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Flow of Viscoelastic Liquid Through Circular Annulus

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ABSTRACT

In this paper the flow of viscoelastic liquid through the region bounded by two concentric circular cylinders, under the influence of exponential decreasing pressure gradient has been investigated. For a slowly decaying pressure gradient when material constants are small the flow properties corresponds to that of a Newtonian fluid, in case of impulsive type pressure gradient the importance of relaxation time cannot be neglected. Velocity of the flow is expressed in terms of Bessel and modified Bessel function.

INTRODUCTION

Non-Newtonian liquids such as blood, thick oils pastes, paints, colloid solutions are highly viscous. Their behaviour cannot be explained by the classical Hydrodynamic stress-rate strain relations. Generalising the stress-rate of strain relations of classical Hydrodynamics, the rhelogical behaviour of the non-Newtonian liquids have been studied by Rivlin [8], Rivlin and Reiner [9]. Langlois and Rivlin [4] have studied slow steady state flow of viscoelastic fluids through non-circular tubes. Rivlin [10] has discussed some exact solutions of viscoelastic fluids. Dutta [1] has obtained the solutions for viscoelastic Maxwell fluid through a circular annulus. Jones and Walters [2,3] have discussed the oscillatory motion of viscoelastic liquid. Nand Lal Singh [6] studied unsteady flow of a viscoelastic fluid between two parallel planes under periodic pressure gradient. In view of the considerable interest being evinced at present in the field, it was considered worthwhile to study the flow of viscoelastic liquid specified by three constants, through circular annulus under the influence of exponential pressure gradient. Expressing velocity of the flow in terms of Bessel and modified Bessel functions, two interesting cases have been studied.

EQUATIONS OF MOTION

The equations of motion together with stress-rate of strain relations of viscoelastic liquids, characterised by three material constants a viscocity coefficient and two relaxation times under the approximation of small rates of strain are given by

$$\tau^{ij} = -p\, g^{ij} + \tau'^{ij} \tag{1}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau'^{ij} = 2\eta_0\left(1 + \lambda_2 \frac{\partial}{\partial t}\right) e^{ij} \tag{2}$$

$$e_{ij} = (V_{i,j} + V_{j,i})/2 \tag{3}$$

$$\rho\left(\frac{\partial V^i}{\partial t} + V^i{}_{,j}\, V^j\right) = \tau^{ij}{}_{,j} \tag{4}$$

$$V^i{}_{,i} = 0 \tag{5}$$

Operating by $(1 + \lambda_1 \frac{\partial}{\partial t})$ on equation (4) and using equations (1) and (2) we have

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial V^i}{\partial t} + V^i{}_{,j}\, V^j\right) = -\frac{1}{\rho}\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \times p\, g^{ij}{}_{,j} + 2\gamma_0\left(1 + \lambda_2 \frac{\partial}{\partial t}\right) e^{ij}{}_{,j} \tag{6}$$

where τ^{ij} and τ'^{ij} denote stress and deviatoric stress tensors, V^i the components of velocity, g^{ij} are contravariant components of metric tensor, e_{ij} the strain rate of deformation, p the pressure ρ the density, γ Kinematic viscosity the coefficients η_0, λ_1, λ_2 are material constants, subject to conditions (such as $\eta_0 > 0$, $\lambda_1 \geqslant \lambda_2 \geqslant 0$) dictated by thermodynamic principles. It was pointed out by Oldroyd [7] that for a liquid at rest any small shearstress decays as e^{-t/λ_1} and in a liquid element free from stress any small rate of strain decays as e^{-t/λ_2}. Michael C. Williams and R. Byron [5] have shown that varies from 1/9 to 2/3.

