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The invariants of the no-slip tensor in wall-bounded flows

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Abstract

The three invariants of the velocity gradient tensor have been used to study turbulent flow structures. For incompressible flow the first invariant P is zero and the topology of the flow structures can be investigated in terms of the second and third invariants. However, the invariants Q and R are zero at a no-slip wall and can no longer be used to identify and study structures at the surface in a wall-bounded flow. At the wall, the velocity field can be described by a no-slip Taylor-series expansion. Like the velocity gradient tensor, it is possible to define the invariants $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ of the no-slip tensor. In this paper flow structures on a no-slip boundary will be studied in terms of these invariants.

Introduction

Free-slip critical points

A critical point is a point in a flow field where the velocity $u_1 = u_2 = u_3 = 0$ and the streamline slope is indeterminate. Close to the critical point the velocity field u_i in x_i space is described by the linear terms of a Taylor-series expansion, i.e.

$$\begin{aligned} u_1 &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ u_2 &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ u_3 &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{aligned}$$

or

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\dot{x_i} = \frac{dx_i}{dt} = A_{ij}x_j \qquad (1)$$

where $A_{ij} = \partial u_i / \partial x_j$ is the velocity gradient tensor.

The invariants of the velocity gradient tensor

The invariants of the velocity gradient tensor have been used to study turbulent flow structures in order to extract information regarding the scales, kinematics and dynamics of these structures (e.g. Davidson 2004, pg. 268 and Elsinga & Marusic 2010). For an observer moving in a non-rotating frame of reference with any particle in a flow field, the flow surrounding the particle is described in terms of the nine components of the velocity gradient tensor A_{ij}.

The characteristic equation of A_{ij} is

$$\lambda_i^3 + P\lambda_i^2 + Q\lambda_i + R = 0, \qquad (2)$$

where λ_i are the eigenvalues and P, Q and R are the tensor invariants which are defined as

$$\begin{aligned} P &= -(A_{11} + A_{22} + A_{33}) \\ Q &= A_{11}A_{22} - A_{12}A_{21} + A_{11}A_{33} - A_{13}A_{31} \\ &\quad + A_{22}A_{33} - A_{23}A_{32} \qquad \text{and} \\ R &= A_{11}A_{23}A_{32} - A_{11}A_{22}A_{33} + A_{12}A_{21}A_{33} \\ &\quad - A_{12}A_{23}A_{31} - A_{13}A_{21}A_{32} + A_{13}A_{22}A_{31} \end{aligned} \qquad (3)$$

The characteristic equation (2) can have (i) all real roots which are distinct, (ii) all real roots where at least two roots are equal, or (iii) one real root and a conjugate pair of complex roots. In the $P-Q-R$ space, the surface which divides the real solutions from the complex solutions is given by

$$27R^2 + (4P^3 - 18PQ)R + (4Q^3 - P^2Q^2) = 0 \qquad (4)$$

For incompressible flow, the first invariant P is zero and hence all free-slip critical points can be described by the second and third invariants, i.e. Q and R respectively. An example in the use of these invariants in the study of turbulence using data from Direct Numerical Simulations of homogeneous isotropic turbulence is given in Ooi, Chong & Soria (1999).

No-slip critical points

In wall-bounded flow, there is no-slip at the wall, i.e. $u_1 = u_2 = u_3 = 0$ when $x_3 = 0$, where x_3 is the wall-normal direction. At the wall where the no-slip condition holds

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $A_{13} = \partial u_1/\partial x_3$ and $A_{23} = \partial u_2/\partial x_3$ [1]. All the invariants of the above tensor are zero at the wall and hence they cannot be used to map the topology of the flow pattern at a no-slip surface.

To satisfy the no-slip condition, the Taylor-series expansion at the wall (i.e. at $x_3 = 0$) can be written as

$$\begin{aligned} u_1 &= A_{13}x_3 + (\mathcal{A}_{11}x_1 + \mathcal{A}_{12}x_2 + \mathcal{A}_{13}x_3)x_3 \\ u_2 &= A_{23}x_3 + (\mathcal{A}_{21}x_1 + \mathcal{A}_{22}x_2 + \mathcal{A}_{23}x_3)x_3 \\ u_3 &= (\mathcal{A}_{31}x_1 + \mathcal{A}_{32}x_2 + \mathcal{A}_{33}x_3)x_3 \end{aligned}$$

Since $u_1 = u_2 = u_3 = 0$ at the wall (at $x_3 = 0$), we cannot integrate the velocity field at the wall to obtain the surface flow pattern or skin friction lines.

