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EVOLUTION CALCULATIONS FOR TURBULENT BOUNDARY LAYERS APPROACHING EQUILIBRIUM SINK FLOW

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ABSTRACT

The evolution of a turbulent boundary layer approaching a smooth wall equilibrium sink flow is considered. The closure problem is described for the general arbitrary pressure gradient flow where the only assumptions made are the use of the classical similarity laws such as Prandtl's law of the wall and Coles' law of the wake together with the mean continuity and mean momentum differential and integral equations. The important parameters are identified and the problem reduces to one semi-empirical input with the assumption that the Reynolds shear stress can be described by a two-parameter family. Good agreement is shown with experimental data.

INTRODUCTION

Perry, Marusic & Li (1994) initially developed a mathematical framework for computing the evolution of turbulent boundary layers using the classical similarity laws such as Prandtl's law of the wall and Coles' law of the wake together with the momentum integral and differential equations. It was found that these equations show that there are 4 parameters which control the streamwise evolution of the layer and the Reynolds shear stress distribution and these are S, Π, β and ζ. $S = U_1/U_\tau$ where U_1 is the local freestream velocity and U_τ is the friction velocity, Π is Coles (1956) wake factor, $\beta = (\delta^*/\tau_0)(dp/dx)$ is the Clauser pressure gradient parameter where δ^* is the displacement thickness, p is the freestream static pressure, τ_0 is the wall shear stress and x is streamwise distance and $\zeta = S\delta_c d\Pi/dx$ where δ_c is the boundary layer thickness. The total shear stress is expressed as

$$\frac{\tau}{\tau_0} = f_1[\eta, \Pi, S] + g_1[\eta, \Pi, S]\zeta + g_2[\eta, \Pi, S]\beta \tag{1}$$

where $\eta = z/\delta_c$ where z is the distance normal to the wall. Here f_1, g_1 and g_2 are known analytical functions. Their precise form depends on the wall-wake formulation. Here we use

$$\frac{U}{U_\tau} = \frac{1}{\kappa}\log\left[\frac{zU_\tau}{\nu}\right] + A - \frac{1}{3\kappa}\eta^3 + \frac{\Pi}{\kappa}2\eta^2(3-2\eta) \tag{2}$$

where $\kappa = 0.41$ is the Karman constant and A is the universal smooth wall constant taken here to be 5.0.

In this initial work of Perry *et al.* (1994) the evolution equations were restricted and the only problems which could be solved were the so called quasi-equilibrium flow cases where it could be assumed that ζ was sufficiently small to neglect its effect even though Π is permitted to vary with x. Thus the problem reduces to considering the relation

$$C[\Pi, \beta, S] = 0 \tag{3}$$

where for a given Π it is assumed that the velocity defect distribution is fixed and the shear stress distribution is fixed (approximately). Hence from data, if we know β at a given S for a fixed Π (i.e. for one experimental data point), then for this fixed Π we can find β versus S for all S using equation (1) to ensure that τ/τ_0 profiles are matched (approximately for all S. Now, it is found that for S sufficiently large, $\beta = \beta_a$ (the asymptotic value of β) and C is no longer a function of S. If this procedure is repeated for different values of Π, a one-to-one relationship between β_a and Π can be found which is based on experiment. This formulation is consistent with a universal relation for eddy viscosity ϵ, i.e. $\epsilon/(\delta_c U_\tau) = \phi[\eta, \Pi]$. Unfortunately, such formulations are known to break down in non-equilibrium flows, i.e. flows with significant ζ contribution - see Marusic & Perry (1995) for an example.

NEW EVOLUTION EQUATIONS

What we are concerned with here is computing the evolution for flows approaching equilibrium sink flow. The restricted formulation of Perry *et al.* (1994) allows approximate calculations for a given fixed upstream condition. We wish to extend this to allow for a choice of an arbitrary initial condition. In order to do this the general problem of non-equilibrium flows must be considered where it is essential to also include the effects of ζ. In this case, the function C in (3) needs to be replaced by,

$$\mathcal{F}[\Pi, S, \beta, \zeta] = 0. \tag{4}$$

Hence in order to describe the state of the layer, we require three of the four variables in the above expression. For quasi-equilibrium flows, Perry *et al.* (1994) relied on a one-parameter family to describe the shear stress. A two parameter family of shear stress profiles of the form

$$\frac{\tau}{\tau_0} = f[\eta, \Pi, \beta_a]. \tag{5}$$

would be closer to the truth and when used in conjunction with (1) some information can be obtained regarding (4) as follows. Consider the $S-\beta$ plane at a fixed Π. If such a plane contains an experimental data point, then S, Π, β and ζ are known for that data point and so also is τ/τ_0 versus η from (1). Trace out a curve for increasing S of fixed shear stress profile shape on the $S-\beta$ plane. By taking $S \to \infty$ we obtain asymptotic values of ζ_a and β_a as shown in figure 1. (Going to $S = \infty$ is simply a convenient curve-fitting procedure and could never be approached experimentally). This process of keeping profile shape fixed will be referred to as "profile matching". Accurate agreement is found by using a least-squares-error criterion, i.e.

