Towards a Statistically Accurate Wall-Model for Large-Eddy Simulation

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Abstract
This study proposes to investigate the development of a new generation of wall-models for large-eddy simulation (LES) based on the observation that large-scale motions occurring in a turbulent boundary layer leave their “footprint” in the vicinity of the wall. Following these observations, Marusic et al. [8, 11] proposed a novel conceptual model able to predict wall shear-stress fluctuation with a high level of fidelity. Making use of a numerical/experimental database covering two decades of Reynolds numbers, a series of a priori tests is proposed here to highlight the benefits of this new approach comparatively to a standard log-law approach. A correction of the model is finally proposed to extend its domain of application with the perspective of implementing it in a LES solver.

Introduction
Despite many decades of study, modelling and predicting the wall shear stress in wall-bounded flows remains an active topic of research. The advent of the standard law-of-the-wall, considerably improved this field of research leading to accurate models for the mean friction. However, many configurations of interest (fatigue/rupture issues on air-plane wings, wind gusts in meteorology and environmental flows) would benefit from an accurate prediction of the wall shear-stress fluctuation.

The use of large-eddy simulation (LES) is very promising for this purpose, but Baggett et al. [2] estimated that the number of grid points necessary to accurately resolve a channel flow with LES scales as \(Re_c^2\) (Reynolds number based on the boundary layer thickness \(\delta\), and the friction velocity, \(\nu = \sqrt{\tau / \rho}\), where \(\tau\) stands for the mean wall shear-stress streamwise component, and \(\rho\) the density at the wall assumed constant here). This consideration helps to understand that the cost of a wall-resolved LES rapidly becomes prohibitive, and approximate boundary conditions must be used to model the near wall region [13]. This big challenge is being to render the simulation cost dependent on an integral length scale, i.e. a fraction of \(\delta\).

From the first attempt of Schumann [16], the use of approximate boundary conditions failed to predict reliably the fluctuating wall-shear stress, mainly because most of the modelling approaches developed to date rely on the standard log-law [17] which provides a good prediction of the mean friction without being able to mimic the spectral content of wall-bounded turbulent flows. Indeed, as depicted in figure 1, it is now well accepted [5] that two main energetic ranges of scales interact within the flow: an inner peak occurring near the wall, around \(y^+ \approx 15\), and an outer peak appearing in the middle of the logarithmic region \((y_m \approx \sqrt{TSRe_c})\) which becomes more prominent as the Reynolds number increases [6, 18, 5]. Note that in this paper \(y\) denotes the wall-normal direction, \(u\) the streamwise velocity, and \(+\) superscript a quantity scaled in wall units.

Many authors analysed the topology of these two prominent scales and their interaction appears to be well described by a superposition mechanisms [1, 5], and an amplitude modulation effect [3, 9].

Recently, Marusic et al. et al. [8, 11] proposed a novel conceptual approach embedding the superposition/modulation mechanisms into an inner-outer scale interaction (IOSI) model able to predict the high-order statistics and the spectral content of wall-bounded turbulent flows.

The purpose of this paper is to verify whether this promising approach is well-posed for developing a new generation of boundary conditions for LES. As such, a high fidelity database covering a wide range of Reynolds numbers (934 \(\leq Re_c \leq 22884\)) is used to perform a priori tests. This database gathers results from direct numerical simulation (DNS) of channel flow by del Alamo et al. [4] (Re\(c = 934\)) and TBL measurements by Kulandaivelu [7] (Re\(c \in \{2740; 3514; 4228; 4816; 5885; 8159; 10111; 13320; 17775; 22884\})

Wall-Model Formulation

The database is first used to analyse the wavelength associated to each peak as a function of Reynolds number. Figure 2 clearly shows that the inner peak wavelength scales in wall units \((\lambda_{x, inner} \approx 1000\) which renders its resolution very demanding when the Reynolds number increases. Contrarily, the outer peak tends to scale with the boundary layer thickness \((\lambda_x, outer \approx 4\delta)\), which is large enough to be resolved by a LES but with the need for a wall modeling procedure that scales with an integral length scale.

