $Z = (P^H(\alpha\Lambda_1 + (1-\alpha)\Lambda_2)P)^{-1}$ and note that Z is positive definite and Hermitian for $0 \leq \alpha \leq 1$. Then (see [8] for matrix differential calculus rules)

$$\begin{aligned} \frac{1}{2}\frac{d^2f(\alpha)}{d\alpha^2} &= \operatorname{tr}\left\{ZP^H(\Lambda_1 - \Lambda_2)PZP^H(\Lambda_1 - \Lambda_2)PZ\right\} && (19) \\ &= \operatorname{tr}\left\{ZP^H\Lambda_1PZP^H\Lambda_1PZ - 2ZP^H\Lambda_1PZP^H\Lambda_2PZ\right. \\ &\qquad \left. + ZP^H\Lambda_2PZP^H\Lambda_2PZ\right\} && (20) \\ &= \operatorname{tr}\left\{((A-B)Z)^H((A-B)Z)\right\} && (21) \\ &\geq 0 && (22) \end{aligned}$$

where $A = Z^{\frac{1}{2}}P^H\Lambda_1P$, $B = Z^{\frac{1}{2}}P^H\Lambda_2P$, and $Z^{\frac{1}{2}}$ is any matrix such that $(Z^{\frac{1}{2}})^HZ^{\frac{1}{2}} = Z$. This proves not only that $f(0)$ and $f(1)$ being finite implies that Z is well-defined[3] for $0 \leq \alpha \leq 1$, but that (18) always holds. $\square$

The practical significance of Theorems 7 and 10 is that the worst performance (6) of a linearly precoded OFDM system can be found by standard convex optimization routines [16]. (The same holds for the best performance.)

The theoretical significance is more profound because the resulting elegant geometry suggests that, for an OFDM system, the natural coordinate system for describing an FIR channel is based on the subchannel attenuation coefficients (5) and not the impulse response $\boldsymbol{h}$.

IV. On the Worst Channel

Channel spectral nulls, defined in Definition 3, are often considered undesirable because they can cause poor performance in communication systems. This section confirms this intuition by proving that the worst performance of any given linearly precoded OFDM system, defined in (6), is achieved by a channel having $L-1$ spectral nulls.

[2] Specifically, Lemma 1 shows that $\Gamma(\omega)$ is always nonnegative and has a well-defined Taylor series about any point. Therefore, the first nonzero term in the Taylor series expansion about a point ω_0 at which $\Gamma(\omega_0) = 0$ must be an even power of $(\omega - \omega_0)$. Moreover, expanding (11) as a trigonometric polynomial shows that the degree of this first term is twice the multiplicity of the zero.

[3] The inverse used to define Z always exists at a point α unless $f(\alpha) = \infty$. Since $f(0)$ and $f(1)$ are finite and (22) holds, $f(\alpha)$ cannot diverge on the interval $[0, 1]$ of interest.

Theorem 11: For any given precoder $P \in \mathbb{C}^{n \times p}$ and channel length L, where $n \geq L-1$, there exists a channel $\boldsymbol{h} \in \mathbb{C}^L$ having $L-1$ spectral nulls (Definition 3) and such that

$$\operatorname{tr}\left\{(P^H \Lambda(\boldsymbol{h}) P)^{-1}\right\} = \sup_{\substack{\boldsymbol{h}' \in \mathbb{C}^L \\ \|\boldsymbol{h}'\|^2 = 1}} \operatorname{tr}\left\{(P^H \Lambda(\boldsymbol{h}') P)^{-1}\right\}.$$

Proof: This immediately follows from the convexity of $\operatorname{tr}\{(P^H \Lambda P)^{-1}\}$ (Theorem 10) and the fact that the extreme points of C all have $L-1$ spectral nulls (Theorem 9). □

The proof of Theorem 11 holds for any convex performance measure, and not just the specific choice (4). Thus, the undesirability of spectral nulls may be expressed by saying that any convex measure of performance of a linearly precoded OFDM system achieves its maximum for some channel having as many spectral nulls as possible. (A length-L FIR channel can have at most $L-1$ spectral nulls.)

Remark: By generating random 3×2 precoder matrices and determining their worst performances (6) over a length $L = 2$ channel, it was observed that the spectral null of the worst channel for any given precoder was not necessarily located at 0, $2\pi/3$, or $4\pi/3$ (corresponding to Λ_{00}, Λ_{11}, or Λ_{22} in (5) being zero). It was, however, always observed to lie close to one of these three angular frequencies.

V. Conclusion

This correspondence explained why channels with spectral nulls are often associated with poor performance; channels with spectral nulls form the faces of the convex set of all subchannel attenuation coefficients. Therefore, any performance measure which is a convex function of the subchannel attenuation coefficients—such as the MSE of the maximum-likelihood estimate of the source symbols—achieves its maximum for some channel having the most spectral nulls.

The second main contribution of this correspondence was to describe completely the geometry of the subchannel attenuation coefficients. It was contended that the subchannel attenuation coefficients form a natural coordinate system in which to study certain aspects of linearly precoded OFDM systems. It is therefore expected that an understanding of the geometry of the subchannel attenuation coefficients will prove useful in other areas of research.

