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Generalized logarithmic law for high-order moments in turbulent boundary layers

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High-Reynolds-number data in turbulent boundary layers are analysed to examine statistical moments of streamwise velocity fluctuations u'. Prior work has shown that the variance of u' exhibits logarithmic behaviour with distance to the surface, within an inertial sublayer. Here we extend these observations to even-order moments. We show that the $2p$-order moments, raised to the power $1/p$, also follow logarithmic behaviour according to $\langle (u'^{+})^{2p}\rangle^{1/p} = B_p - A_p \ln(z/\delta)$, where u'^{+} is the velocity fluctuation normalized by the friction velocity, δ is an outer length scale and B_p are non-universal constants. The slopes A_p in the logarithmic region appear quite insensitive to Reynolds number, consistent with universal behaviour for wall-bounded flows. The slopes differ from predictions that assume Gaussian statistics, and instead are consistent with sub-Gaussian behaviour.

Key words: turbulent flows, turbulent boundary layers

1. Introduction

High-Reynolds-number boundary layers are one of the most practically relevant turbulent flows since they occur whenever inertia-dominated flows interact with solid boundaries. Applications range from airflow in the vicinity of aircraft wings to turbulent flow in wind farms. The most well-known universal feature of such flows is the logarithmic law (Prandtl 1925; von Kármán 1930) for the mean velocity profile as a function of distance to the wall (z) in the inertial region,

$$\langle u\rangle/u_\tau \equiv \langle u^+\rangle = \kappa^{-1}\ln(zu_\tau/\nu) + B, \tag{1.1}$$

for which there is significant empirical evidence (see recent reviews of Smits, McKeon & Marusic (2011) and Jimenez (2012)). Here, u_τ is the friction velocity based on the wall stress τ_w according to $u_\tau = \sqrt{\tau_w/\rho}$ (ρ is the fluid density), ν is the kinematic viscosity, κ is the von Kármán constant, and B is another constant. The existence of such a basic property of wall-bounded turbulent flows has proven to be immensely

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useful to provide predictions in many applications, as well as to calibrate model parameters, test simulations and guide theoretical understanding.

More recently, and motivated by model predictions based on the 'attached eddy hypothesis' by Townsend (1976), Perry & Chong (1982) and Perry, Henbest & Chong (1986), a logarithmic behaviour has also been observed in the inertial region for the variance of the fluctuations in the streamwise velocity component. Specifically, there has been growing evidence in recent years (Marusic & Kunkel 2003; Hultmark *et al.* 2012; Marusic *et al.* 2013) for a universal law of the form

$$\langle (u'^{+})^2 \rangle = B_1 - A_1 \ln(z/\delta), \tag{1.2}$$

where $u'^{+} = (u - \langle u \rangle)/u_\tau$ is the normalized streamwise velocity fluctuation and δ is an outer length scale, which for boundary layers is the boundary-layer thickness, but for pipes can be their radius, or the height of a channel. Recently, Hultmark (2012) showed that a logarithmic profile could also be obtained from a matching procedure for differentiated profiles of the variance within an overlap region. The prior data are consistent with a value (the Townsend–Perry constant) of $A_1 \approx 1.25$ (Marusic & Kunkel 2003; Smits *et al.* 2011; Hultmark *et al.* 2012; Marusic 2012; Marusic *et al.* 2013), whereas B_1 is dependent upon flow conditions and geometry, and is thus not expected to be universal.

Equation (1.2) is consistent with the attached eddy hypothesis in which the length scales of eddies are proportional to the distance to the wall and their population density is inversely proportional to distance. A simple heuristic argument is to say that the streamwise velocity fluctuations in the inertial layer at some distance z from the wall consist of the sum of independent velocity increments, each associated with the occurrence of a wall-attached eddy. At a distance z, the number of such summands is proportional to $\sim\ln(\delta/z)$, and if the summands are statistically independent (e.g. if the eddies are non-interacting) the variance will also be proportional to the number of summands, and hence decrease with wall distance according to $\langle (u'^{+})^2 \rangle \propto -\ln(z/\delta)$. As the wall is approached, more and more velocity increments must be added, each of them having the same characteristic velocity scale proportional to u_τ, and thus the variance increases.