By defining $\overset{o}{x_i}$ as

$$\overset{o}{x} = \frac{dx_i}{d\tau} \qquad (5)$$

where $d\tau = x_3 dt$. the no-slip velocity field can be expressed as

$$\begin{aligned} \frac{u_1}{x_3} &= \overset{o}{x_1} = \frac{dx_1}{d\tau} = A_{13} + (\mathcal{A}_{11}x_1 + \mathcal{A}_{12}x_2 + \mathcal{A}_{13}x_3) \\ \frac{u_2}{x_3} &= \overset{o}{x_2} = \frac{dx_2}{d\tau} = A_{23} + (\mathcal{A}_{21}x_1 + \mathcal{A}_{22}x_2 + \mathcal{A}_{23}x_3) \\ \frac{u_3}{x_3} &= \overset{o}{x_3} = \frac{dx_3}{d\tau} = (\mathcal{A}_{31}x_1 + \mathcal{A}_{32}x_2 + \mathcal{A}_{33}x_3) \end{aligned}$$

[1] Wall shear stress or skin friction is defined as the normal derivative of the velocity vector at the wall. Hence the direction of the near-wall velocity vector when projected normal to the wall is in the same direction as the wall shear stress vector, i.e. surface streamlines are known as skin friction lines.

A critical point occurs on the wall when $A_{13} = A_{23} = 0$ and at this critical point

$$\begin{bmatrix} \overset{o}{x}_1 \\ \overset{o}{x}_2 \\ \overset{o}{x}_3 \end{bmatrix} = \begin{bmatrix} dx_1/d\tau \\ dx_2/d\tau \\ dx_3/d\tau \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\overset{o}{x}_i = \frac{dx_i}{d\tau} = \mathcal{A}_{ij} x_j \tag{6}$$

where $\mathcal{A}_{ij}$ will be referred to as the 'no-slip tensor'. The velocity field $\overset{o}{x}_i$ at the wall ($x_3 = 0$) is no longer zero and can be integrated in τ to generate surface streamlines (limiting streamlines or skin-friction lines).

The invariant of the no-slip tensor

To satisfy boundary conditions on a no-slip surface, the no-slip tensor is given by

$$\mathcal{A}_{ij} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & 0 & -\frac{1}{2}(\mathcal{A}_{11} + \mathcal{A}_{22}) \end{bmatrix} \tag{7}$$

By substitution of the above expansion into the Navier-Stokes equation, it can be shown that $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{21}$ and $\mathcal{A}_{22}$ are related to the vorticity gradients and $\mathcal{A}_{13}$ and $\mathcal{A}_{23}$ are related to pressure gradients. Like the velocity gradient tensor, the no-slip tensor has three invariants $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. For incompressible flow, the first invariant $\mathcal{P}$ is no longer zero and is given by

$$\mathcal{P} = -\frac{1}{2}\left(\mathcal{A}_{11} + \mathcal{A}_{22}\right). \tag{8}$$

It can also be shown that the relationship between the three invariants of the no-slip tensor is given by

$$2\mathcal{P}^3 + \mathcal{P}\mathcal{Q} + \mathcal{R} = 0. \tag{9}$$

TAYLOR-SERIES EXPANSIONS OF VELOCITY FIELDS

To investigate the fluid flow topology using the invariants of the velocity gradient tensor or the no-slip tensor, the technique described in Perry & Chong (1986) will be used to generate smooth three-dimensional vector fields which are local solutions of the Navier-Stokes equations. A brief description of the technique is given in the next section.

Taylor-series expansion solutions of the Navier-Stokes equations

To describe a complex three-dimensional flow field where there are a number of critical points, the velocity field can be represented by Taylor-series expansions of arbitrary order N as

$$u_i = \sum_{n=0}^{N} S(a_i, b_i, c_i) x_1^{a_i} x_2^{b_i} x_3^{c_i} \tag{10}$$

where $i = 1, 2, 3$, and x_i are the orthogonal spatial coordinates. (a_i, b_i, c_i) uniquely specify the terms in the expansion and the powers of x_1, x_2 and x_3 respectively.

