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# The Attached Eddy Hypothesis and von Kármán's Constant

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### Abstract

Townsend's attached eddy hypothesis states that the flow in the logarithmic region of wall-bounded turbulent flows will be dominated at the energy-containing scales by a hierarchy of eddies, whose corresponding velocity fields extend to the wall [20]. These eddies are assumed to be geometrically self-similar, differing from each other only in their size, which scales with their distance from the wall. The hypothesis has subsequently gained significant support from high Reynolds number experiments and from numerical simulations [17].

Recently, a more rigorous physical and mathematical basis for the attached eddy hypothesis has been put forward by Marusic and Woodcock [12]. In this present work, we utilise this analysis to investigate the predicted nature of von Kármán's constant ($\kappa$), which has been a source of controversy, particularly since Townsend [20] argued that $\kappa$ should change at very high Reynolds numbers. We show that strictly applying the attached eddy hypothesis results in von Kármán's constant rapidly approaching a constant value as the Reynolds number increases.

### Introduction

The great complexity of turbulent flows has always been a huge barrier to the development of practical physical models of the phenomenon. Furthermore, the direct numerical simulation of turbulent flows is limited by Reynolds number due to the increasing multitude of scales that need to be resolved.

One prominent model for wall-bounded flows stems from the so-called attached eddy hypothesis of A. A. Townsend [19]. Townsend's hypothesis, states that the flow in the log-region consists of a series of geometrically self-similar eddies, which scale with their distance from the wall, and whose corresponding velocity fields extend to the wall. In this way, the study of a highly complicated phenomenon involving many scales of motion is effectively reduced to the study of a single representative eddying motion.

In order to produce statistical predictions from the attached eddy hypothesis, Townsend adopted a distribution of eddy sizes specifically in order to obtain a constant Reynolds shear stress [20]. Using this model, Townsend was able to derive the second-order moments of the velocity as a function of the distance from the wall. If $u$, $v$ and $w$ represent the velocity fluctuations in the streamwise, spanwise and wall-normal directions respectively, he obtained

$$\left\langle u^2 \right\rangle^+ = B_1 - A_1 \log(z/\delta), \tag{1}$$

$$\left\langle v^2 \right\rangle^+ = B_{1,v} - A_{1,v} \log(z/\delta), \tag{2}$$

$$\left\langle w^2 \right\rangle^+ = B_{1,w}, \tag{3}$$

$$-\langle uw \rangle^+ = 1, \tag{4}$$

where the angled brackets represent ensemble averages. Here, $\delta$ denotes the maximum distance from the wall at which the flow is dominated by the presence of the attached eddies (i.e. the boundary layer thickness), and the superscript + indicates that the quantities have been scaled according to wall variables, that is with $U_\tau$, the friction velocity or $\nu/U_\tau$, the viscous length scale. All of the As and Bs above are constants. This result only applies where the flow is sufficiently close to the wall to be affected by its presence and yet sufficiently far from the wall that the effect of viscosity is negligible. The above equations have subsequently been vindicated by high Reynolds number experiments [3, 4, 9, 11, 13, 21] and direct numerical simulations [17].

Various authors have reviewed the nature of the log-region in recent years [1, 5, 6, 10, 18]. While there remain differing interpretations of the causal relationships between the coherent structures in the log-region, a consensus has emerged that the region contains a hierarchy of eddies, whose behaviours and distribution concur with Townsend's hypothesis. It has been shown that such a self-similar hierarchical structure is consistent with invariant solutions associated with the leading order dynamics [7, 8].

Townsend's result was extended by Perry and coworkers [14, 15, 16], who also used the attached eddy hypothesis along with a prescribed distribution of eddy sizes to obtain the classical logarithmic law of the wall:

$$\langle U \rangle^+ = \frac{1}{\kappa} \log\left(z^+\right) + C, \tag{5}$$

where $\kappa$ is von Kármán's constant, and $C$ depends on the roughness of the surface, but is otherwise constant.

It is noted that the both of these derivations (for equations 1-5) were predicated on the adoption of a prescribed distribution of eddy sizes, and also on the assumption that there are no correlations between eddies of different sizes.

Recently, Woodcock & Marusic [12, 22] formulated a new derivation of the attached eddy model, which minimised the number of assumptions. They avoided specifying either a prescribed distribution of eddy sizes or a constant Reynolds shear stress. In order to do this, they presented an extended form of Campbell's theorem (originally a method used to account for the random arrival of electrons at an anode, and now applied to the random placement of eddies on a wall). Using this, they were able to derive all of the moments of the velocity fluctuations.