Another consequence of statistical independence among the summands, from the central limit theorem, is tendency for Gaussian behaviour and for even moments to behave according to $\langle (u'^{+})^{2p} \rangle \to (2p-1)!!\langle (u'^{+})^2 \rangle^p$ (where $n!! \equiv n(n-2)(n-4)\ldots 1$ is the double factorial). This expectation suggests that the pth root of moments of velocity fluctuations follows

$$\langle (u'^{+})^{2p} \rangle^{1/p} = B_p - A_p \ln(z/\delta) = D_p(Re_\tau) - A_p \ln z^{+}, \tag{1.3}$$

i.e. a generalized logarithmic law for high-order moments. When expressed in terms of distance to the wall in viscous units $z^{+} = z/z_\nu = z u_\tau/\nu$, logarithmic behaviour would also be expected, with the same slope $-A_p$ but with a Reynolds-number-dependent offset $D_p = B_p + A_p \ln Re_\tau$, where $Re_\tau = u_\tau \delta/\nu$. For Gaussian statistics, asymptotically one expects $A_p = A_1\,[(2p-1)!!]^{1/p}$.

Recently, the logarithmic scaling of the variance was used by Pullin, Inoue & Saito (2013) to examine the turbulence intensity (the ratio of the root-mean-square (r.m.s.) to the mean velocity) of turbulent boundary layers at very large Reynolds numbers. It was concluded that a probable consequence of the log law for mean and variance is that turbulence intensity tends to zero at asymptotically high Reynolds number. In fact, examining higher-order moments enables one to make such analysis stronger, in

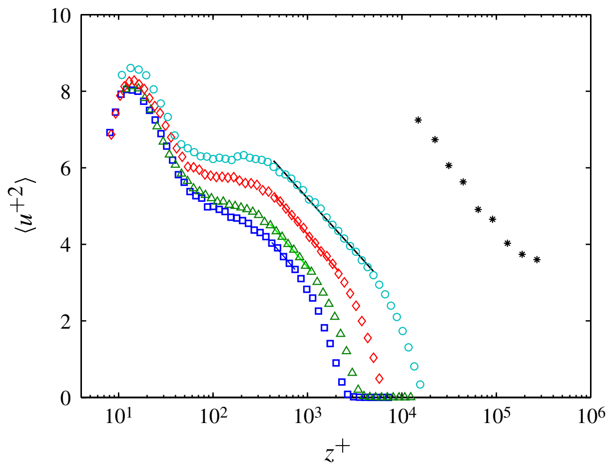

FIGURE 1. Variance of streamwise velocity in turbulent boundary layers at several Reynolds numbers $Re_\tau = 2800$ (□), 3900 (△), 7300 (◇) and 19 030 (○), respectively, and SLTEST data ($*$). The solid lines indicate fits to the logarithmic region, performed in the range $z^+ > 400$ and $z/\delta < 0.3$.

the sense that one may even examine the limit of the *maximum* individual fluctuation, since $u'^2|_{max} = \lim_{p\to\infty} \langle [(u')^2]^p \rangle^{1/p}$.

In this paper, we use experimental data in turbulent boundary layers at various high Reynolds numbers to determine whether there is empirical evidence for a generalized logarithmic law such as (1.3), and whether the prediction from Gaussian statistics holds.

2. Description of data sets

For our analysis, we use the datasets from Hutchins *et al.* (2009) at Reynolds numbers $Re_\tau = 2800$, 3900, 7300 and 19 030 measured in the large Melbourne wind tunnel (HRNBLWT) with a 2.5 µm diameter hot wire, and from Hutchins *et al.* (2012) for measurements at the SLTEST site in Utah's Western Desert using a wall-normal array of nine sonic anemometers under nominally neutrally buoyant conditions. As a baseline reference, figure 1 reproduces known results for the streamwise velocity variance as a function of height for the four laboratory Reynolds numbers, as well as for the SLTEST data at much higher Reynolds number.

