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Upstream Influence and Separation in Flow Past Obstacles on a β-Plane

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ABSTRACT

Flow past a cylindrical obstacle in an enclosed channel is examined when the entire configuration is rotating rapidly about an axis which is aligned with that of the obstacle. When viewed from a frame of reference which is rotating with the channel, Coriolis forces dominate and act to constrain the motion so that it is two-dimensional. In this paper a uniform flow is forced past the obstacle in a channel which has depth varying linearly across its width. The latter, which is equivalent to the so-called β-plane approximation for geophysical flows, permits waves to travel away from the obstacle, forming a lee-wave train behind it. If the dissipation in the system is sufficiently small, some waves can also travel large distances upstream and modify the oncoming uniform flow. Numerical solutions are presented and compared with previous theoretical results which are available in certain limits of the governing parameters. The possibility of flow separation off certain types of obstacles is also considered.

INTRODUCTION

The study of flows observed from a rotating frame of reference is clearly of relevance to large-scale motion in the oceans and atmosphere, where Coriolis forces due to the earth's rotation play a significant role in the dynamics. Although both the ocean and atmosphere are density-stratified, much can be gained from studying the motion of homogeneous fluid under similar circumstances, because this can model the depth-averaged motion of those bodies. The next step in the study of large-scale geophysical flows is to include effects due to the latitudinal variation of apparent rotation rate, and this is usually modelled by the so-called β-plane approximation which represents the variation linearly in mid-latitudes. The resulting spatial gradient of Coriolis force permits a form of wave motion which travels horizontally within the system; these waves known as Rossby waves, are dispersive and have the unusual property that the longitudinal component of the phase velocity is always directed westwards. The group velocity can, however, travel in any direction and some care is required on 'open' boundaries of the flow domain.

In this paper several important features of β-plane flows will be examined by considering the relatively simple model of an uniform flow past a cylindrical obstacle. Two different shaped obstacles will be studied: a circular cylinder and a symmetric aerofoil, both with a uniform cross-section over the depth of the fluid. The properties of rotating flows, under appropriate conditions, ensure that the fluid motion is dominantly two-dimensional and depth-independent, and this leads to significant simplifications of the analysis. As is common in laboratory experiments, the β-plane effects will be modelled by a linear variation in depth of the channel across its width. This can be shown to be equivalent, analytically, to a linear variation in rotation rate across the channel (Greenspan, 1970) and therefore the flows in this study are equivalent to zonal flows in the oceans and atmosphere. The channel is also considered to be of finite width and for computational reasons, outlined later, the channel in this study will be wider near the obstacle than at the inflow and outflow regions. In the laboratory experiments by Boyer and Davies (1982) the channel is of uniform width and the obstacle is a circular cylinder. The extension to an infinite-width channel is feasible in principle, but a continuous set of allowable wave-modes must be accounted for. Note that the channel walls in both this study and the experiments can reflect waves.

The flow is studied over a wide parameter range in which inertial, β-plane and Ekman-suction effects are important. The last of these introduces a dissipative mechanism, through friction in the Ekman layers on the top and bottom boundaries. The complete flow can be described by two dynamical parameters, with the effects of the geometrical parameters being generally less important provided the channel is not too narrow.

As mentioned earlier, the presence of β-effects allows Rossby waves to propagate within the system and these are apparent in an unsteady flow past an obstacle. The oncoming stream tends to advect most of these downstream but there will always be some waves with a large enough group velocity to propagate upstream of the obstacle. Thus, when the flow becomes steady, as it always will after a sufficient time, the only waves remaining are the stationary waves with their phase speed equal to the local flow speed. These form a lee-wave train behind the obstacle, similar to that shown in Miles and Huppert (1968), and they can also significantly modify the oncoming uniform stream. The latter effect is similar to that described for internal waves by Baines (1977), and in the absence of dissipation it can have a profound effect on the upstream boundary conditions. In particular, the results in Miles and Huppert (1968), which are based on the so-called Long's hypothesis, are deficient in that regard.