Figure 1: Example of a premultiplied energy spectrogram of streamwise velocity fluctuation in a turbulent boundary layer.

Figure 2: Wavelength associated to the inner peak (top) and outer peak (bottom) as a function of Reynolds number.
This implies that a viable model should somehow embed the spectral content of the inner peak, which is strongly feasible as many studies suggest that the inner peak region \( (\lambda_x < 7000; y^+ < 30) \) presents a universal pattern for any kind of wall-bounded flows \([12]\). The model for wall shear stress fluctuation formulated by Mathis et al. \([11]\) relies on this assumption. It is formulated as follows:

\[
\tau^+ = \tau^+ \left[ 1 + \alpha \left( u''_1^+ \right) \right] + \alpha \left( u''_1 \right) + \tau^+ \left( \right),
\]

where prime denotes a fluctuating quantity (i.e. whose mean value was subtracted), 1-subscript quantities are evaluated at the 1st off-wall point \( i.e. u''_1^+ \equiv \tau''(y^+_1) \), \( (\cdot) \) represents a filtered quantity (subgrid scale filtering in the sense of LES), \( \alpha = 0.1 \) the superposition/modulation coefficient, and \( \tau^+ \) the so-called universal fluctuating wall shear-stress embedding the inner peak features. The time-wise signal used for \( \tau^+ \) in what follows was extracted from the DNS by del Álamo et al. \([4]\) making use of the procedure advocated by Mathis et al. \([11]\).

Nevertheless, this model is limited to the fluctuating wall shear-stress and an estimate of the mean, \( \tau \), is also needed. The Reichardt \([14]\) equation is used to this purpose:

\[
\bar{u}_1^+ = \frac{1}{k} \ln \left( 1 + \kappa y^+_1 \right) + R_1 \left( 1 - e^{-y^+_1 / R_2} - \frac{y^+_1}{R_2} e^{-y^+_1 / R_3} \right),
\]

where \( u_1^+ \) is the mean streamwise velocity evaluated at the 1st off-wall grid point, \( \kappa = 0.41 \) the von Kármán constant, \( R_1 = 7.375, R_2 = 10 \) and \( R_3 = 3 \). In the context of wall-model for LES, the only unknown of eq. \( (2) \) is hidden behind \( \text{ quantities, namely the mean wall friction } \tau \). Note that the original Reichardt law was developed with \( R_1 = 7.8 \) and \( R_2 = 11 \) but the modifications proposed here give a better collapse on the recent database generated in the last decade.

In the forthcoming sections, the designation “standard wall-model” will refer to as the use of equation \( (2) \) only, and “new wall-model” to the combination of equations \( (1) \) and \( (2) \).

**Wall-Model Performance**

**Potential and Optimal Prediction**

In their original study, Mathis et al. \([11]\), developed the model in an ideal framework, namely: 1) the filter cutoff length is chosen to be exactly between the two peaks, \( \lambda_{x,c} = 7000 \); 2) the outer time-wise signal is taken exactly in the middle of the logarithmic layer, where the peak of large-scale streamwise turbulent energy is maximum \([10]\), \( y^+_1 = \sqrt{\tau \Lambda \tau} \). Using this optimal set of parameters, Mathis et al. \([11]\) have shown that this new model was able to reproduce the standard deviation and the autospectral density of the wall shear-stress fluctuation.

Here, the same set of parameters are used to highlight the optimal potential of the model. As shown in figure 3, the model is not only able to reproduce the standard deviation, but the full probability density function (PDF). As the wall friction is very difficult to measure experimentally with a sufficient spectral accuracy at very high Reynolds numbers, the detailed PDF of reference is only plotted for the available DNS. Note that a standard wall-model is not able to reproduce the expected PDF. Moreover, recalling that the 2nd-order moment and the autospectral density are related together (the integral of the the autospectral density corresponds to the variance), it is interesting to note in figure 4 that the spectral content of the wall shear-stress is also predicted with a good level of fidelity. By definition, a standard model cannot rebuild information whose wavelength is lower than the filter cutoff, and this is why no energy is predicted under \( \lambda_{x,c} = 7000 \). Finally the last test illustrating the potential of this novel model is proposed in figure 5 where the autospectral density is shown as a function of Reynolds number. The model appears to capture the emergence of the outer peak “footprint” near the wall.