$$\frac{\partial}{\partial \beta_a}\left\{\int_0^1 \left[\left(\frac{\tau}{\tau_0}\right)_{S\to\infty} - \left(\frac{\tau}{\tau_0}\right)\right]^2 d\eta\right\} = 0$$

$$\frac{\partial}{\partial \zeta_a}\left\{\int_0^1 \left[\left(\frac{\tau}{\tau_0}\right)_{S\to\infty} - \left(\frac{\tau}{\tau_0}\right)\right]^2 d\eta\right\} = 0$$

If this process is repeated often enough for different Π's then we obtain a $\Pi - \beta_a$ diagram with distributions of extrapolated data points corresponding to different values of ζ_a. By a surface fit to ζ_a on the the $\Pi - \beta_a$ plane, contours of ζ_a can be mapped out and we thus have a known function ψ

$$\psi[\Pi, \beta_a, \zeta_a] = 0. \tag{6}$$

By shear stress profile matching we can then map out isosurfaces of ζ in $\Pi - \beta - S$ space and thus (4) is known. Calculating the evolution of the boundary layer is then possible by solving a coupled set of ODE's which come from the momentum integral equation, the law of the wall, law of the wake and the definitions of ζ and β, i.e.

$$\frac{dS}{dR_x} = \frac{\chi[R_x, K] R[S, \Pi, \zeta, \beta]}{S E[\Pi] \exp[\kappa S]}, \tag{7}$$

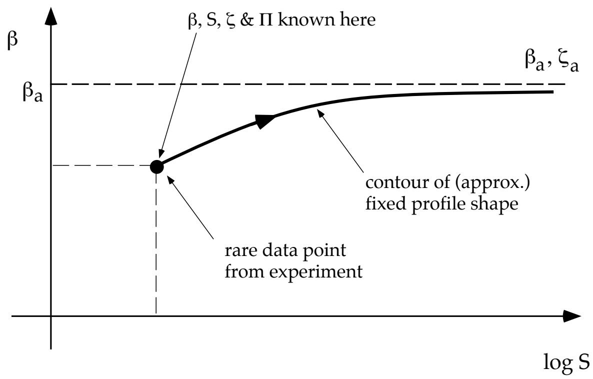

Figure 1: Contour of fixed shear stress profile shape for fixed Π.

$$\frac{d\Pi}{dR_x} = \frac{\zeta \chi[R_x, K]}{S^2 E[\Pi] \exp[\kappa S]}, \tag{8}$$

where using (2)

$$E[\Pi] = \exp[-\kappa A - 2\Pi + 1/3],$$

$$\begin{aligned} R &= \frac{S}{\kappa S^2 C_1 - \kappa S C_2 + C_2} \\ &+ \frac{\beta(2SC_1 - C_2)}{C_1(\kappa S^2 C_1 - \kappa S C_2 + C_2)} \\ &+ \frac{\zeta(\frac{dC_2}{d\Pi} - S\frac{dC_1}{d\Pi} - 2(C_2 - SC_1))}{\kappa S C_1 - \kappa S C_2 + C_2}, \end{aligned}$$

where

$$C_1[\Pi] = \int_0^\infty \frac{U_1 - U}{U_\tau} d\eta,$$

and

$$C_2[\Pi] = \int_0^\infty (\frac{U_1 - U}{U_\tau})^2 d\eta.$$

Here $R_x = xU_0/\nu$, $K = \nu/(LU_0)$ (often referred to as an acceleration parameter), $\chi[R_x, K] = U_1/U_0$ where U_0 is the value of the freestream velocity at some initial point $R_x = 0$ or $x = 0$ and U_1 is the freestream velocity at some general value of x or R_x.

The Clauser parameter β is known given K and the law of the wall, law of the wake formulation:

$$\beta = -C_1[\Pi] S^2 E[\Pi] \exp[\kappa S] K. \tag{9}$$

Therefore, in summary, the evolution of the flow can be computed using equations (7) and (8) where equations (4) and (9) are the necessary auxilary equations.