Another factor which should be taken into account, especially when comparing the experimental results with those obtained analytically, is that the viscous boundary layers which are neglected in the analysis can separate from the obstacle and significantly distort the flow. This is similar to the flow separation seen in non-rotating flows at high Reynolds number, and it has been studied extensively in uniformly rotating, or f-plane, flows where β-effects are neglected. In that situation it is found that once inertial effects begin to dominate Ekman-dissipation, the $E^{\frac{1}{4}}$ layers on the obstacle can separate and enclose a finite region of stagnant fluid (Page, 1986). Once a β-plane approximation is included the same situation can arise, but the critical ratio of inertia to dissipation will depend on the magnitude of the β-effect, rather like it depended on the height of the topography in Page (1982). The effects of $E^{\frac{1}{4}}$ layer separation are not included explicitly in this study, but the criterion for separation will be examined to determine whether they might be important. For bodies with a bluff trailing-edge, such as the circular cylinder, they will be found to be significant.

The plan of this paper is to first introduce the governing equations and the important flow parameters, then outline the theory which has previously covered certain parameter regimes. Numerical results for two different shaped obstacles will be shown and compared with that theory, then the possibility of flow separation will be examined.

GOVERNING EQUATIONS

Consider a cylindrical obstacle of typical length- and width-scale l^*, which is mounted vertically in a closed channel of infinite length, typical width w^* and average depth d^*. The channel is filled with a homogeneous fluid of density ρ^* and kinematic viscosity ν^*, which is forced through one end of the channel with a typical velocity U^*. If the entire configuration is rotating about a vertical axis with a uniform angular velocity Ω^*, and viewed from that rotating frame, then two important dynamical parameters arise

$$Ro = \frac{U^*}{\Omega^* l^*} \quad \text{and} \quad E = \frac{\nu^*}{\Omega^* d^{*2}}, \tag{1}$$

known as the Rossby and Ekman numbers, respectively. For quasi-steady flow with Ro and E small the Taylor-Proudman theorem applies and the flow is dominantly two-dimensional and depth-independent, with the pressure acting as a streamfunction.

Nondimensionalising velocities with U^*, lengths with l^*, times with $(\Omega^*)^{-1}$ and reduced pressure with $\rho^* U^* \Omega^* l^*$, the equations for the horizontal components of the velocity become (Greenspan, 1968)

$$\frac{\partial u}{\partial t} + Ro\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] - 2v = -\frac{\partial P}{\partial x} + d^2 E \nabla^2 u \tag{2}$$

$$\frac{\partial v}{\partial t} + Ro\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] + 2u = -\frac{\partial P}{\partial y} + d^2 E \nabla^2 v \tag{3}$$

with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \; . \tag{4}$$

As a result of the Taylor-Proudman theorem, terms involving w or $\partial/\partial z$ can be neglected to leading order and this leads to the definition of a streamfunction ψ with

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \tag{5}$$

so that $\psi \equiv \frac{1}{2}P$. Eliminating P from (4) and (5), and neglecting the diffusion terms leads to the equation

$$\frac{\partial \zeta}{\partial t} + Ro\left[u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right] = 2\frac{\partial w}{\partial z} \tag{6}$$

for the vertical component of vorticity, $\zeta = \nabla^2 \psi$. The term on the right-hand side can be determined from detailed analysis of the Ekman layers on $z=\tan\beta y$ and $z=d$, where β is the slope of the channel bottom and $d=d^*/l^*$, and this gives that

$$\frac{\partial w}{\partial z} = -E^{\frac{1}{2}}\zeta - \frac{\tan\beta}{d} v \tag{7}$$

(see, for example, Roberts and Soward, 1978). Putting this all together leads to the equation

$$\tau\frac{\partial \zeta}{\partial t} + \lambda\left[\frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x}\right] = -\zeta - 2\alpha\frac{\partial \psi}{\partial x} \tag{8}$$

where $\lambda = Ro/2E^{\frac{1}{2}}$ and $\alpha = \tan\beta/dE^{\frac{1}{2}}$ are the two important scaled parameters. Both are taken to be $O(1)$ numbers in this study. Note that τ in (8) is the Ekman-dissipation timescale $\tau = \frac{1}{2}E^{-\frac{1}{2}}$.