![Figure 3: Probability density function of wall shear-stress fluctuation predicted by the new wall-model and a standard approach.](image1)

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![Figure 4: Premultiplied autospectral density of wall shear-stress fluctuation as a function of Reynolds number and streamwise wavelength.](image2)

**Figure 4:** Premultiplied autospectral density of wall shear-stress fluctuation as a function of Reynolds number and streamwise wavelength

**Influence of the Subgrid Scale Filter Size**

The wall normal distance of the input signal is now fixed to its optimum \( y^+_1 = \sqrt{\tau \Lambda \tau} \) and the influence of the filter wavelength is presented in figure 6 for the standard deviation, skewness, and kurtosis (the definition used here is such that a Gaussian distribution has a kurtosis of 3). As expected from the results shown in figure 3, the standard model is not able to capture properly the Reynolds number dependency. Surprisingly, the new model is slightly sensitive to the change in the cutoff wavelength, especially when it remains consistent with the model formulation, i.e. chosen between the inner and outer peak wavelengths \( (1000 \leq \lambda_{x,c} \leq 4 \tau \Lambda \tau) \). The Reynolds number trend

![Figure 5: Map of premultiplied energy spectra of wall shear-stress fluctuation predicted by the new wall-model and a standard approach.](image3)
seems to be preserved, particularly for the standard deviation. A break in the trend of the skewness and kurtosis can be observed but conclusions cannot be easily drawn at such Reynolds numbers because of the lack of accurate data of reference.

Finally, figure 7 depicts the error between the predictions of both the new model and the standard one, relatively to the trend proposed by Schlatter & Örlü [15] for standard deviation. The error committed by the standard model is striking contrarily to the new one. This has two important implications for using the model in a LES: 1) the grid spacing does not have to perfectly respect the optimal value \( \lambda_{c,x} = 7000 \); 2) even if a good cutoff wavelength should remain between the two peaks (\( \lambda_{c,x} \leq 20000 \)), it is still an achievable number in terms of grid spacing for high Reynolds number simulations.

**Figure 6:** Standard deviation, skewness and kurtosis of wall shear-stress fluctuation predicted by the new model and a standard approach (wall normal location fixed to \( y^+ = \sqrt{15} Re_c \)). Available DNS data by Schlatter & Örlü [http://www.mech.kth.se/pschllatt/DATA/] and del Alamo et al. [4] are plotted as reference. The function proposed by Schlatter & Örlü [15] for standard deviation is also shown.

**Figure 7:** Error in the prediction of the wall shear-stress standard deviation \( \langle \tau_{rms,predicted}^+ \rangle \) relatively to the function proposed by Schlatter & Örlü [15] \( \langle \tau_{rms,SO}^+ \rangle = 0.018 \ln(Re_c) + 0.298 \). The relative error is defined as \( \langle \tau_{rms,predicted}^+ \rangle / \langle \tau_{rms,SO}^+ \rangle \).

### Influence of the 1st Off-Wall Grid Point Location

Here we fix the cutoff wavelength to its optimal value (\( \lambda_{c,x} = 7000 \)) and vary the location of the input signal inside the boundary layer. The results shown in figure 8, indicate that the optimal prediction occurs at the middle of the log-layer as expected by Mathis et al. [11]. When the Reynolds number increases, the discrepancies become more obvious and this could potentially cause some issues for LES as it is often difficult to control the location of the 1st off-wall grid point, especially in complex configurations.