References

[1] M. Alard and R. Lassalle, "Principles of modulation and channel coding for digital broadcasting for mobile receivers," *EBU Tech. Rev.*, vol. 224, pp. 168–190, Aug. 1987.
[2] K. J. Astrom, *Introduction to Stochastic Control Theory*. New York: Academic, 1970.
[3] R. W. Chang, "Synthesis of band-limited orthogonal signals for multichannel data transmission," *Bell Syst. Tech. J.*, Dec. 1966.
[4] G. B. Giannakis, "Filterbanks for blind channel identification and equalization," *IEEE Signal Processing Lett.*, vol. 4, no. 6, pp. 184–187, June 1997.
[5] W. C. Jakes, Ed., *Microwave Mobile Communications*. New York: Wiley, 1974.
[6] B. Le Floch, M. Alard, and C. Berrou, "Coded orthogonal frequency division multiplex," *Proc. IEEE*, vol. 83, pp. 982–996, June 1995.
[7] B. Le Floch, R. H. Lassalle, and D. Castelain, "Digital sound broadcasting to mobile receivers," *IEEE Trans. Consumer Electron.*, vol. 35, Aug. 1989.
[8] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus With Applications in Statistics and Econometrics*. New York: Wiley, 1994.
[9] J. H. Manton, "Dissecting OFDM: The independent roles of the cyclic prefix and the IDFT operation," *IEEE Commun. Lett.*, vol. 5, pp. 474–476, Nov. 2001.
[10] J. H. Manton and Y. Hua, "A frequency domain deterministic approach to channel identification," *IEEE Signal Processing Lett.*, vol. 6, pp. 323–326, Dec. 1999.
[11] F. Mueller-Roemer, "Directions in audio broadcasting," *J. Audio Eng. Soc.*, vol. 41, no. 3, pp. 158–173, Mar. 1993.
[12] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton Univ. Press, 1970.
[13] W. D. Rummler, R. P. Coutts, and M. Liniger, "Multipath fading channel models for microwave digital radio," *IEEE Commun. Mag.*, vol. 24, pp. 30–42, Nov. 1986.
[14] B. R. Saltzberg, "Performance of an efficient parallel data transmission system," *IEEE Trans. Commun.*, vol. COM-15, pp. 805–811, Dec. 1967.
[15] A. Scaglione, G. B. Giannakis, and S. Barbarossa, "Redundant filterbank precoders and equalizers—Part I: Unification and optimal designs," *IEEE Trans. Signal Processing*, vol. 47, pp. 1988–2006, July 1999.
[16] H. Tuy, *Convex Analysis and Global Optimization*. Norwell, MA: Kluwer, 1998.
[17] S. B. Weinstein and P. M. Ebert, "Data transmission by frequency division multiplexing using the discrete Fourier transform," *IEEE Trans. Commun.*, vol. COM-19, pp. 628–634, Oct. 1971.

Blind Equalization of MIMO–FIR Channels Driven by White but Higher Order Colored Source Signals

Ruey-wen Liu, *Life Fellow, IEEE*, and Yujiro Inouye, *Member, IEEE*

***Abstract*—The goal of the present correspondence is to find a minimal amount of statistical information (or a much weaker condition) about source signals for which blind equalization is possible for multiple-input multiple-output finite-impulse response (MIMO-FIR) channels. First, a sufficiently broad framework is set up within which such a theoretical problem is well posed. Within this framework, it is shown that second-order statistics (SOS) alone are not sufficient for blind equalization when the source signals are white. Additional higher order statistics (HOS) are needed. Then we show that the only additional higher order statistical information needed is spatial fourth-order cumulants. Though it has not yet been proved to be minimal, it is interesting to note that this is the same as the weakest known condition on the source signals even for an instantaneous mixture. We then show a necessary and sufficient condition for blind equalization when the source signals are white and spatially fourth-order uncorrelated. Based on this condition, criterion (A) for blind equalization of MIMO-FIR channels is developed by exploiting the temporal fourth-order statistics. Finally, based on this criterion, a new necessary and sufficient condition in terms of cumulants for the blind equalization of MIMO-FIR channels is obtained.**

***Index Terms*—Blind, convolutive mixtures, deconvolution, direct equalization, higher order statistics (HOS), multiple-input multiple-output (MIMO), source separation.**

I. Introduction

The subject of blind signal separation was started in 1985 by Herault, Jutten, and Ans [1]. In the late 1980s and early 1990s, the efforts have been on the case of instantaneous mixtures. See [2] and references

Manuscript received October 1, 1998; revised November 1, 2001. The material in this correspondence was presented in part at ASSP'99, Phoenix, AZ, May 1999, and at ISCAS'99, Orlando, FL, May 1999.

R.-w. Liu is with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA (e-mail: liu.1@nd.edu).

Y. Inouye is with the Department of Electronic and Control Systems Engineering, Shimane University, 1060 Nishikawatsu, Matsue, Shimane 690-8504, Japan (e-mail: inouye@riko.shimane-u.ac.jp).

Communicated by U. Madhow, Associate Editor for Detection and Estimation.

Publisher Item Identifier S 0018-9448(02)02798-0.