The factor[2] S is given by

$$S = \frac{(a_i + b_i + c_i)!}{a_i! b_i! c_i!} \tag{11}$$

and

$$n = a_i + b_i + c_i \tag{12}$$

The above representation of the expansion for the velocity field is ideal for generating algorithms so that computer programs can be written to obtain Taylor-series expansion for the velocity, to arbitrary orders. Once generated, the expansions in the above notation can be translated to a more conventional notation. For example, the third order Taylor-series expansions for the velocity field in a conventional notation is given by

$$\begin{aligned}
u_1 &= P_1 + P_2x_1 + P_3x_2 + P_4x_3 \\
&+ P_5x_1^2 + P_6x_2^2 + P_7x_3^2 + 2P_8x_1x_2 + 2P_9x_1x_3 \\
&+ 2P_{10}x_2x_3 + P_{11}x_1^3 + P_{12}x_2^3 + P_{13}x_3^3 + 3P_{14}x_1^2x_2 \\
&+ 3P_{15}x_1^2x_3 + 3P_{16}x_1x_2^2 + 3P_{17}x_1x_3^2 + 3P_{18}x_2^2x_3 \\
&+ 3P_{19}x_2x_3^2 + 6P_{20}x_1x_2x_3 \\
u_2 &= Q_1 + Q_2x_1 + Q_3x_2 + Q_4x_3 \\
&+ Q_5x_1^2 + Q_6x_2^2 + Q_7x_3^2 + 2Q_8x_1x_2 + 2Q_9x_1x_3 \\
&+ 2Q_{10}x_2x_3 + Q_{11}x_1^3 + Q_{12}x_2^3 + Q_{13}x_3^3 + 3Q_{14}x_1^2x_2 \\
&+ 3Q_{15}x_1^2x_3 + 3Q_{16}x_1x_2^2 + 3Q_{17}x_1x_3^2 + 3Q_{18}x_2^2x_3 \\
&+ 3Q_{19}x_2x_3^2 + 6Q_{20}x_1x_2x_3 \\
u_3 &= R_1 + R_2x_1 + R_3x_2 + R_4x_3 \\
&+ R_5x_1^2 + R_6x_2^2 + R_7x_3^2 + 2R_8x_1x_2 + 2R_9x_1x_3 \\
&+ 2R_{10}x_2x_3 + R_{11}x_1^3 + R_{12}x_2^3 + R_{13}x_3^3 + 3R_{14}x_1^2x_2 \\
&+ 3R_{15}x_1^2x_3 + 3R_{16}x_1x_2^2 + 3R_{17}x_1x_3^2 + 3R_{18}x_2^2x_3 \\
&+ 3R_{19}x_2x_3^2 + 6R_{20}x_1x_2x_3
\end{aligned} \tag{13}$$

where the coefficients of the expansion for u_1, u_2 and u_3 are $P's$, $Q's$ and $R's$ respectively[3].

It can be shown that the number of unknown coefficients for a N^{th}-order expansion is given by

$$N_c = 3 \sum_{K=0}^{N} \sum_{J=0}^{K+1} J \tag{14}$$

The Navier-Stokes equation and continuity equation in tensor notation[4] is given by

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{15}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{16}$$

where p is 'kinematic' pressure $= p/\rho$ and ν is the kinematic viscosity.

Differentiating the velocity expansions (13), substitution into the continuity equation (16), and equating like powers of x_1, x_2 and x_3, relationships between the coefficients can be generated. Examples of such relationships are:

$$\begin{aligned}
P_2 + Q_3 + R_4 &= 0 \\
P_5 + Q_8 + R_9 &= 0 \\
P_8 + Q_6 + R_{10} &= 0 \\
&\dots \text{ etc}
\end{aligned} \tag{17}$$

[2] S was introduced in the original technique for generating the Taylor-series expansions and is retained in the analysis given in this paper - for details, see Perry & Chong (1986).

[3] Note that P's, Q's and R's are coefficients of the expansion for the velocity field and not to be confused with the invariants of the velocity gradient tensor.