FORMULATION AND SOLUTION OF PROBLEM

We shall investigate the flow of viscoelastic liquid described by (1) to (5) equations through the region bounded by two concentric circular cylinders of radii a and b ($b < a$) under the influence of exponential pressure gradient. Using the cylindrical coordinates system (r, θ, z), the components of velocity are given by

$$V_1 = 0,\ V_2 = 0,\ V_3 = V_z(r,t) \tag{7}$$

Using equations (3), (4), (6) and (7) it follows that

$$-\frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \tag{8}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial V_z}{\partial t} = -\frac{1}{\rho}\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial z} + \gamma_0\left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right] \tag{9}$$

where $\gamma_0 = \frac{\eta_0}{\rho}$ is kinematic viscosity. From equations (8) and (9) it follows that

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \phi(t) \tag{10}$$

Since we have assumed the pressure gradient to be exponential, we can take

$$\phi(t) = \alpha e^{-m^2 t} \tag{11}$$

where a and m are real constants. In this case we can assume

$$V_z(r,t) = f(r)\, e^{-m^2 t} \tag{12}$$

where $b \le r \le a$

Using relations (11) and (12) in equation (9) we have

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \frac{m^2}{\gamma_0}\frac{(1-\lambda_1 m^2)}{1-\lambda_2 m^2}\left(f + \frac{\alpha}{m^2}\right) = 0 \tag{13}$$

Taking r/a = R equation (13) can be expressed in the form

$$\frac{d^2 f}{dR^2} + \frac{1}{R}\frac{df}{dR} + \eta^2\left(f + \frac{\alpha}{m^2}\right) = 0 \tag{14}$$

where

$$\eta^2 = \frac{m^2 a^2}{\gamma_0}\left(\frac{1-\lambda_1 m^2}{1-\lambda_2 m^2}\right) \tag{14a}$$

the solution of differential equations (14) is given by

$$f(R) = A\, J_0(\eta R) + B\, Y_0(\eta R) - \frac{\alpha}{m^2} \tag{15}$$

where J_0 and Y_0 are respectively, Bessel functions of first and second kind and of order zero [11]. A and B are constants to be determined subject to the following boundary conditions.

$$f(1) = 0; \quad f(\sigma) = 0 \tag{16}$$

where

$\sigma = b/a$

Using the boundary conditions (16) we have,

$$A = \frac{\alpha}{m^2}\left[\frac{Y_0(\eta\sigma) - Y_0(\eta)}{Y_0(\eta\sigma)\, J_0(\eta) - Y_0(\eta)\, J_0(\eta\sigma)}\right]$$

$$B = \frac{-\alpha}{m^2}\left[\frac{J_0(\eta\sigma) - J_0(\eta)}{Y_0(\eta\sigma)\, J_0(\eta) - Y_0(\eta)\, J_0(\eta\sigma)}\right] \tag{17}$$

Substituting for A and B in equation (15) we have,

$$f(R) = \frac{\alpha}{m^2}\left[J_0(\eta R)\left\{\frac{Y_0(\eta\sigma) - Y_0(\eta)}{Y_0(\eta\sigma)J_0(\eta) - Y_0(\eta)J_0(\eta\sigma)}\right\} - Y_0(\eta R)\left\{\frac{J_0(\eta\sigma) - J_0(\eta)}{Y_0(\eta\sigma)\, J_0(\eta) - Y_0(\eta)J_0(\eta\sigma)}\right\} - 1\right] \tag{18}$$

Now we shall discuss two cases of small and large values of η.

Case (1): For slowly decaying pressure gradient m is small, it follows from (14a) η is small and for small values of λ_1, λ_2 the flow corresponds to unsteady viscous incompressible Newtonian flow. We have the following asymptotic expansions [11].

$$J_0(\eta R) \simeq 1 - \frac{\eta^2 R}{4}; \quad Y_0(\eta R) = \left(1 - \frac{\eta^2 R^2}{4}\right)\log(\eta R) + \frac{\eta^2 R^2}{4}$$

Using the above relations, equation (18) simplifies to

$$f(R) = -\frac{\eta^2 \alpha}{m^2}\left[\frac{(\sigma^2-1)\log R + (1-R^2)\log\sigma}{\eta^2(\sigma^2-1) - \eta^2(1+\sigma^2)\log\sigma + \log\sigma^4}\right] \tag{19}$$

Therefore the velocity of the flow in this case, using (12) is given by

$$V_z(r,t) = \frac{\eta^2\alpha}{m^2}\left[\frac{(\sigma^2-1)\log R + (1-\sigma^2)\log\sigma}{\eta^2(\sigma^2-1) - \eta^2(1+\sigma^2)\log\sigma + \log\sigma^4}\right] \times e^{-m^2 t} \tag{20}$$

