The Attached Eddy Hypothesis and von Kármán’s Constant

J. D. Woodcock and I. Marusic

Department of Mechanical Engineering
The University of Melbourne, Victoria 3010, Australia

Abstract

Townsend’s attached eddy hypothesis states that the flow in the logarithmic region of wall-bounded turbulent flows will be dominated at the energy-containing scales by a hierarchy of eddies, whose corresponding velocity fields extend to the wall [20]. These eddies are assumed to be geometrically self-similar, differing from each other only in their size, which scales with their distance from the wall. The hypothesis has subsequently gained significant support from high Reynolds number experiments and from numerical simulations [17].

Recently, a more rigorous physical and mathematical basis for the attached eddy hypothesis has been put forward by Marusic and Woodcock [12]. In this present work, we utilise this analysis to investigate the predicted nature of von Kármán’s constant (κ), which has been a source of controversy, particularly since Townsend [20] argued that κ should change at very high Reynolds numbers. We show that strictly applying the attached eddy hypothesis results in von Kármán’s constant rapidly approaching a constant value as the Reynolds number increases.

Introduction

The great complexity of turbulent flows has always been a huge barrier to the development of practical physical models of the phenomenon. Furthermore, the direct numerical simulation of turbulent flows is limited by Reynolds number due to the increasing multitude of scales that need to be resolved.

One prominent model for wall-bounded flows stems from the so-called attached eddy hypothesis of A. A. Townsend [19]. Townsend’s hypothesis, states that the flow in the log-region consists of a series of geometrically self-similar eddies, which scale with their distance from the wall, and whose corresponding velocity fields extend to the wall. In this way, the study of a single representative eddying motion.

In order to produce statistical predictions from the attached eddy hypothesis, Townsend adopted a distribution of eddy sizes specifically in order to obtain a constant Reynolds shear stress [20]. Using this model, Townsend was able to derive the second-order moments of the velocity as a function of the distance from the wall. If $u$, $v$, and $w$ represent the velocity fluctuations in the streamwise, spanwise and wall-normal directions respectively, he obtained

$$
\left\langle u^2 \right\rangle^+ = B_1 - A_1 \log (z/\delta),
$$

$$
\left\langle v^2 \right\rangle^+ = B_{1,v} - A_{1,v} \log (z/\delta),
$$

$$
\left\langle w^2 \right\rangle^+ = B_{1,w},
$$

and

$$
\left\langle uv \right\rangle^+ = 1,
$$

where the angled brackets represent ensemble averages. Here, $\delta$ denotes the maximum distance from the wall at which the flow is dominated by the presence of the attached eddies (i.e. the boundary layer thickness), and the superscript + indicates that the quantities have been scaled according to wall variables, that is with $U_\tau$, the friction velocity or $v/U_\tau$, the viscous length scale. All of the $A$s and $B$s above are constants. This result only applies where the flow is sufficiently close to the wall to be affected by its presence and yet sufficiently far from the wall that the effect of viscosity is negligible. The above equations have subsequently been vindicated by high Reynolds number experiments [3, 4, 9, 11, 13, 21] and direct numerical simulations [17].

Various authors have reviewed the nature of the log-region in recent years [1, 5, 6, 10, 18]. While there remain differing interpretations of the causal relationships between the coherent structures in the log-region, a consensus has emerged that the region contains a hierarchy of eddies, whose behaviours and distribution concur with Townsend’s hypothesis. It has been shown that such a self-similar hierarchical structure is consistent with invariant solutions associated with the leading order dynamics [7, 8].

Townsend’s result was extended by Perry and coworkers [14, 15, 16], who also used the attached eddy hypothesis along with a prescribed distribution of eddy sizes to obtain the classical logarithmic law of the wall:

$$
\left\langle U \right\rangle^+ = \frac{1}{\kappa} \log (z^+) + C,
$$

where $\kappa$ is von Kármán’s constant, and $C$ depends on the roughness of the surface, but is otherwise constant.

It is noted that the both of these derivations (for equations 1-5) were predicated on the adoption of a prescribed distribution of eddy sizes, and also on the assumption that there are no correlations between eddies of different sizes.