The coupled equation $\nabla^2\psi = \zeta$ and (8) can be solved once initial values of ψ and ζ are specified, and in this study $\psi = -y$ and $\zeta = 0$ at $t=0$. The equations are then integrated numerically with the boundary condition ψ=constant on solid surface and radiation conditions on open boundaries.

LINEAR THEORIES

For $\lambda \ll 1$ the equation (8) is linear and relatively easy to solve exactly for the steady flow past a cylindrical obstacle (Johnson and Page, 1987). For $\alpha=0$ the flow is irrotational and as α is increased the velocities tend to decrease on the upstream side of the cylinder and increase on the downstream side. As $\alpha\to\infty$ this becomes particularly pronounced, with a stagnant region of fluid forming in front of the cylinder, extending a distance of order $\alpha^{\frac{1}{2}}$ upstream, and an narrow jet, with width of $O(1/\alpha)$, moving around the cylinder on the lee-side. Outside of these regions, the flow is uniform, as is illustrated in Figure 1. Details of the asymptotics for this flow are given in

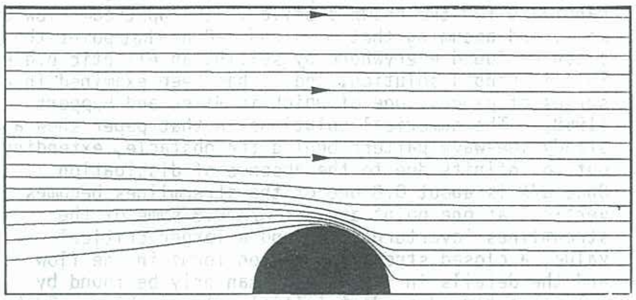


Figure 1. Streamlines for flow past a circular cylinder for $\lambda=0$ and $\alpha=20$ showing the blocked region on the upstream side and 'western-boundary current' on the downstream side.

Foster (1985) and these are in accord with both the exact and numerical solutions presented in Johnson and Page (1987). One important feature apparent from the asymptotics is that the tangential velocity on the downstream side of the cylinder is $v = -\frac{1}{2}\alpha\sin 2s$ for $\frac{1}{2}\pi \leqslant s \leqslant \pi$, where s is the arclength measured from the front stagnation point, so that the velocities in the 'western-boundary current' are of order α. As a result the derivative dv/ds will be large and negative near the rear stagnation point, and once λ is of order $1/\alpha$ the possibility of $E^{\frac{1}{4}}$-layer separation, such as occurs in Page (1982), must be considered. Foster (1985) incorrectly concludes that separation will be suppressed in that limit but in fact a closer examination of his solution reveals that it will be enhanced.

Another feature apparent in the solution in Figure 1 is that there is significant, even dominant, upstream influence in the flow-field. The 'stagnant' region extends a large distance upstream for $\alpha \gg 1$, and this region was formed through the propagation of Rossby waves away from the cylinder. Substituting a harmonic solution into (8) for $\lambda=0$ indicates that Rossby waves do form, but that they decay in amplitude on a time scale of order τ through Ekman dissipation.

Similar effects to those described above for the circular cylinder can be expected for flow past an aerofoil, and in general they are less problematic, particularly near the rear of the obstacle. Details are given in Johnson and Page (1986).

NONLINEAR THEORIES

Some progress can be made with nonlinear theories in special cases of the parameters λ and α. The first of these is for α large and λ of order $1/\alpha$, examined to some extent by Foster (1985). In this limit nonlinear effects are restricted to the 'western-boundary current' region in the lee of the cylinder and Foster shows that the flow in this layer breaks down once $\alpha\lambda > 1/8$ for $\alpha \gg 1$. The same restriction is extended to finite values of α in Johnson and Page (1987), by examination of the flow near the rear stagnation point, but it is not yet apparent what form of solution can be expected for $\alpha\lambda > 1/8$. Since this is a detail which is restricted to the class of bluff-ended bodies it is not pursued here; in particular, those difficulties are not apparent for an aerofoil-shaped obstacle.

