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Quasilinear Unsaturated Soil-Water Movement: Scattering Analogue, Infiltration and Watertightness of Cavities and Tunnels

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ABSTRACT

For a given moisture range, the quasilinear analysis involves only two soil parameters, α, the sorptive number, and, in most practical applications, K_0, the saturated conductivity. The formulation preserves integral capillary flow properties of the soil, yielding relatively simple solutions amply accurate for engineering applications in the field. The scattering analogue augments the power of the method. It makes available to the soil-water context powerful techniques and many extant scattering results developed in fields such as electromagnetics and acoustics, and yields solutions of many multidimensional infiltration problems. Other applications concern the penetration, or otherwise, of unsaturated seepage water into cavities and tunnels. The analysis has been extended to periodic and unsteady flows.

1. QUASILINEAR ANALYSIS

The concepts underlying the modern physical theory of water flow in unsaturated soil are due to a keen disciple of Willard Gibbs, Edgar Buckingham (1907). The general flow equation which follows (Richards, 1931) is a highly nonlinear Fokker-Planck equation, central to the treatment of unsaturated water movement as a quantitative predictive field of physical science (Philip, 1969, 1970, 1972a).

Philip (1968) showed that representing the dependence of hydraulic conductivity K (dimensions [length][time]$^{-1}$) on moisture potential Ψ (dimensions [length]) in the form

$$K(\Psi) = K_0 e^{\alpha(\Psi-\Psi_0)} \tag{1}$$

reduces the steady flow equation to the *linear*

$$\nabla_*^2\Theta = \alpha\partial\Theta/\partial z_* , \tag{2}$$

where the Kirchhoff potential

$$\Theta = \int_{\Psi_1}^{\Psi} K\, d\Psi . \tag{3}$$

Here z_* is the vertical physical space coordinate, positive downward, and ∇_*^2 is the Laplacian in the physical coordinates. This formulation applies to flow systems with $-\infty \leqslant \Psi_1 \leqslant \Psi \leqslant \Psi_0 \leqslant 0$. We call the analysis based on (2) *quasilinear* because it embodies with reasonable accuracy the nonlinearity of $K(\Psi)$ and also accommodates the nonlinear dependence of Ψ on moisture content.

The prescription

$$\alpha = [K(\Psi_0)-K(\Psi_1)] \left\{ \int_{\Psi_1}^{\Psi_0} K(\Psi)d\Psi \right\}^{-1} \tag{4}$$

[exact when (1) is exact] is optimal for computing α for infiltration from arbitrary three-dimensional cavities into an infinite region (Philip, 1985a, 1987).

Note from (1) that α has the dimensions [length]$^{-1}$; and (4) shows that α^{-1} is a K-weighted mean moisture potential. Thus α^{-1} is a potential characteristic of the unsaturated flow process. The quantity $2\alpha^{-1}$ is the *sorptive length* and α is the *sorptive number*. Note (Philip, 1987) that α is directly proportional to a characteristic capillary pore radius R. In fact

$$R = 7.4 \times 10^{-6}\alpha \tag{5}$$

when R is in m and α in m^{-1}. The value of α ranges from 0.2 m^{-1} or less in fine-textured soils to 5 m^{-1} or more in coarse ones.

The quasilinear analysis needs only two soil parameters, K_0 and α. In many practical problems $\Psi_0 = 0$ so K_0 is the saturated conductivity, and α tends to be virtually independent of Ψ_1, so long as it is large and negative, as in dry landscapes. Direct field determination of K_0 and α is under active study (Philip, 1985b, 1986a; White et al., 1986).

2. CONVENTIONAL METHODS OF SOLVING (2)

Solving (2) for buried and surface point and line sources (Philip, 1969, 1971; Raats, 1971) gave Green's functions leading to solution of many problems involving flux boundary conditions. Often, however, practical infiltration problems involve water supply at $\Psi = \Psi_0$, and we seek the resulting flow rate and moisture distribution. Other than an approximate result based on the point source (Philip, 1968), the first solution of this class, involving dual integral equations, was due to Wooding (1968).

Separation of variables has been used to obtain exact solutions of (2) for infiltration from cylindrical and spherical cavities (Philip, 1984a,b). Typical is the solution for the sphere

$$H = \left[\frac{\pi}{sr}\right]^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n(2n+1)I_{n+\frac{1}{2}}(s) \frac{K_{n+\frac{1}{2}}(sr)}{K_{n+\frac{1}{2}}(s)} P_n(\cos\theta) . \tag{6}$$