The familiar logarithmic region can be discerned, and solid lines show fits to the logarithmic law (Hultmark *et al.* 2012; Marusic *et al.* 2013) $\langle u^{+2} \rangle = B_1 - A_1 \ln(z/\delta)$ for each case. For the current Reynolds numbers, the range $z^+ > 400$ and $z/\delta < 0.3$ is chosen to perform the fits. However, we note that, for the lower-Reynolds-number datasets, it would appear that one could also have chosen a lower z^+ limit. This would be consistent with a Reynolds-number dependence in the lower limit of the logarithmic layer, such as the $\sim 3Re_\tau^{1/2}$ dependence that has been reported previously (Klewicki, Fife & Wei 2009; Alfredsson, Segalini & Örlü 2011; Marusic *et al.* 2013). For the present analysis, we have opted to use the simpler and more 'conservative' (shorter) range with a fixed $z^+ > 400$. The resulting fits would be almost unchanged

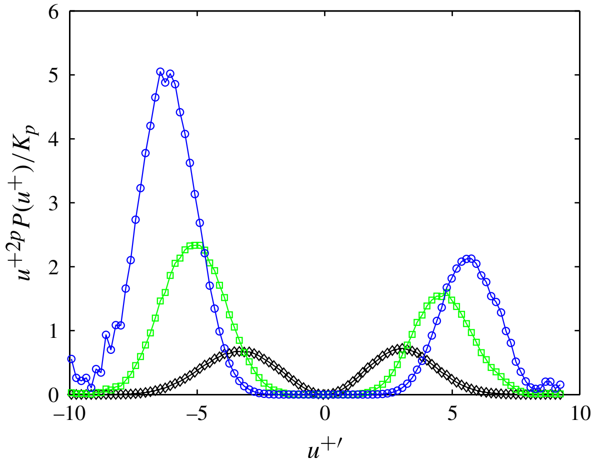

FIGURE 2. Premultiplied p.d.f. of normalized velocity fluctuations $(u'^+)^{2p} P(u^{+\prime})$ for the $Re_\delta = 19\,030$ dataset, at a height of $z/\delta = 0.08$. Different moments are represented as $2p = 2$ ($\diamond$), $2p = 6$ ($\square$) and $2p = 10$ ($\bigcirc$). Curves are divided by an (arbitrary) factor K_p for clarity.

in the present range of Reynolds numbers. For $Re_\tau = 19\,030$, the fitted constants are $B_1 = 1.71$ and $A_1 = 1.19$. For the other three lower Reynolds numbers, $Re_\tau = 2800$, 3900 and 7300, the fitted values are $(A_1, B_1) = (1.25, 1.68)$, (1.23, 1.73) and (1.27, 1.72), respectively. These are shown as solid lines in the figure. We thus confirm that A_1 and B_1 appear to be (approximately) independent of Reynolds number, with a nearly universal slope of $A_1 \approx 1.25$, with an accuracy of $\sim$5 %. Owing to the scatter in the SLTEST dataset, we do not perform fits through the data, but show it to illustrate that the logarithmic trend and slope appear to be fairly consistent with those of the laboratory data. The value $A_1 = 1.25$ is essentially the same as the values reported by Hultmark *et al.* (2012) and Marusic *et al.* (2013) (the latter including similar datasets, as well as others).

Next, we evaluate higher-order moments of the velocity fluctuations as a function of height. Before proceeding, the convergence of high-order moments is an issue that needs to be verified with care. Here we evaluate the pre-multiplied velocity fluctuation probability density function (p.d.f.) in order to determine whether the moments can be considered statistically converged. As a representative case, we consider the high-Reynolds-number case $Re_\tau = 19\,030$ from the laboratory data, and select data at a height $z/\delta = 0.08$ ($z^+ = 1520$), near the core of the logarithmic region that will be focused on in the subsequent analysis. Premultiplied p.d.f.s for $2p = 2$, 6 and 10 are shown in figure 2. They show acceptable 'closure' of the premultiplied p.d.f. in the sense that the moment, which is the area under the curve, is well captured by the amount of data available, at least up to the highest order $2p = 10$ considered here. Similar analysis for the SLTEST dataset at $z^+ \approx 4 \times 10^4$ (not shown) confirms that moments up to $2p = 6$ can be considered well converged, with a 10 % statistical convergence error estimate for the moment $\langle (u'^+)^6 \rangle$, leading to about a 3 % error estimate for our quantity of interest, $\langle (u'^+)^6 \rangle^{1/3}$. For the $2p = 10$ case, the error estimate is close to 50 % for $\langle (u'^+)^{10} \rangle$, and thus of 10 % for $\langle (u'^+)^{10} \rangle^{1/5}$.