[4] Using Einstein's notation where repeated indices implies summation over indices 1, 2 and 3.

For an N^{th}-order expansion, the number of these relationships is given by

$$E_c = \sum_{n=0}^{N} \sum_{J=0}^{n} J \tag{18}$$

By equating cross-derivatives of pressure of the Navier-Stokes equations, and by collecting terms for like powers of x_1, x_2 and x_3, the Navier-Stokes relationships can be generated. A typical example of a Navier-Stokes relationship for a fifth order expansion is given below:

$$\begin{aligned}
\dot{R_8} &- \dot{P_{10}} \\
+R_2P_8 + R_3Q_8 + R_4R_8 + R_5P_3 &+ R_6Q_2 \\
+R_8P_2 + R_8Q_3 + R_9R_3 + R_{10}R_2 &+ 3R_{14}P_1 \\
+3R_{16}Q_1 + 3R_{20}R_1 - P_2P_{10} - P_3Q_{10} &- P_4R_{10} \\
-P_6Q_4 - P_7R_3 - P_8P_4 - P_9P_3 &- P_{10}Q_3 \\
-P_{10}R_4 - 3P_{18}Q_1 - 3P_{19}R_1 - 3P_{20}P_1 & \\
-12\nu(R_{24} + R_{26} + R_{35} - P_{33} - P_{28} - P_{29}) &= 0
\end{aligned} \tag{19}$$

For an N^{th}-order expansion, the number of these relationships is given by

$$E_{NS} = \sum_{n=3}^{N} \sum_{J=2}^{n-1} (2J-1) \tag{20}$$

Note that for time-dependent flow, these relationships are ordinary differential equations (hence determining the dynamics of the flow). For steady flow problems these relationships, like the continuity relationships, are algebriac (kinematic) relationships.

The number of unknown coefficients for different order expansions and the number of continuity and Navier-Stokes relationships generated is given in the following table.

N	N_c	E_c	E_{NS}
0	3	0	0
1	12	1	0
2	30	4	0
3	60	10	3
4	105	20	11
5	168	35	26
.	.	.	.
.	.	.	.
15	2448	680	1001

In all cases the total number of relationships generated exceeds the total number of unknown coefficients of a given order expansion N, and hence further equations are needed for closure.

Example of a third order Taylor-series expansion solution

A third order solution for separated flow above a no-slip surface will be used to illustrate how a solution can be generated. In the following solution, x_1 is the flow direction, x_2 is the spanwise direction and x_3 is the wall-normal direction.

Applying the no-slip condition (all coefficients without x_3 are zero) and continuity, the third-order Taylor-series expansions for the velocity field given in (13) expansion are reduced to

$$\begin{aligned}
u_1 &= P_4x_3 + P_7x_3^2 + 2P_9x_1x_3 + 2P_{10}x_2x_3 + P_{13}x_3^3 \\
&+ 3P_{15}x_1^2x_3 + 3P_{17}x_1x_3^2 + 3P_{18}x_2^2x_3 + 3P_{19}x_2x_3^2 \\
&+ 6P_{20}x_1x_2x_3 \\
u_2 &= Q_4x_3 + Q_7x_3^2 + 2Q_9x_1x_3 + 2Q_{10}x_2x_3 + Q_{13}x_3^3 \\
&+ 3Q_{15}x_1^2x_3 + 3Q_{17}x_1x_3^2 + 3Q_{18}x_2^2x_3 + 3Q_{19}x_2x_3^2 \\
&+ 6Q_{20}x_1x_2x_3 \\
u_3 &= R_7x_3^2 + R_{13}x_3^3 + 3R_{17}x_1x_3^2 + 3R_{19}x_2x_3^2
\end{aligned} \tag{21}$$