Here I_n and K_n are modified Bessel functions of the first and second kinds of order n, and P_n the Legendre polynomial of the first kind of degree n; r is the spherical radial coordinate, normalized with respect to cavity radius a, θ is colatitude, and $r \cos\theta = z = z_*/a$. We have also that

$$s = \tfrac{1}{2}\alpha a,\ H = \frac{\Theta}{\Theta_0} e^{-sz},\ \Theta_0 = \int_{\Psi_1}^{\Psi_0} K d\Psi , \tag{7}$$

with Ψ_0, Ψ_1 respective values of Ψ at the cavity surface and at infinity. The dimensionless parameter s is a measure of the relative importance of capillarity and gravity. As $s \to 0$, capillarity dominates the flow process; and as $s \to \infty$, gravity dominates. Results following from solutions like

(6) include Q_*, the total cavity discharge. For the sphere we have the typical result

$$Q_* = 4\pi^2\alpha^{-1}\Theta_0 \sum_{n=0}^{\infty} (-1)^n(2n+1) \frac{I_{n+\frac{1}{2}}(s)}{K_{n+\frac{1}{2}}(s)} \tag{8}$$

Superficially these results are satisfactory. Problems arise, however, in summing the series entering equations such as (7) and (8). Convergence is rapid and summation simple for $0 \leqslant s \leqslant 2$; summation is still possible for $s = 5$, but the practical limit is about $s = 10$. The range $0 \leqslant s \leqslant 10$ covers many applications, but considerable interest attaches to the distribution of Θ, and the value of Q_* as gravity dominates the flow process ($s \to \infty$). Numerical work suggested asymptotic results for s large typified by that for the sphere

$$Q_* \approx 4\pi\alpha^{-1}\Theta_0 s^2(1+2s^{-2/3}). \tag{9}$$

This approach, however, fails to explain the (typical) remarkable result that the second term on the right of (9) is just $2s^{-2/3}$ times the first; and it gives no information on the horizontal transition from (near) saturation to (near) dryness in regions below the cavity.

The corresponding problems for elliptic-cylindrical (including strip-shaped) and spheroidal (including disc-shaped) cavities lead to solutions involving series of Mathieu functions, modified Mathieu functions with negative parameter, and spheroidal wave functions either with imaginary parameter or imaginary argument. Putting aside the formidable tasks of evaluating many of these, we are still left with the problem that exact solutions for these geometries lead to convergence difficulties as severe as those for the circular cylinder and the sphere.

These various solutions thus left unanswered the question, important in many engineering contexts, of the role of capillary effects in seepage problems with large length scales. Conventional means of solving (2) had thus led to something of an *impasse*.

3. THE SCATTERING ANALOGUE

These obstacles were largely removed when Waechter and Philip (1985) recognized the exact analogue between steady quasilinear flows and the scattering of plane waves. The analogy is a striking example of the economical approach to nature afforded by mathematical physics. Beyond its aesthetic appeal, the analogue leads to saving of effort and to new hydrological insights. A considerable body of established results on wave scattering (e.g. Bowman et al., 1969), and the associated specialized mathematical methods, became immediately available to the soil-water field. Expansions for small s prove useful, but the asymptotic methods and results for large s are of greatest importance. These, developed in contexts such as acoustics and electromagnetics, yield remarkably accurate results "even far outside the range where they should be good", in the words of Joe Keller.

Substituting (7) transforms (2) to the dimensionless

$$\nabla^2 H - s^2 H = 0 , \tag{10}$$

similar in form to the dimensionless reduced wave equation

$$\nabla^2 F + k^2 F = 0 . \tag{11}$$

Here ∇^2 is the Laplacian in normalized coordinates, F is the normalized potential, and k is the dimensionless wave number. The boundary conditions on (11) governing the scattering of plane waves incident in the z-direction on an acoustically soft obstacle are simply *minus* those on (10) governing infiltration from a geometrically identical cavity. The solutions of the two systems are thus identical provided

$$F \equiv -H, \; k \equiv is . \tag{12}$$

This is the basis of the analogue.

Far-field scattering functions are useful in scattering theory (Bowman et al., 1969). The 3-dim. function is

$$\underline{S}(\theta,\psi) = \lim_{r\to\infty} kr\, e^{-ikr} F(r,\theta,\psi) , \tag{13}$$

with ψ the longitude. An important theorem (Van de Hulst, 1949) connects the total extinction cross-section σ_e with the *forward scattering function* – in 3-dim. $\underline{S}_0$ (i.e. $\underline{S}$ for $\theta = 0$). The 3-dim. theorem is

$$\sigma_e = \frac{4\pi}{k} \operatorname{Im} \underline{S}_0 . \tag{14}$$

The analogous *far-field wetting functions* in the soil-water context (Philip, 1985c) are also useful. The 3-dim. function

$$S(\theta,\psi) = \lim_{r\to\infty} sr\, e^{sr} H(r,\theta,\psi). \tag{15}$$

Philip (1985c) showed that, analogous to the scattering theorem, the physical cavity flow Q_* is connected to the *downward wetting function* – in 3-dim. S_0 (i.e. S for $\theta = 0$). The 3-dim. theorem yields