In this work, we look at the implications of this new derivation of the attached eddy model for von Kármán's constant. Previously, Townsend [20] and Davidson [2] have predicted that the attached eddy hypothesis should result in a von Kármán's constant that continually changes with the Reynolds number. Townsend argued that von Kármán's constant will increase as the ratio of the energy present in the fluctuations to that present within the mean flow increases. He therefore concluded that any such variations would be unlikely to be detectable under ordinary circumstances, but would become important at extremely large Reynolds numbers. Conversely however, we find that von Kármán's constant initially increases with the Reynolds number, but rapidly converges to a constant.

### Mathematical Formulation

Following the attached eddy hypothesis, the velocity distribution is modelled as the superposition of the velocity fields corresponding to each of the eddies present. The eddies are all of identical shape and relative dimensions, and differ only in their heights.

Each individual eddy can therefore be seen as a separate system. Its defining characteristics are its height, $h$, and its location on the wall, $\mathbf{x}_e$. The length scale of the eddy will therefore be $h$, while the *friction velocity* will be its velocity scale.

Therefore, if $\mathbf{Q}$ is the velocity field at $\mathbf{x}$ corresponding to an individual eddy, then its spatial and height dependence will be

$$\mathbf{Q} = \mathbf{Q}\left(\frac{\mathbf{x}-\mathbf{x}_e}{h}\right). \tag{6}$$

The total velocity, $\mathbf{U}(\mathbf{x})$, is then simply the superposition of the velocity fields corresponding to each of the individual eddies. However, we could never postulate the locations and sizes of all eddies present, and so we must instead consider only the statistical properties of the entire flow. (We apply the method of images at the wall, $z=0$, in order to determine the boundary conditions. This will become important subsequently.)

The distribution of eddy sizes follows from the observation that $h$ is the system's only natural length scale. From dimensional analysis, for eddies that are space-filling we can see that if $\rho_h$ denotes the density of eddies of size $h$, then

$$\rho_h \propto \frac{1}{h^3}. \tag{7}$$

The probability that an eddy has size $h$, which we denote by $P(h)$, will clearly be proportional to $\rho_h$. Therefore, if the heights of the eddies range from $h_{min}$ and $h_{max}$, then the probability can be determined via normalisation to be

$$P(h) = 2\left(h_{min}^{-2} - h_{max}^{-2}\right)^{-1}\frac{1}{h^3}. \tag{8}$$

To simplify the equations, particularly at higher orders we introduce a new set of functions, $\lambda_{k,l,m}$, known as the cumulants of the velocity:

$$\lambda_{k,l,m}(z) = \beta\int_{h_{min}}^{h_{max}} I_{k,l,m}\left(\frac{z}{h}\right)h^2 P(h)\,\mathrm{d}h, \tag{9}$$

where the coefficient $\beta$ represents the density of eddies on the wall and $I_{k,l,m}(z/h)$ is called the eddy contribution function, and is given by

$$I_{k,l,m}(Z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_x^k(\mathbf{X})\,Q_y^l(\mathbf{X})\,Q_z^m(\mathbf{X})\,\mathrm{d}X\,\mathrm{d}Y. \tag{10}$$

The capital $\mathbf{X}$, and its components $X$, $Y$ and $Z$, represent the location scaled by $h$. That is, $(X,Y,Z) = (x/h, y/h, z/h)$. Using cumulants, the mean velocity can be expressed as

$$\langle U\rangle = \lambda_1, \qquad \langle V\rangle = \lambda_{0,1,0}, \qquad \langle W\rangle = \lambda_{0,0,1}, \tag{11}$$

where we have used the shorthand

$$\lambda_n \equiv \lambda_{n,0,0} \tag{12}$$

for purely streamwise quantities. If we denote velocity fluctuations by $\mathbf{u}$, so that

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) - \langle\mathbf{U}(\mathbf{x})\rangle, \tag{13}$$

then the moments of these velocity fluctuations are given by

$$\langle u^2\rangle = \lambda_2, \tag{14}$$
$$\langle u^3\rangle = \lambda_3, \tag{15}$$
$$\langle u^4\rangle = \lambda_4 + 3\lambda_2^2, \tag{16}$$
$$\langle u^6\rangle = \lambda_6 + 15\lambda_2\lambda_4 + 10\lambda_3^2 + 15\lambda_2^3, \tag{17}$$
$$\begin{aligned}\langle u^8\rangle = {} & \lambda_8 + 28\lambda_2\lambda_6 + 56\lambda_3\lambda_5 + 35\lambda_4^2 \\ & + 210\lambda_2^2\lambda_4 + 280\lambda_2\lambda_3^2 + 105\lambda_2^4,\end{aligned} \tag{18}$$
$$\langle uw\rangle = \lambda_{1,0,1}, \tag{19}$$

and similarly for $\langle v^n\rangle$ and $\langle w^n\rangle$.