The second situation in which some analytical progress can be made is when Ekman dissipation is relatively small i.e. when both λ and α are large and of the same order. In that case (8) for a steady flow can be written as

$$J(\psi, \zeta + 2\alpha y/\lambda) = 0 \quad , \tag{9}$$

where J is the two-dimensional Jacobian, and it follows that $(\zeta + 2\alpha y/\lambda)$ is a function of ψ only. Once profiles for ψ and ζ known at one value of x, for example, this function of ψ can be determined and in principle the solution can be found everywhere. An obvious candidate for the known profile is the upstream flow at $x=-\infty$, and assuming that $\psi=-y$ and $\zeta=0$ at that point then ψ can be found everywhere by solving an elliptic p.d.e. This is Long's solution, and it has been examined in a series of papers, one of which is Miles and Huppert (1968). The numerical solutions in that paper show a steady lee-wave pattern behind the obstacle, extending out to infinity due to the absence of dissipation. Once α/λ is about 0.8 one of the streamlines becomes vertical at one point and for $\alpha/\lambda>0.8$ some of the streamlines 'overturn'. Beyond a larger critical value, a closed streamline region forms in the flow and the details in that region can only be found by solving a time-dependent initial-value problem. Such details are, however, irrelevant because Long's model is deficient for the reason previewed in the introduction, namely that upstream influence through Rossby-wave propagation has been neglected; this means that ψ will not be equal to $-y$ upstream and that it is only really possible to specify the average velocity at $x=-\infty$, with the details of the profile determinable through appropriate radiation conditions on the time-dependent problem. This requirement is true for all values of α/λ, but it only really becomes a significant influence once α/λ is no longer small. Experiments which show significant upstream influence in the analogous case of a stratified flow are described in Baines (1977).

Important modifications to the above occur when α and λ are no longer small. No complete analysis is available in that case but the dissipative effect of the ζ term in (8) tends to reduce the amplitude of both the lee waves and the upstream influence with distance from the obstacle. The balance of the terms $\lambda u\partial\zeta/\partial x$ and ζ in (8) indicates that vorticity decays over distance of $O(\lambda)$ due to this effect, so that providing the upstream boundary is sufficiently far away then a uniform flow is a suitable boundary condition at that point. In the next section some numerical solutions will be presented which take advantage of this feature.

NUMERICAL SOLUTIONS

To examine solutions of (8) over a range of $O(1)$ values of α and λ, the equation was solved numerically for two different shaped obstacles: a circular cylinder and a symmetric aerofoil. Although the steady solution was of primary interest, the equation was integrated as a time-dependent problem starting from the $\lambda=\alpha=0$ potential solution at $t=0$. The integrations were continued until the vorticity fields at successive time steps differed by less than 0.1%.

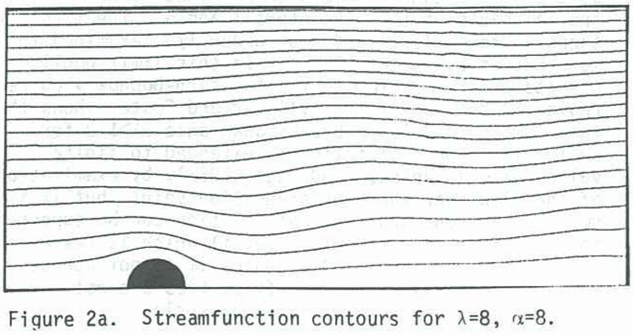


Figure 2a. Streamfunction contours for $\lambda=8$, $\alpha=8$.