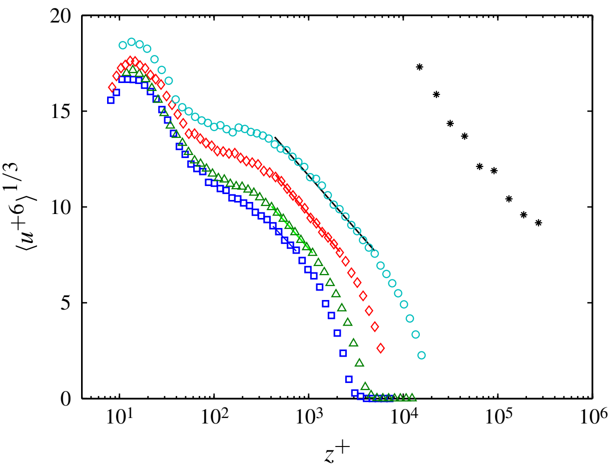

FIGURE 3. Moments of order $2p = 6$ of streamwise velocity as a function of wall-normal distance, for several different Reynolds numbers (symbols as in figure 1). The lines show the fits in the range $z^+ > 400$ and $z/\delta < 0.3$, with slopes denoted by A_3. Since the experimental uncertainty of the variance of u^+ is of the order of 4 % (Hutchins *et al.* 2009; Ng *et al.* 2011), it can be estimated that the uncertainty on the sixth-order moments is of the order of $(6/2) \times 4 = 12\,\%$ (but on the cube root of the moment, the uncertainty is again of the order of 4 %, and is not indicated in the figure in order to avoid clutter).

3. Results

Higher-order moments for $2p = 4$, 6, 8 and 10 are evaluated as a function of wall distance for the different Reynolds-number data sets. Results are shown in figure 3 for $2p = 6$. The results are consistent with a 'generalized logarithmic law' for the sixth-order moments, in the same inertial layer in which the variance displayed logarithmic behaviour. Figure 4 shows the results for the highest laboratory Reynolds-number data considered, for all the even-order moments up to $2p = 10$. The results are consistent with a 'generalized logarithmic law' within the inertial layer for all the moments considered.

The next question is whether for a given order there is a continuing dependence on Re_τ even at high Re_τ, or whether the slope A_p asymptotes to a certain value that becomes independent of Re_τ, at large Re_τ. To that end, we perform fits of (1.3) to the data within the same portion of the logarithmic regions described above, namely $z^+ > 400$ and $z/\delta < 0.3$. Results of the fits are shown in figures 1, 3 and 4 as solid lines. Thus we determine the slopes A_p for each of the Reynolds-number cases for the laboratory data. The results are plotted in figure 5 as symbols. The dashed line shows the prediction based on Gaussian statistics $A_p = A_1\,[(2p-1)!!]^{1/p}$, and appears (almost) linear due to the asymptotic behaviour of the double factorial function ($n!! \sim n^{(n+1)/2}\mathrm{e}^{-n/2}$ and so $(n!!)^{2/n} \sim n$).

It appears that A_p tends to values that fall below the Gaussian values (i.e. displaying 'sub-Gaussian statistics'), once high Reynolds numbers are reached. Further evidence of Reynolds-number independence is that the slope of the SLTEST data, as seen from the asterisks in figure 3, is quite consistent with the slope of the largest Reynolds

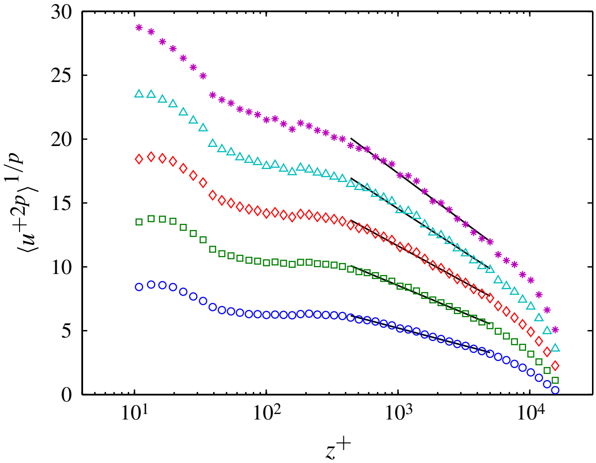

FIGURE 4. Moments of order $2p = 2$ (○), 4 (□), 6 (◇), 8 (△) and 10 ($*$) of streamwise velocity fluctuation as a function of wall-normal distance z^+, for the Reynolds-number case $Re_\tau = 19\,030$. The lines show the fits in the range $z^+ > 400$ and $z/\delta < 0.3$, with (negative) slopes A_p.