### General Flow Properties

In order to derive the flow profiles from the attached eddy hypothesis, we must recognise that the velocity field corresponding to a single eddy will only be non-negligible for a finite distance from the wall. Mathematically, we can therefore say that there must exist some $\alpha$ such that

$$\mathbf{Q}\left(\frac{\mathbf{x}}{h}\right) \approx 0, \qquad \text{for } z > \alpha h \quad (\alpha > 1). \tag{20}$$

By substituting (8) into (9) and rearranging it is possible to show that, so long as $z > \alpha h_{min}$,

$$\lambda_{k,l,m}(z) = 2\beta\left(h_{min}^{-2} - h_{max}^{-2}\right)^{-1}\int_{z/h_{max}}^{\alpha} I_{k,l,m}(Z)\frac{\mathrm{d}Z}{Z}. \tag{21}$$

This takes into account the fact that where $\mathbf{Q}$ is zero, $I_{k,l,m}$ will also be zero. The implication of the fact that $I_{k,l,m}$ will diminish at higher $Z$, is that a significant portion of $\lambda_{k,l,m}$ will emanate from around $Z \simeq 0$. It is therefore reasonable to expand $I_{k,l,m}(Z)$ in a Taylor series around $Z=0$. This results in

$$I_{k,l,m}(Z) = I_{k,l,m}(0) + \tilde{I}_{k,l,m}(Z), \qquad \text{so that } \tilde{I}_{k,l,m}(0) = 0, \tag{22}$$

where $\tilde{I}_{k,l,m}(Z)$ contains all of the higher order terms in the Taylor series expansion. Substituting this into (21) and integrating where possible gives

$$\lambda_{k,l,m}(z) = A_{k,l,m}\log\left(\frac{z}{h_{max}}\right) + B_{k,l,m}, \qquad \text{for } z \ll h_{max}, \tag{23}$$

where $A_{k,l,m}$ and $B_{k,l,m}$ are constants given by

$$A_{k,l,m} = -\frac{2\beta}{h_{min}^{-2} - h_{max}^{-2}} I_{k,l,m}(0), \tag{24}$$

$$B_{k,l,m} = \frac{2\beta}{h_{min}^{-2} - h_{max}^{-2}}\left[I_{k,l,m}(0)\log\alpha + \int_0^{\alpha}\frac{\tilde{I}_{k,l,m}(Z)}{Z}\mathrm{d}Z\right]. \tag{25}$$

It is important to note that since the the fluid cannot flow through the wall at $Z=0$, all $I_{k,l,m}$ will be zero at $Z=0$ if $m \neq 0$ (that is if the eddy contribution function has a wall-normal component). This results in a constant $\lambda_{k,l,m}$ wherever $m \neq 0$.

### Implications for von Kármán's Constant

It is now possible to determine the Reynolds number dependence of von Kármán's constant through the above results for the mean velocity. However, first we need to define the Reynolds number, $Re_\tau$, in terms of the range of eddy sizes present. Accordingly, we adopt

$$Re_\tau = 100\frac{h_{max}}{h_{min}}. \tag{26}$$

It is clear from (23) and (24) that von Kármán's constant will be given by

$$\frac{1}{\kappa} = -\frac{2\beta}{h_{min}^{-2} - h_{max}^{-2}} I_{1,0,0}(0). \tag{27}$$

We would, however, prefer to not express $\kappa$ in terms of $\beta$, the density of eddies per unit area, since $\beta$ will depend upon the range of scales present within the flow. We will therefore re-express $\kappa$ in terms of universal quantities.

To this end, we introduce $N$, representing the number of eddies present. Because the placement of the eddies is a Poisson process, $N$ can be inferred from the expected distance from a single eddy to its nearest neighbour in the positive $x$ and $y$ directions. The spatial self-similarity of the eddies affects not only their heights and intensities, but also the average distances between them. Furthermore, it follows from the fact that the placement of eddies is a Poisson process that the expected distance to each subsequent eddy will depend only on the height of the subsequent eddy (and not the previous eddy).

We now define a new constant $k_x$, such that if the height of the next-closest eddy were known to be $h$, then the expected distance to the next-closest eddy in the positive $x$-direction will be $k_x h$. (More specifically, $k_x h$ represents the distance in a strip of height $h'$ to the nearest eddy of height between $h$ and $h + \mathrm{d}h$ divided by $h'$ and $\mathrm{d}h$.) For the spanwise direction, we define an analogous constant $k_y$.