Case (2): When m is large and as and are different

$$\eta^2 \simeq \frac{-m^2 a^2}{\gamma_0}\,\frac{\lambda_1}{\lambda_2} = -\eta'^2$$

The effect of relaxation times cannot be neglected. The solution of differential equation (14) can be expressed as

$$f(R) = C\, I_0\,(\eta' R) + D\, K_0\,(\eta' R) - \alpha/m^2 \qquad (21)$$

Where I_0 and K_0 are modified Bessel functions of first and second kind of zero order [11]. C and D are constants to be determined using the boundary conditions (16). Using the boundary conditions we obtain

$$C = -\frac{\alpha}{m^2}\left[\frac{K_0(\eta'\sigma) - K_0(\eta')}{I_0(\eta')K_0(\eta'\sigma) - I_0(\eta'\sigma)\,K_0(\eta')}\right]$$

$$D = -\frac{\alpha}{m^2}\left[\frac{I_0(\eta'\sigma) - I_0(\eta')}{I_0(\eta')\,K_0(\eta'\sigma) - I_0(\eta'\sigma)\,K_0(\eta')}\right] \qquad (22)$$

when $\eta' \gg 1$ we have the following asymptotic expressions [11]

$$I_0(\eta' R) \approx \frac{e^{\eta' R}}{\sqrt{2\pi\,\eta' R}}\,; \quad K_0(\eta' R) \approx e^{-\eta' R}\sqrt{\pi/2R\eta'}$$

Substituting for C and D from equation (22) and using the above asymptotic values of modified Bessel functions equation (21) becomes

$$f(R) = -\frac{\alpha}{m^2}\left[\frac{\mathrm{Sinh}\,\eta'(\sigma-R) + \sqrt{\sigma}\,\mathrm{Sinh}\,\eta'(R-1)}{\sqrt{R}\,\mathrm{Sinh}\,\eta'\,(\sigma-1)} - 1\right] \qquad (23)$$

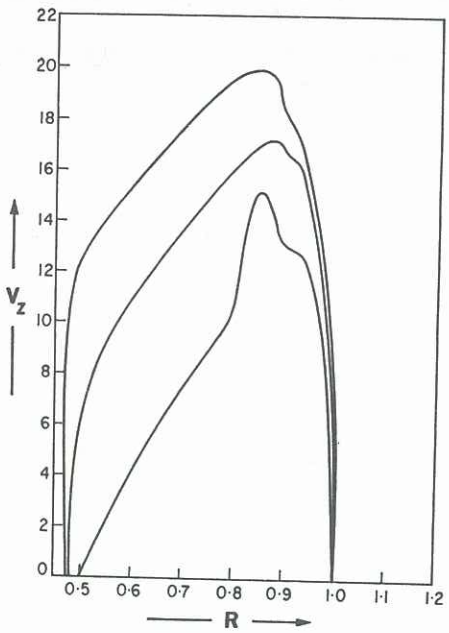
Therefore the velocity V_z is given by

$$V_z(R,t) = \frac{\alpha}{m^2}\left[\frac{\mathrm{Sinh}\,\eta'(\sigma-R) + \sqrt{\sigma}\,\mathrm{Sinh}\,\eta'(R-1)}{\sqrt{R}\,\mathrm{Sinh}\,\eta'(\sigma-1)} - 1\right] e^{-m^2 t} \qquad (24)$$

This is the velocity of the fluid particle in the present case. If we take $\lambda_1 \to 0$, $\lambda_2 \to 0$ in the equations (20) and (24), the solution of the problem of the flow of an ordinary viscous liquid through circular annulus can be deduced as a special case of this investigation.

GRAPH

A graph for different annular regions is plotted for $\lambda_1 = 1/100$, $\lambda_2 = 1/300$, $\sigma = 1/2$, $1/3$, $1/4$, $\eta_0 = 10^{-6}$, m = 1 and $\alpha = 1$. It is found that as the ratio of cross sectional radii decreases the velocity increases, obtaining maximum round about 0.85.



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