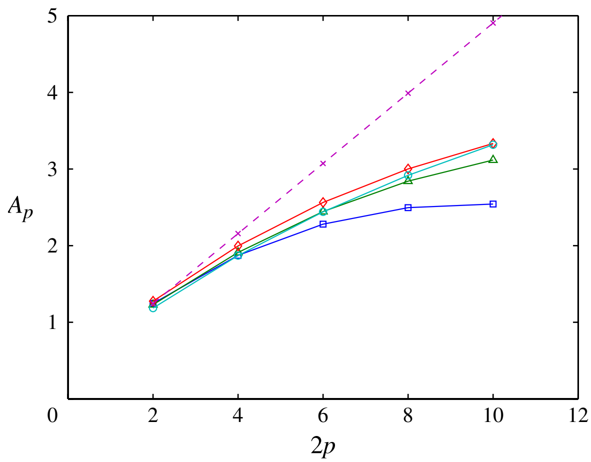

FIGURE 5. Coefficients A_p in logarithmic law for moments as a function of moment order $2p$, for various Reynolds numbers $Re_\tau = 2800$ (□), 3900 (△), 7300 (◇) and 19 030 (○). Given the ∼4 % uncertainty in the scaled moments, the uncertainty in the determined slopes A_p can be considered to be less than 10 %. The crosses and dashed line show the results expected for Gaussian statistics, $A_p = A_1\,[(2p-1)!!]^{1/p}$.

number in the laboratory data. It thus appears that the sub-Gaussian scaling of moments does not become more Gaussian at increasing Reynolds numbers.

To further explore the Reynolds-number dependences of sub-Gaussian statistics, we plot in figure 6 the flatness $F_4(z/\delta)$ in outer units. These data agree well with previously reported flatness results in zero-pressure-gradient boundary layers, as

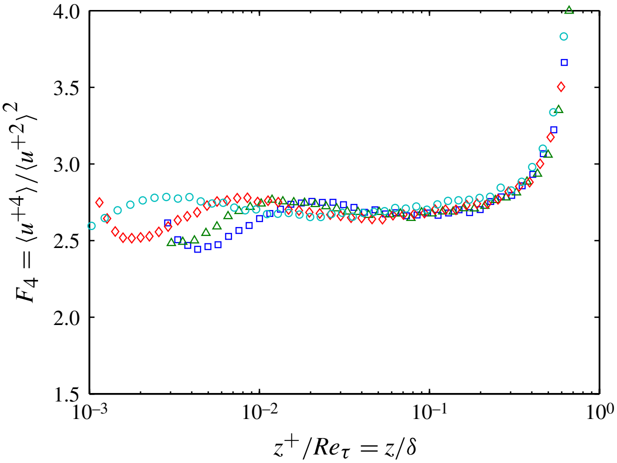

FIGURE 6. Flatness factor F_4 as a function of wall distance, for various Reynolds numbers $Re_\tau = 2800$ (□), 3900 (△), 7300 (◇) and 19 030 (○).

reviewed by Fernholz & Finley (1996). Flatness in the inertial region is seen to be below the Gaussian value of $F_4 = 3$, and there does not appear to be a discernible Reynolds-number dependence in terms of sub-Gaussianity.

A related issue is the near-wall peak value of the variance and higher-order moments. From the plots of moments, we observe that the peaks are located near $z^+_{m1} = 15$, approximately independent of moment order or Reynolds number. The present data suggest a small dependence on Reynolds number, as has been argued by Hutchins *et al.* (2009), although we note that it has been proposed that, for the variance, the peak may become independent of Reynolds number (Hultmark *et al.* 2012). Conversely, if the inner peak arises purely from an extrapolation of the inertial-layer motions, which themselves evolve with increasing Reynolds number, one would expect

$$\langle (u'^+)^2\rangle (z^+_{m1}) \sim A_1 \ln(\delta/z_{m1}) \sim A_1 \ln Re_\tau, \tag{3.1}$$

using the usual scaling that $\delta/z_{m1} \propto Re_\tau$. If so, one would expect high-order moments to obey

$$[\langle (u'^+)^{2p}\rangle (z^+_{m1})]^{1/p} \sim A_p \ln Re_\tau. \tag{3.2}$$