If the nearest eddy in the positive $x$-direction were known to be of height $h_1$ and the nearest eddy in the positive $y$-direction were known to be of size $h_2$, then we would know that the number of eddies present, on a plane of area $L^2$, would be expected to be

$$N_{h_1,h_2} = \frac{L^2}{(k_x h_1)(k_y h_2)}. \tag{28}$$

We can now infer $N$ from the probability density of the eddy heights via

$$N = \int_{h_{min}}^{h_{max}} \int_{h_{min}}^{h_{max}} N_{h_1,h_2} P(h_1) P(h_2)\, \mathrm{d}h_1\, \mathrm{d}h_2. \tag{29}$$

By substituting (8) into the above, and integrating, we find that

$$N = \frac{4}{9}\frac{L^2}{k_x k_y} \frac{\left(1 - \left(\frac{h_{min}}{h_{max}}\right)^3\right)^2}{h_{min}^2 \left(1 - \left(\frac{h_{min}}{h_{max}}\right)^2\right)^2} \tag{30}$$

By using the fact that $\beta \equiv N/L^2$ we can rewrite (27) as

$$\frac{1}{\kappa} = -\frac{8 I_{1,0,0}(0)}{9(k_x k_y)} \frac{\left(1 - \left(\frac{h_{min}}{h_{max}}\right)^3\right)^2}{\left(1 - \left(\frac{h_{min}}{h_{max}}\right)^2\right)^3}. \tag{31}$$

By substituting the Reynolds number for the eddy size ratios using (26), this becomes

$$\frac{1}{\kappa} = -\frac{8 I_{1,0,0}(0)}{9(k_x k_y)} \frac{\left(1 - 10^6 Re_\tau^{-3}\right)^2}{\left(1 - 10^4 Re_\tau^{-2}\right)^3}. \tag{32}$$

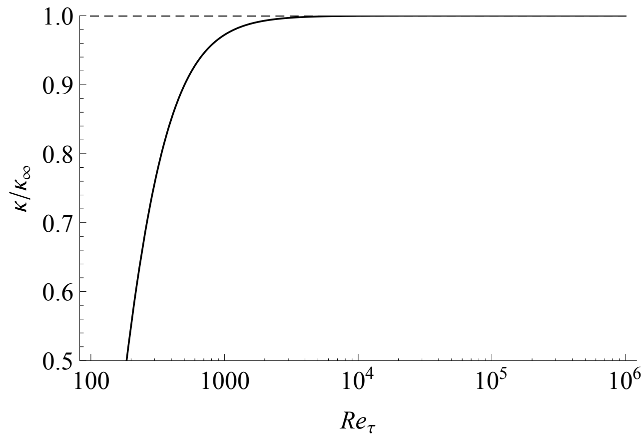


Figure 1: Graph showing the dependence of von Kármán's constant on the Reynolds number.

As $Re_\tau$ increases, this asymptotes to a constant, which we denote by $\kappa_\infty$. It is given by

$$\frac{1}{\kappa_\infty} = -\frac{8 I_{1,0,0}(0)}{9(k_x k_y)}. \tag{33}$$

The ratio of von Kármán's constant to its asymptote is simply

$$\frac{\kappa}{\kappa_\infty} = \frac{\left(1 - 10^4 Re_\tau^{-2}\right)^3}{\left(1 - 10^6 Re_\tau^{-3}\right)^2}, \tag{34}$$

A plot of the above function can be seen in figure 1. There it can clearly be seen that while $\kappa$ increases very slowly with the Reynolds number at low $Re_\tau$, it rapidly asymptotes to a constant. This contradicts the predictions of Townsend [20] and indicates that the attached eddy hypothesis does indeed predict a universal log-law at high Reynolds numbers.

## Conclusions

Townsend's attached eddy hypothesis states that the flow in the log-region is dominated by a hierarchy of geometrically self-similar eddies, the velocity fields corresponding to each of which extend to the wall. Townsend [20] himself predicted that the attached eddy hypothesis would produce a von Kármán's constant that varied significantly with the Reynolds number at high Reynolds number, but should vary little at low Reynolds number. This would imply that the flow profile would not obey a log-law at very high Reynolds numbers, and this is discussed further by Davidson [2].

However, we have demonstrated here that according to the attached eddy hypothesis von Kármán's constant will rapidly approach a constant as the Reynolds number increases, and is in clear contrast to the argument of Townsend. The important implication of this result is that the log-law should be expected to hold at all sufficiently high Reynolds numbers.

While previous applications of this hypothesis were predicated upon a series of physical and mathematical assumptions, in this work we seek to minimise the number of assumptions necessary in applying the attached eddy hypothesis.

As with earlier applications of the attached eddy model, we effectively model the flow in the inviscid log-region through a single eddying motion. This we achieve by modelling the flow as a random distribution of self similar eddies. To this end, we have extended Campbell's theorem to apply to the random distribution of eddies on a wall.

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