To further simplify the problem, the flow is assumed to be symmetrical about the $x_1 - x_3$ plane, i.e. u_1 is even in x_2, hence $P_{10} = P_{19} = P_{20} = 0$ and u_2 is odd in x_2, hence $Q_4 = Q_7 = Q_9 = Q_{13} = Q_{15} = Q_{17} = Q_{18} = 0$. From continuity

$$R_{19} = -(P_{20} + Q_{18}) = 0$$

and the expansions simplify to

$$\begin{aligned}
u_1 &= P_4x_3 + P_7x_3^2 + 2P_9x_1x_3 + P_{13}x_3^3 \\
&+ 3P_{15}x_1^2x_3 + 3P_{17}x_1x_3^2 + 3P_{18}x_2^2x_3 \\
u_2 &= 2Q_{10}x_2x_3 + 3Q_{19}x_2x_3^2 + 6Q_{20}x_1x_2x_3 \\
u_3 &= R_7x_3^2 + R_{13}x_3^3 + 3R_{17}x_1x_3^2
\end{aligned} \tag{22}$$

The above expansion has to satisfy continuity, i.e.

$$\begin{aligned}
P_9 + Q_{10} + R_7 &= 0 \\
P_{15} + Q_{20} + R_{17} &= 0 \\
P_{17} + Q_{19} + R_{13} &= 0
\end{aligned} \tag{23}$$

For a steady solution of the Navier-Stokes equation,

$$R_{17} - P_{15} - P_{18} - P_{13} = 0 \tag{24}$$

The no-slip velocity field is given by

$$\begin{aligned}
\overset{o}{x}_1 = u_1/x_3 &= P_4 + P_7x_3 + 2P_9x_1 + P_{13}x_3^2 \\
&+ 3P_{15}x_1^2 + 3P_{17}x_1x_3 + 3P_{18}x_2^2 \\
\overset{o}{x}_2 = u_2/x_3 &= 2Q_{10}x_2 + 3Q_{19}x_2x_3 + 6Q_{20}x_1x_2 \\
\overset{o}{x}_3 = u_3/x_3 &= R_7x_3 + R_{13}x_3^2 + 3R_{17}x_1x_3
\end{aligned} \tag{25}$$

Further equations can be generated by specifying the location

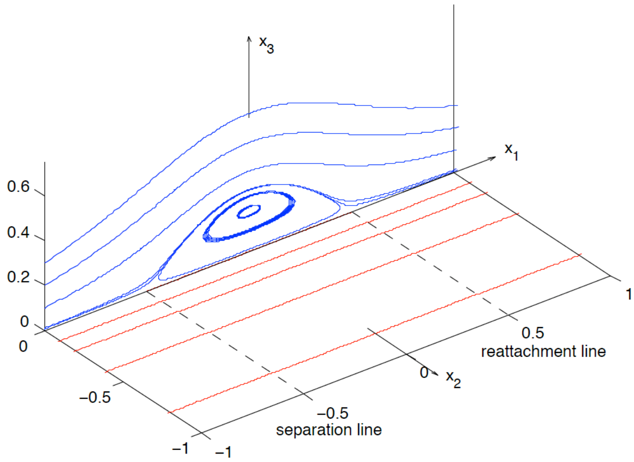

Figure 1: Classical two-dimensional separation bubble

and properties of critical points in the flow field. For example, in generating a separated flow pattern, the properties of separation/reattachment points can be expressed as

$$\begin{bmatrix} \overset{o}{x}_1 \\ \overset{o}{x}_2 \\ \overset{o}{x}_3 \end{bmatrix} = \begin{bmatrix} a & 0 & B \\ 0 & na & 0 \\ 0 & 0 & -\frac{1}{2}a(n+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{26}$$

For an angle of separation/separation θ, B is given by

$$B = \frac{-\frac{1}{2}a(n+3)}{tan\theta} \tag{27}$$

Solutions

Using a few simple parameters, various three-dimensional flow patterns can be generated. For example, if we assume that the factor n is zero for the separation and reattachment points, the classical two-dimensional separation bubble as shown in figure 1 can be generated. In this solution, the separation point is located on the surface ($x_3 = 0$) at $x_1 = -0.5$ and the reattachment point is located at $x_1 = 0.5$. In this solution, $K = 0.5$. Note that $u_2 = 0$ and on the surface, we have a separation/reattachment line at $x_1 = \pm 0.5$, i.e. along these lines the no-slip velocity is zero and all points along these lines are critical points.

By changing the values of n for the separation and reattachment points, we can generate three-dimensional separated flow such as that shown in figure 2. This has been called a 'U-separation' by Hornung & Perry (1984).