Recently, Woodcock & Marusic [12, 22] formulated a new derivation of the attached eddy model, which minimised the number of assumptions. They avoided specifying either a prescribed distribution of eddy sizes or a constant Reynolds shear stress. In order to do this, they presented an extended form of Campbell’s theorem (originally a method used to account for the random arrival of electrons at an anode, and now applied to the random placement of eddies on a wall). Using this, they were able to derive all of the moments of the velocity fluctuations.

In this work, we look at the implications of this new derivation of the attached eddy model for von Kármán’s constant. Previously, Townsend [20] and Davidson [2] have predicted that the attached eddy hypothesis should result in a von Kármán’s constant that continually changes with the Reynolds number. Townsend argued that von Kármán’s constant will increase as the ratio of the energy present in the fluctuations to that present within the mean flow increases. He therefore concluded that any such variations would be unlikely to be detectable under ordinary circumstances, but would become important at extremely large Reynolds numbers. Conversely however, we find that von Kármán’s constant initially increases with the Reynolds num-
ber, but rapidly converges to a constant.

**Mathematical Formulation**

Following the attached eddy hypothesis, the velocity distribution is modelled as the superposition of the velocity fields corresponding to each of the eddies present. The eddies are all of identical shape and relative dimensions, and differ only in their heights.

Each individual eddy can therefore be seen as a separate system. Its defining characteristics are its height, $h$, and its location on the wall, $x$. The length scale of the eddy will therefore be $h$, while the friction velocity will be its velocity scale.

Therefore, if $Q$ is the velocity field at $x$ corresponding to an individual eddy, then its spatial and height dependence will be

$$Q = Q \left( \frac{x - x_c}{h} \right).$$

(6)

The total velocity, $U(x)$, is then simply the superposition of the velocity fields corresponding to each of the individual eddies. However, we could not postulate the locations and sizes of all eddies present, and so we must instead consider only the statistical properties of the entire flow. (We apply the method of images at the wall, $z = 0$, in order to determine the boundary conditions. This will become important subsequently.)

The distribution of eddy sizes follows from the observation that $h$ is the system’s only natural length scale. From dimensional analysis, for eddies that are space-filling we can see that if $p_h$ denotes the density of eddies of size $h$, then

$$p_h \approx \frac{1}{h^3}.$$  

(7)

The probability that an eddy has size $h$, which we denote by $P(h)$, will clearly be proportional to $p_h$. Therefore, if the heights of the eddies range from $h_{min}$ and $h_{max}$, then the probability can be determined via normalisation to be

$$P(h) = 2 \left( h_{min}^{-1} - h_{max}^{-1} \right)^{-1} \int_0^h P(h') dh'.$$

(8)

To simplify the equations, particularly at higher orders we introduce a new set of functions, $I_{k,l,m}$, known as the cumulants of the velocity:

$$\lambda_{k,l,m}(z) = \beta \int_{h_{min}}^{h_{max}} I_{k,l,m} \left( \frac{z}{h} \right) h^2 P(h) dh,$$

(9)

where the coefficient $\beta$ represents the density of eddies on the wall and $I_{k,l,m}(z/h)$ is called the eddy contribution function, and is given by

$$I_{k,l,m}(Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_k(X) Q_l(Y) Q_m(Z) dX dY.$$

(10)

The capital $X$, and its components $X$, $Y$, and $Z$, represent the location scaled by $h$. That is, $(X, Y, Z) = (x/h, y/h, z/h)$. Using cumulants, the mean velocity can be expressed as

$$\langle U \rangle = \lambda_1, \quad \langle V \rangle = \lambda_{0,1,0}, \quad \langle W \rangle = \lambda_{0,0,1},$$