The measured peaks are shown in figure 7 as a function of Reynolds number for various moment orders. The dashed lines provide the extrapolated inertial-layer prediction using the measured values of A_p. Clearly, the measurements fall below these possible trends, indicating that the near-wall peak in high-order moments grows much slower with Reynolds number than predicted by extrapolation of the generalized logarithmic law in the inertial layer, down to a fixed z^+_{m1}. Therefore, it appears more likely that the peaks could be related to the logarithmic law extrapolated down only to a value $z^+_{m2} \sim Re^\gamma_\tau$ with $\gamma > 0$. As mentioned before, $\gamma = 1/2$ has been suggested in the literature (Klewicki *et al.* 2009; Alfredsson *et al.* 2011; Marusic *et al.* 2013) as the lower limit of the logarithmic region for the second-order moment. Then the

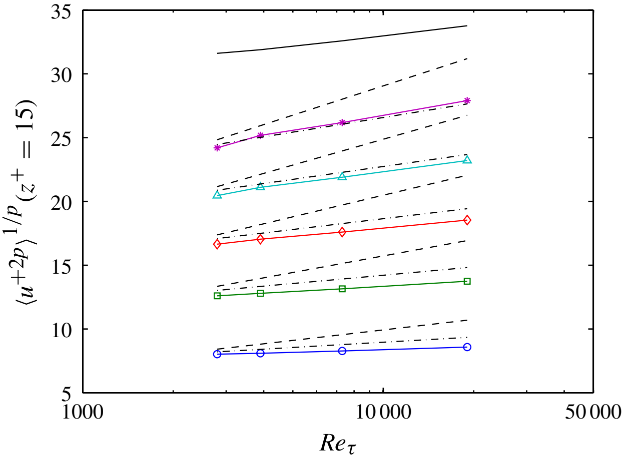

FIGURE 7. Scaling of peak of velocity moments as a function of Reynolds number, for various moment orders $2p = 2$ (○), 4 (□), 6 (◇), 8 (△) and 10 ($*$). Dashed and dot-dashed lines are $\langle (u'^+)^{2p}\rangle_{peak}^{1/p} \sim A_p \ln(Re_\tau) + K_p$ and $\langle (u'^+)^{2p}\rangle_{peak}^{1/p} \sim 0.5A_p \ln(Re_\tau) + K_p'$, respectively, where A_p is the measured slope in the logarithmic regions, and the K_p are arbitrary offsets. The solid line is the Gaussian prediction $[(2p-1)!!]^{1/p}\,\langle (u'^+)^2\rangle_{peak}$, for $2p = 10$.

peak moments would scale as $\langle (u'^+)^{2p}\rangle (z^+_{m2})]^{1/p} \sim A_p(1-\gamma)\ln Re_\tau$. The case $\gamma = 1/2$ is shown with dot-dashed lines in figure 7, and can be seen to give good agreement for the moments of high orders $2p = 8$ and 10, but not for the lower orders $2p = 2$ and 4, for which the data increase with Re_τ even slower than with a slope of $0.5A_p$.

Finally, if one simply took the measured variance as a function of Reynolds number (circles in figure 7) and assumed Gaussian statistics, one would predict $\langle (u'^+)^{2p}\rangle_{z^+_{m1}}^{1/p} \sim [(2p-1)!!]^{1/p}\,\langle (u'^+)^2\rangle_{z^+_{m1}}$, shown in the figure for $2p = 10$ as a solid line at the top. The slope appears to be close to the observations (which are shown as asterisks), but a significant offset (of $\sim$7) is visible.

4. Concluding remarks

In conclusion, observations of a new generalized logarithmic law for high-order moments in high-Reynolds-number turbulent boundary layers have been presented. The results point to a sequence of possibly universal coefficients A_p, of which A_1 is the Townsend–Perry constant. Higher-order moments are consistent with sub-Gaussian statistics, which implies that A_p cannot be trivially related to A_1 and a theory for A_p is still missing.