The invariants of the vector field

Figure 3 shows the second and third invariants of the velocity gradient tensor, Q and R respectively, at various points located in the streamwise-spanwise plane ($x_1 - x_2$ plane, $x_3 = 0.7$) for the flow pattern shown in figure 2. Here the first invariant P is zero. Since the invariants are zero at the wall, Q and R must diminish as the wall is approached. At the wall ($x_1 - x_2$ plane, $x_3 = 0$) , when Q and R are zero, the topology must be described by the invariants $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ of the no-slip tensor $\mathcal{A}_{ij}$. The second and third invariants, $\mathcal{Q}$ and $\mathcal{R}$ at various points at the wall ($x_1 - x_2$ plane, $x_3 = 0$) are shown in figure 4. Here $\mathcal{P}$ is not zero.

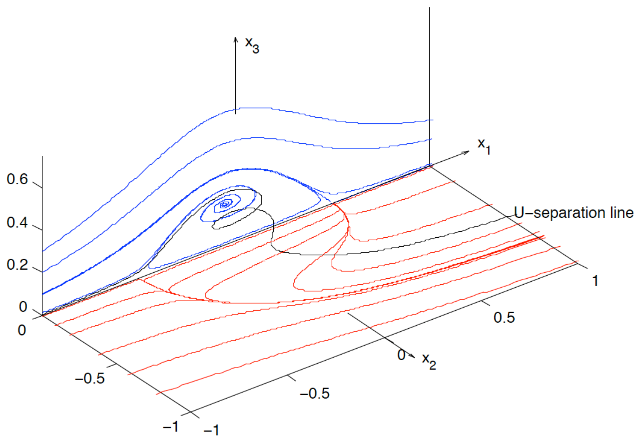

Figure 2: Streamlines for a U-separation flow.

Conclusions

A general method for studying no-slip critical points has been outlined. Other flow patterns can be generated and provide an excellent method for studying the topology of flow patterns close to a no-slip wall. This is the focus of future work.

Acknowledgements

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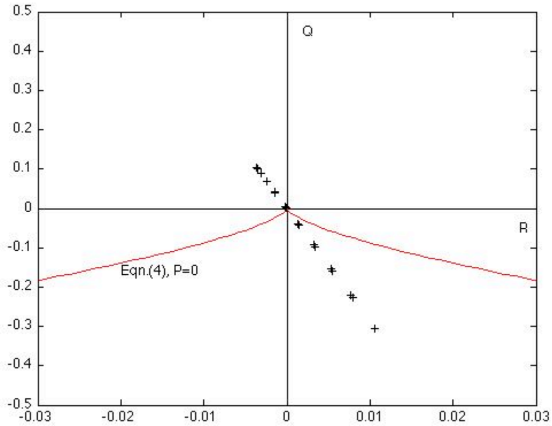

Figure 3: The second and third invariants of the velocity gradient tensor A_{ij}, Q and R respectively, at various points on a $x_1 - x_2$ plane, $x_3 = 0.7$. The first invariant P is zero.

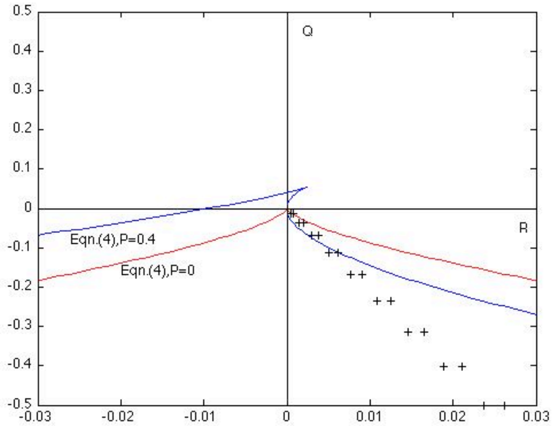

Figure 4: The second and third invariants of the no-slip tensor $\mathcal{A}_{ij}$, $\mathcal{Q}$ and $\mathcal{R}$ respectively, at various points on a $x_1 - x_2$ plane, $x_3 = 0$. The first invariant $\mathcal{P}$ is not zero for all these points. In the above figure P, Q and R should be labelled as $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ respectively.

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