### Improvement of the Wall-Model

Equation (1) can be rewritten in order to separate the wall shear-stress and the outer signal contributions:

\[
\langle \tau^+ - \tau^o \rangle / (1 + \tau^o) = \alpha \langle \tau^+ \rangle.
\]

(3)

From this equation, it is easy to understand that the energy of the predicted wall shear-stress fluctuation will be preserved whatever is the wall normal location of the input signal if:

\[
\alpha \sqrt{\langle u^+ \rangle^2} = \text{constant, } \forall y^+, \quad (4)
\]

where the constant can be evaluated from the optimal value taken at the middle of the logarithmic layer, \( \alpha \sqrt{\langle u_{ml}^1 \rangle^2} \). \( \alpha \) being the original coefficient proposed by Mathis et al. [11], and \( u_{ml}^1 \), the fluctuating streamwise velocity at the middle of the log-layer. Hence, one can build up an equation for \( \alpha \) that embeds the wall-normal variation of the input large-scale signal:

\[
\alpha_i = C \alpha, \quad \text{with } C = \frac{\sqrt{\langle u_{ml}^1 \rangle^2}}{\sqrt{\langle u_{ml}^1 \rangle^2}}. \quad (5)
\]

The evaluation of the correction coefficient \( C \) is shown in figure 9 for a wall-normal distance situated in the logarithmic layer \( (100 \leq y^+_i \leq 0.15 Re_c) \).

**Figure 8:** Wall shear-stress standard deviation predicted by the new approach as a function of the 1st off-wall grid point location (filter cutoff set to \( \lambda_{c,x} = 7000 \)). All predictions are made with the optimal cutoff wavelength, \( \lambda_{c,x} = 7000 \), and shown for the whole experimental dataset \((2740 \leq Re_c \leq 22884)\).

**Figure 9:** Evolution of the correction parameter as a function of the wall-normal distance normalised by the location of the middle of the logarithmic layer, \( y_{ml} = 1.15 Re_c \).
Over the range of Reynolds numbers investigated, a common trend stands for $C$. This allows us to reformulate the original model in order to incorporate the wall-normal distance of the input signal:

$$\tau^+ = \tau^* \left[ 1 + C \alpha \langle u_1^+ \rangle + C \alpha \langle u_1^+ \rangle \right],$$  \hspace{1cm} (6)

where

$$C = 1 + 0.05 \left( \ln(y_1^+)/\nu \right)^2,$$  \hspace{1cm} (7)

and $\alpha$ is the original coefficient proposed by Mathis et al. [11]. Figure 10 shows that this correction allows us to extend the domain of application as long as the $1^\text{st}$ off-wall grid point is situated inside the logarithmic layer. This also improves the prediction of higher order moments (not shown here).

![Figure 10: Wall shear-stress standard deviation with and without correction of the superposition/modulation coefficient $\alpha$. All the predictions are made with the optimal cutoff wavelength, $\lambda_{cml}^+ = 7000$, and shown for the whole experimental dataset ($2740 \leq \text{Re}_\tau \leq 22884$). The first point of the profile corresponds to the beginning of the logarithmic layer ($y_1^+ = 100$).](image)

**Conclusions**

This paper discusses the potential of an inner-outer scale interaction (IOSI) model for turbulent boundary layers and puts into perspective its implementation in a LES solver. Contrarily to a standard wall-model, this novel approach is not only able to reproduce the high-order moments of the fluctuating wall-shear-stress with a good level of fidelity, but also their Reynolds number dependency. *A priori* tests over two decades of Reynolds numbers highlight the well-posed nature of the model for LES, and an improvement is proposed to extend its range of application. Still, the conclusions drawn in this study must be balanced by two remarks. First, there is a lack of spectrally accurate data of wall shear-stress fluctuation at high Reynolds number which could inform/confirm the $\text{Re}_\tau$-trends observed. Second, in the framework of wall-models for LES, conclusions brought by *a posteriori* tests will always be more trustful than the ones from *a priori* tests. Even so, the *a priori* tests presented here suggest that this novel approach exhibits promising properties to develop new post-processing tools for wall-bounded turbulent flows.

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**References**