(11)

where we have used the shorthand

$$\lambda_n \equiv \lambda_{n,0,0}$$

(12)

for purely streamwise quantities. If we denote velocity fluctuations by $u$, so that

$$u(x) = U(x) - \langle U(x) \rangle,$$

(13)

then the moments of these velocity fluctuations are given by

$$\langle u^2 \rangle = \lambda_2,$$

(14)

$$\langle u^1 \rangle = \lambda_3,$$

(15)

$$\langle u^3 \rangle = \lambda_4 + 3\lambda_2^2,$$

(16)

$$\langle u^5 \rangle = \lambda_6 + 15\lambda_2\lambda_4 + 10\lambda_2^3 + 15\lambda_2^2,$$

(17)

$$\langle u^3 \rangle = \lambda_8 + 28\lambda_2\lambda_6 + 56\lambda_3\lambda_5 + 35\lambda_2^4,$$

(18)

$$\langle u^n \rangle = \lambda_{1,0,1},$$

(19)

and similarly for $\langle v^n \rangle$ and $\langle w^n \rangle$.

**General Flow Properties**

In order to derive the flow profiles from the attached eddy hypothesis, we must recognise that the velocity field corresponding to a single eddy will only be non-negligible for a finite distance from the wall. Mathematically, we can therefore say that there must exist some $\alpha$ such that

$$Q \left( \frac{x}{h} \right) \approx 0, \quad \text{for } z > \alpha h \quad (\alpha > 1).$$

(20)

By substituting (8) into (9) and rearranging it is possible to show that, so long as $z > \alpha h_{min}$,

$$\lambda_{k,l,m}(z) = 2\beta \left( h_{min}^{-1} - h_{max}^{-1} \right)^{-1} \int_0^\alpha \frac{K_{k,l,m}(Z) dZ}{Z}.$$

(21)

This takes into account the fact that where $Q$ is zero, $I_{k,l,m}$ will also be zero. The implication of the fact that $I_{k,l,m}$ will diminish at higher $Z$, is that a significant portion of $\lambda_{k,l,m}$ will emanate from around $Z \approx 0$. It is therefore reasonable to expand $I_{k,l,m}(Z)$ in a Taylor series around $Z = 0$. This results in

$$I_{k,l,m}(Z) = I_{k,l,m}(0) + I_{k,l,m}'(Z), \quad \text{so that } I_{k,l,m}(0) = 0,$$

(22)

where $I_{k,l,m}(Z)$ contains all of the higher order terms in the Taylor series expansion. Substituting this into (21) and integrating where possible gives

$$\lambda_{k,l,m}(z) = A_{k,l,m} \log \left( \frac{z}{h_{max}} \right) + B_{k,l,m}, \quad \text{for } z \ll h_{max},$$

(23)

where $A_{k,l,m}$ and $B_{k,l,m}$ are constants given by

$$A_{k,l,m} = \frac{-2\beta}{h_{min} - h_{max}} I_{k,l,m}(0),$$

(24)

$$B_{k,l,m} = \frac{-2\beta}{h_{min} - h_{max}} \left[ I_{k,l,m}(0) \log \alpha + \int_0^\alpha \frac{I_{k,l,m}(Z) dZ}{Z} \right].$$

(25)

It is important to note that since the the fluid cannot flow through the wall at $Z = 0$, all $I_{k,l,m}$ will be zero at $Z = 0$ if $m \neq 0$ (that is if the eddy contribution function has a wall-normal component). This results in a constant $\lambda_{k,l,m}$ wherever $m \neq 0$.

**Implications for von Kármán’s Constant**

It is now possible to determine the Reynolds number dependence of von Kármán’s constant through the above results
It is clear from (23) and (24) that von Kármán’s constant will be
Reynolds number, $Re$ for the mean velocity. However, first we need to define the
height of eddies present. Accordingly, we adopt
$$Re = 10^6 \frac{h_{\text{max}}}{h_{\text{min}}}.$$ (26)

We would, however, prefer not to express $\kappa$ in terms of $\beta$, the
density of eddies per unit area, since $\beta$ will depend upon the
range of scales present within the flow. We will therefore re-express $\kappa$ in terms of universal quantities.