SINK FLOW

A sink flow turbulent boundary layer is one whose pressure gradient follows that of a two-dimensional potential sink. The flow is shown schematically in figure 2. Townsend (1956) and Rotta (1962) identified sink flow as the only smooth wall boundary layer that may evolve to a state of precise equilibrium for flows which are two-dimensional in the mean. A precise equilibrium layer is one where all mean and turbulence measurements are invariant with the streamwise direction, when they are scaled with the correct velocity and length scale.

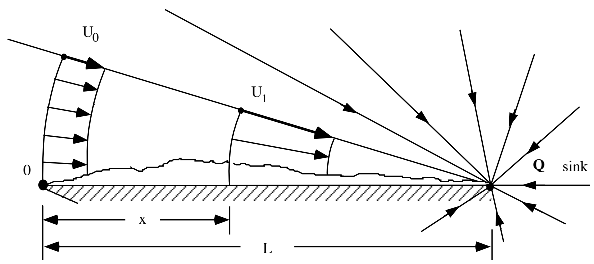

Figure 2: Sink flow.

We will be considering the experiment results which are described in the companion article by Jones *et al.* (1998) which appears in these conference proceedings. In this experimental study, 3 acceleration parameters (K) were investigated with a total of 62 mean-flow stations. The shear-stress profile matching technique as shown in figure 1 was used to find the corresponding β_a and ζ_a values for each experimental station. Figure 3 shows the corresponding β_a versus Π data.

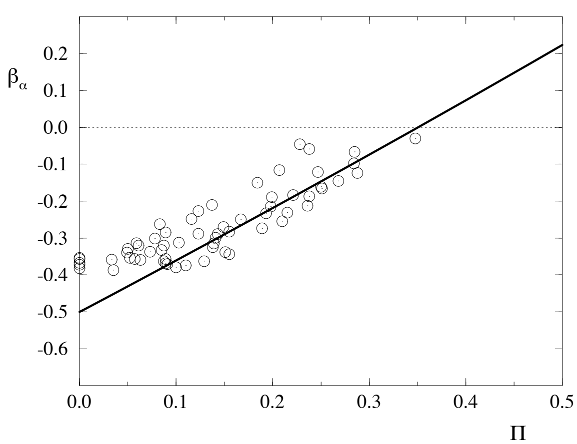

Figure 3: Data of Jones *et al.* (1998). Each point has a different value of ζ_a. Solid line corresponds to equation (10) for $\zeta_a = 0$.

The solid line shown in figure 3 corresponds to a curve fit for the $\zeta_a = 0$ data points with the functional form

$$\beta_{ae} = -0.5 + 1.38\Pi + 0.13\Pi^2. \qquad (10)$$

In order to calculate the evolution of this flow a functional relationship for (6) needs to be established. Using the sparse data available a tentative localised surface curve-fit is proposed:

$$\zeta_a = (0.85 - 6.9\Pi + 8\Pi^2)\Delta\beta_a \qquad (11)$$

where $\Delta\beta_a = \beta_a - \beta_{ae}$.

RESULTS AND DISCUSSION

Using (11), (9) and the least-squares-error shear-stress profile matching, formulation (4) can be described and thus equations (7) and (8) can now be used to compute the evolution of the boundary layer given any initial station where Π, S, β and ζ are known. Good agreement is seen between the experimental data and computation as shown in figure 4.

Although the calculation relies on equation (11), the good agreement seen in figure 4 indicates at least that the mathematical machinery is working correctly. It also indicates that equation (11) seems to be a good estimate for (6) in this restricted area of $\Pi - \zeta_a - \beta_a$ functional space. Solving for flows with arbitrary pressure gradient would rely on a more general and robust expression for (6). More experimental data is undoubtably needed to achieve this. The authors (Perry, Marusic & Jones 1998) have also investigated how theories such as the attached eddy hypothesis of Townsend (1976) might be used to help in the mapping out of $\Pi - \zeta_a - \beta_a$. This work is continuing.

To help determine the range in which equation (11) might be valid, a series a initial conditions were tried for the evolution of sink flow for a given K value. The results are shown in figure 5. Since the acceleration parameter K is constant for sink flow, equations (7) and (8) become autonomous, i.e. R_x need not appear explicitly with appropriate change of independent variable. This means that $S - \Pi$ is a phase plane and solution trajectories can only cross at critical points. As can be seen in the figure, solution trajectories converge to an equilibrium solution as expected, but we observe two such critical points (and a saddle point exists between the two stable nodes). The $\Pi = 0$ solution is the expected one and the other would seem to result when using equation (11) beyond its range of applicabliity.