We point out that prior work has examined sub-Gaussian statistics in turbulence (Jimenez 1998; Wilczek, Daitche & Friedrich 2011). In particular, Jimenez (1998) shows that, for sufficiently steep spectra (steeper than k^{-1}), there is no contradiction between the central limit theorem and sub-Gaussian statistics for velocity fluctuations. However, in the boundary layer's logarithmic region and at scales that dominate the variance, the spectra display a scaling that is (at least approximately) not far from k^{-1} spectral scaling. Since the spectra are thus close to the cross-over k^{-1} scaling, it is unclear whether the analysis of Jimenez (1998) can be applied to explain the present observations. The analysis of Wilczek *et al.* (2011) uses the p.d.f. equation to show that sub-Gaussian p.d.f.s are to be expected in isotropic homogeneous turbulence, but the analysis has not been applied to non-homogeneous boundary-layer turbulence. The work of Tsuji, Lindgren & Johansson (2005) is also of relevance in the present context. They quantified the differences between the shapes of the velocity fluctuation p.d.f.s and Gaussian p.d.f.s using the Kullback–Leibler divergence and found that the deviations are nearly constant across the inertial layer. While they did not observe logarithmic behaviour for the moments, their results are consistent with the present results.

Referring back to the considerations of Pullin *et al.* (2013), it is possible to conclude, based on the present results, that even the maximum (instantaneous) value of u' may be asymptotically small compared to the free stream velocity at increasingly large Reynolds numbers. This conclusion is based on the following arguments (Pullin *et al.* 2013): assuming that the lower limit of the logarithmic region for the even moments of u' follows $z_{m2}^+ \sim Re_\tau^\gamma$, i.e. $z_{m2}/\delta \sim Re_\tau^{\gamma-1}$, the asymptotic behaviour of the scaled moments in the boundary layer would be $\langle u'^+ (z_{m2})^{2p}\rangle^{1/p} \sim B_p - A_p(\gamma-1)\ln Re_\tau \rightarrow A_p(1-\gamma)\ln Re_\tau$ at $Re_\tau \rightarrow \infty$. Combined with the estimate $(u_\tau/U_\infty)^2 \rightarrow \kappa^2/(\ln Re_\tau)^2$ at $Re_\tau \rightarrow \infty$ (i.e. the Coles–Fernholz relation; Pullin *et al.* (2013)), it is clear that the ratio

$$\frac{\langle u' (z_{m2})^{2p}\rangle^{1/p}}{U_\infty^2} \sim \frac{\kappa^2 A_p(1-\gamma)\ln Re_\tau}{(\ln Re_\tau)^2} \rightarrow 0 \quad \text{when } Re_\tau \rightarrow \infty. \tag{4.1}$$

Now consider the random variable g formed by squaring the fluctuating streamwise velocity, $g = u'^2$. It is known that the maximum value of a random variable can be obtained from its infinity norm as $g_{max} = \lim_{p\rightarrow\infty}\langle g^p\rangle^{1/p}$. Hence, we conclude that the maximum value of u'^2 follows

$$\frac{u'^2{}_{max}}{U_\infty^2} = \lim_{p\rightarrow\infty} \frac{\kappa^2 A_p(1-\gamma)\ln Re_\tau}{(\ln Re_\tau)^2} \rightarrow 0 \quad \text{when } Re_\tau \rightarrow \infty, \tag{4.2}$$

provided the generalized Townsend–Perry constants A_p remain finite for any moment order p. Thus, if the generalized logarithmic behaviour we observe in this paper extends to any order moment (and we provide evidence that it holds up to tenth-order moments), even the maximum individual velocity fluctuation would become vanishingly small compared to the free stream velocity (strictly speaking, the difference in velocity between the wall and the free stream) at asymptotically high Reynolds numbers. Hence, the present observations should motivate further theoretical developments.

In terms of empirical models, e.g. those based on a superposition of attached eddies, the sub-Gaussian statistics and non-trivial dependences on wall distance may imply particular correlations among eddies across different scales. Further work is needed to uncover such correlations and also to show how they may relate to modulation effects between the large- and small-scale motions in the flow (Mathis, Hutchins & Marusic 2009). Lastly, the generalized logarithmic law for high-order moments should prove useful to test the accuracy of large eddy simulation models beyond the usual tests that are mostly based on low-order statistics.

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