$$Q_* = 8\pi\alpha^{-1}\Theta_0 S_0 . \tag{16}$$

The scattering functions appropriate to many 2- and 3-dim. obstacle shapes are available in the literature, not only their exact forms but also as expansions appropriate to small and large values of k. The expansions are especially useful when the exact solutions involve exotic functions. The asymptotic expansions are needed in all cases with s so large that summation of the exact solution is impractical. The connexions between scattering and wetting functions then enable us to exploit the numerous known scattering results to yield results on cavity flows in the soil-water context. Typically, we find, applying (12), (14), and (16) to the (corrected) asymptotic expansion for $\underline{S}_0(k)$ for the sphere due to Wu (1956) that the rigorous version of (9) is

$$\begin{aligned} Q_*(s) = 4\pi\alpha^{-1}\Theta_0 s^2[&1 + 1.99230638\, s^{-2/3} \\ &+ 0.71529966\, s^{-4/3} - 0.06093150\, s^{-2} \\ &+ 0.0145506\, s^{-8/3} - 0.014886\, s^{-10/3} \\ &+ O(s^{-4})] . \end{aligned} \tag{17}$$

We see that the term in $s^{-2/3}$ is not an empirical accident, but is rooted in the asymptotics based on Watson transforms. Jones (1963) gave physical arguments indicating that asymptotic expansions for all elliptical-cylindrical and ellipsoidal cavities involve $s^{-2/3}$ (though *not* strips and discs). Note that the exact coefficient of $s^{-2/3}$ is not 2, but is about 0.4% less.

The analogue enables us to identify various regions of the soil-water distribution about a cavity corresponding to the optical regions about an illuminated opaque obstacle. Figure 1 is a schematic diagram of flow from circular cylindrical and spherical cavities in which we use the terminology of the optical analogy. The illuminated region is the analogue of that which remains dry. The umbra (or deep shadow) corresponds to the essentially saturated region of the soil, and the penumbra corresponds to the region where infiltration from the cavity significantly wets the soil. It is convenient to define the umbra as the region with $0.99 \leqslant \Theta/\Theta_0 \leqslant 1$ and the penumbra as that with $0.01 \leqslant \Theta/\Theta_0 \leqslant 0.99$.

Figure 1 deals specifically with circular cylinders and spheres, but similar results hold for cavities of other shapes.

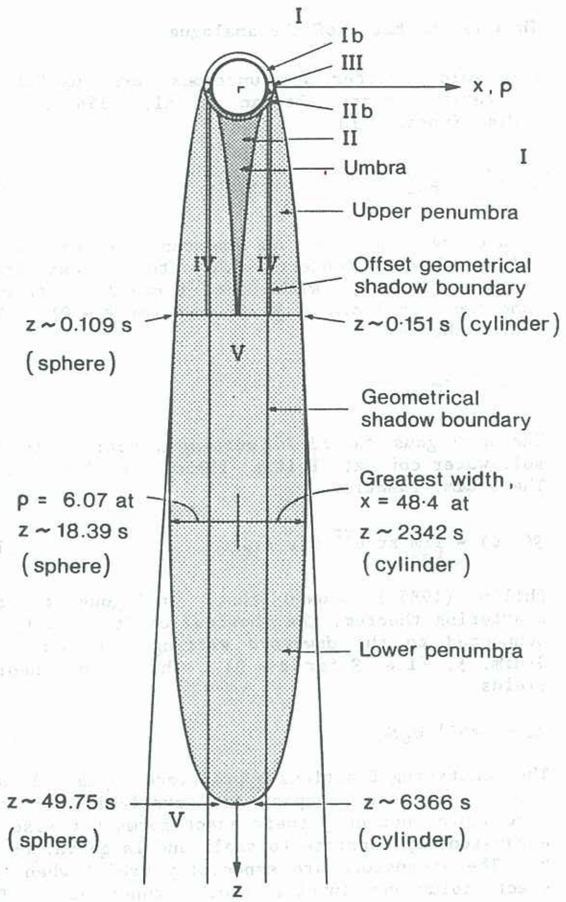

Figure 1: Steady infiltration from cylindrical and spherical cavities. Schematic diagram illustrating the regions of the flow field, using the terminology of the optical analogue. See Waechter and Philip (1985) for details.

Engineers frequently ignore capillary effects in seepage problems with large length scales (i.e. with large s). In terms of the optical analogy, ignoring capillarity is equivalent to using simple geometrical optics, in which an opaque obstacle illuminated by plane waves casts a coherent shadow projected to infinity without scattering. Note that for steady infiltration from a cavity, the "gravity-only" solution consists simply of a saturated column extending to infinity vertically beneath the cavity, with no wetting outside the column. Such a solution is singular in the sense that it gives a wholly misleading picture of the extent and character of the wetted region.