CONCLUSIONS

A framework is described for formulating closure for a turbulent boundary layer evolving in an arbitrary streamwise pressure gradient. This involves using Prandtl's law of the wall and Coles' law of the wake together with the momentum integral and differential equations. Good agreement is seen with the experimental sink flow study of Jones *et al.* (1998).

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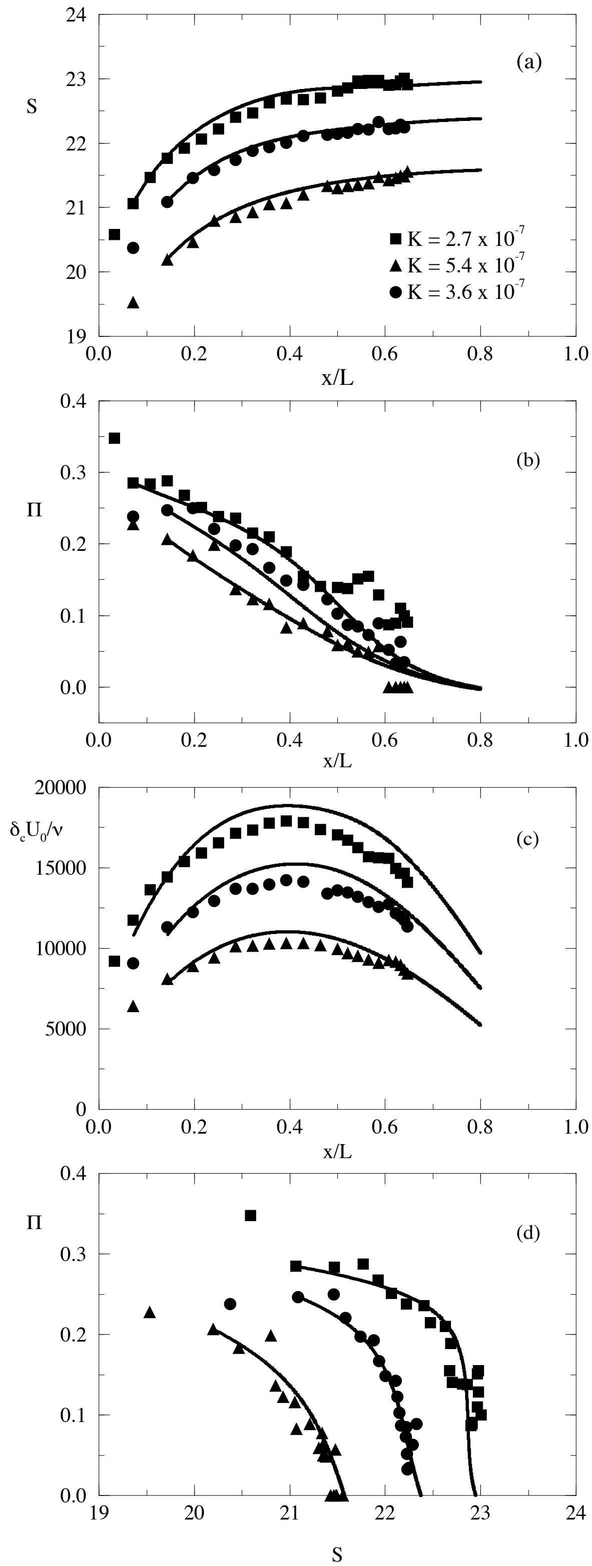

Figure 4: Evolution of mean flow parameters. Symbols represent data; solid lines correspond to calculation.

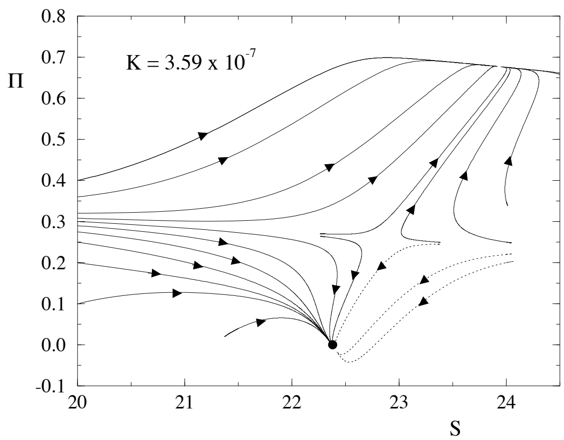

Figure 5: Solution trajectories for sink flow ($K = 3.59\times10^{-7}$) starting from different initial conditions.

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