To this end, we introduce $N$, representing the number of eddies
present. Because the placement of the eddies is a Poisson process,
$N$ can be inferred from the probability density of the eddy
heights and intensities, but also the average distances between
them. Furthermore, it follows from the fact that the placement of eddies is a Poisson process that the expected distance to each
subsequent eddy will depend only on the height of the subse-
quent eddy (and not the previous eddy).

We now define a new constant $k_s$, such that if the height of the
next-closest eddy were known to be $h$, then the expected
distance to the next-closest eddy in the positive $x$-direction will be $k_s h$. (More specifically, $k_s h$ represents the distance in a strip of
height $h'$ to the nearest eddy of height between $h$ and $h + dh$
divided by $h'$ and $dh$.) For the spanwise direction, we define an
analogous constant $k_t$.

If the nearest eddy in the positive $x$-direction were known to be of
height $h_1$ and the nearest eddy in the positive $y$-direction were
known to be of size $h_2$, then we would know that the number of
eddies present, on a plane of area $L^2$, would be expected to be
$$N_{h_1, h_2} = \frac{L^2}{(k_s h_1)(k_t h_2)}. \quad (28)$$

We can now infer $N$ from the probability density of the eddy
heights via
$$N = \int_{h_{\text{min}}}^{h_{\text{max}}} \int_{h_{\text{min}}}^{h_{\text{max}}} N_{h_1, h_2} P(h_1) P(h_2) \, dh_1 \, dh_2. \quad (29)$$

By substituting (8) into the above, and integrating, we find that
$$N = \frac{4}{9} \frac{L^2}{k_s k_t} \left( 1 - \left( \frac{h_{\text{min}}}{h_{\text{max}}} \right)^3 \right)^2 \left( 1 - \left( \frac{h_{\text{min}}}{h_{\text{max}}} \right)^2 \right)^2 \quad (30)$$

By using the fact that $\beta \equiv N/L^2$ we can rewrite (27) as
$$\frac{1}{\kappa} = -\frac{2\beta}{h_{\text{min}} - h_{\text{max}}} I_{1.0.0}(0). \quad (27)$$

By substituting the Reynolds number for the eddy size ratios using (26), this becomes
$$\frac{1}{\kappa} = -\frac{8I_{1.0.0}(0)}{9(k_s k_t)} \left( 1 - 10^6 Re^{-3} \right)^2 \left( 1 - 10^4 Re^{-3} \right)^3. \quad (34)$$

A plot of the above function can be seen in figure 1. There it
can clearly be seen that while $\kappa$ increases very slowly with the
Reynolds number at low $Re$, it rapidly asymptotes to a con-
stant. This contradicts the predictions of Townsend [20] and
indicates that the attached eddy hypothesis does indeed predict
a universal log-law at high Reynolds numbers.

**Conclusions**

Townsend’s attached eddy hypothesis states that the flow in the
log-region is dominated by a hierarchy of geometrically
self-similar eddies, the velocity fields corresponding to each of
which extend to the wall. Townsend [20] himself predicted that
the attached eddy hypothesis would produce a von Kármán’s
constant that varied significantly with the Reynolds number at
high Reynolds number, but should vary little at low Reynolds
number. This would imply that the flow profile would not obey
a log-law at very high Reynolds numbers, and this is discussed
further by Davidson [2].

However, we have demonstrated here that according to the at-
tached eddy hypothesis von Kármán’s constant will rapidly
approach a constant as the Reynolds number increases, and is in
clear contrast to the argument of Townsend. The important
implication of this result is that the log-law should be expected
to hold at all sufficiently high Reynolds numbers.

While previous applications of this hypothesis were predicated
upon a series of physical and mathematical assumptions, in this
work we seek to minimise the number of assumptions necessary
in applying the attached eddy hypothesis.

As with earlier applications of the attached eddy model, we ef-
fectively model the flow in the inviscid log-region through a
single eddying motion. This we achieve by modelling the flow

---

**Figure 1:** Graph showing the dependence of von Kármán’s con-
stant on the Reynolds number.
as a random distribution of self similar eddies. To this end, we have extended Campbell’s theorem to apply to the random distribution of eddies on a wall.

The authors gratefully acknowledge the financial support of the Australian Research Council.

References