The analogue has been used to analyze steady infiltration from buried disc-shaped (Philip, 1986a) and spheroidal (Philip, 1986b) cavities. It may be shown, both physically and mathematically, that the importance of capillarity relative to gravity in producing the flow from a spheroidal source increases monotonically with aspect ratio (ratio of vertical to horizontal dimensions). This result leads to design principles for permeameters used in unsaturated soils. Measurements with the borehole permeameter (Stephens and Neuman, 1982; Philip, 1985b), with large aspect ratio, are confused by strong capillarity effects. On the other hand, these are minimal with the surface disc permeameter (Philip, 1986a; White et al., 1986), with aspect ratio zero.

4. WATER ENTRY INTO CAVITIES AND TUNNELS

A second class of unsaturated flow problems of practical concern is also amenable to the quasilinear analysis. The common picture of the effect of holes on unsaturated soil-water flow is that they take no part in the flow except when they extend to a source of free water. Otherwise, the conventional argument goes, they are in regions of negative water pressure, and water from the surrounding soil cannot enter them. J.H. Knight, R.T. Waechter, and I have embarked on studies which reveal that this is not necessarily so. We have been analyzing steady downward seepage in a uniform soil, interrupted at some level by a hole. The dry hole behaves, of course, as an obstacle to the flow. We have calculated the build-up of water pressure at the walls of the hole. The criterion for seepage of water into the hole is that the pressure at some point on the walls reaches that of the local soil atmosphere. The larger the hole, the more prone it is to have water seeping through it. This is rather obvious, but contrary to the conventional picture where the larger the hole, the less it can be expected to have water in it.

The applications are numerous. Firstly, the solutions illuminate the role of macropores in unsaturated flow. Secondly, they apply to caves, tunnels, and underground storage cavities. It is of especial interest that the results lead to design criteria for most efficient prevention of seepage into tunnels and cavities in unsaturated zones. The engineering contexts will be obvious. They include optimal design of underground repositories for nuclear wastes in deep unsaturated seepage zones in arid areas. Essentially the same mathematical problem arises when we replace the hole by a buried impermeable obstacle such as a stone, a rock, or an underground structure. The solutions yield the distribution of flow velocity and moisture content about such obstacles.

Solutions for the circular-cylindrical and spherical cavities show that, in these cases, the crucial point of water entry is the topmost point of the cavity roof. I am grateful to J.H. Knight for allowing me to present Figure 2, a map of the distribution of Θ and of streamlines, computed for the cylinder for s = 4.

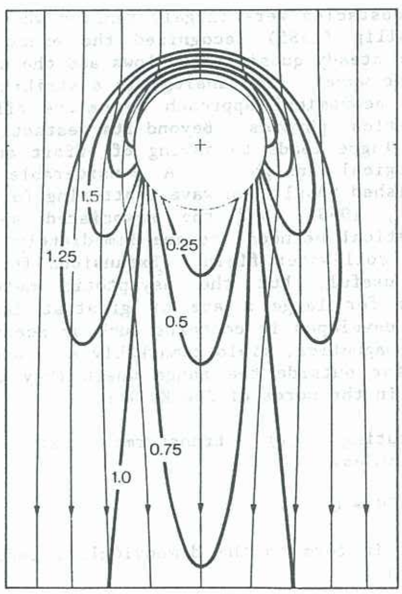

Figure 2: Steady unsaturated seepage about cylindrical cavities. Map of Kirchhoff potential Θ and stream function for dimensionless radius s = 4. Bold curves are equipotentials, with the numerals values of Θ/Θ_0 with Θ_0 the potential at infinity. Unnumbered equipotentials are for Θ/Θ_0 = 2, 3, 5. Fine curves with arrows are streamlines.

The solutions for the parabolic-cylindrical and paraboloidal cavities are of especial interest. They are in simple exact form and numerical values are readily calculable for the full parameter range. A notable feature is that the cavity wall is, for these geometries, not only a stream surface, but also a surface of constant Θ. These shapes constitute separatices between "sharper" shapes where water entry, if any, is near the base of the cavity, and "blunter" ones (like the circular cylinder and the sphere) where water entry is initiated at the roof apex. Our results indicate that flat-topped cavities are inefficient and readily admit water from unsaturated seepage; and that v-shaped roofs perform even worse.

5. EXTENSIONS

Various extensions of the quasilinear analysis have been made. Limited generalizations to heterogeneous (Philip, 1972b, 1974) and anisotropic (Philip, 1986b) soils are possible; and the analysis also forms a useful basis for study of periodic (Philip, 1987) and unsteady (Philip, 1969, 1986c; Warrick, 1974) unsaturated flows.

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