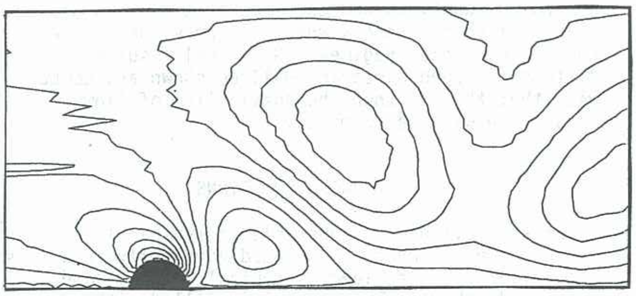
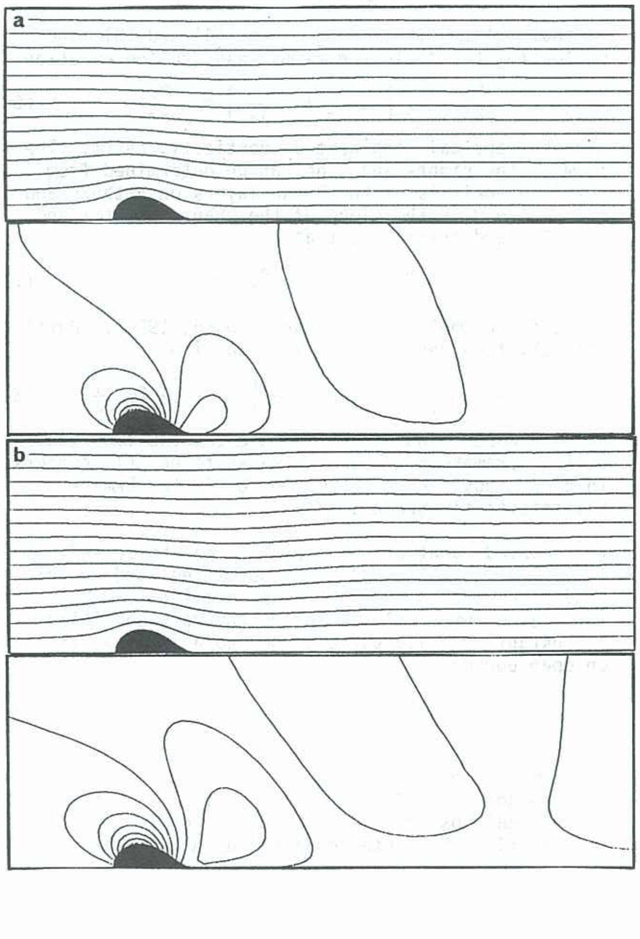


Figure 2b. Vorticity contours for $\lambda=8$, $\alpha=8$.

The numerical solutions for the streamfunction and vorticity past a cylinder when $\lambda=8$ and $\alpha=8$ are shown in Figure 2, and these fall roughly within the $\lambda=O(\alpha)$, $\alpha\gg1$ parameter range described in the previous section. Some upstream influence is apparent in front of the cylinder, but this decays towards the left-hand boundary, which is twenty radii upstream. The standing lee waves behind the cylinder show features similar to those in Miles and Huppert (1968), but none of the streamlines are vertical at any point. The wavenumber of the standing waves can be obtained from (8) through a simple argument, described in Johnson and Page (1986, 1987), and it is equal to $b=(2\alpha/\lambda)^{\frac{1}{2}}=1.4$ in this case. This is broadly confirmed by the numerical solutions. Also apparent in the lee-waves is the dissipation effect of Ekman suction which leads to a decay of vorticity over a length-scale of order λ downstream. The rear stagnation point presents some problems in terms of numerical resolution, due to the appearance of a singularity there, and this can be seen in the vorticity plot close to that point.

The avoid the difficulties at the rear stagnation point, flow past a symmetric Joukowski aerofoil was sought. This should have similar upstream influence and lee-wave effects to the circular obstacle, and it also has the advantage that $E^{\frac{1}{4}}$-layer separation is largely suppressed (see next section). In Figure 3 the streamfunction and vorticity fields are plotted for four pairs of



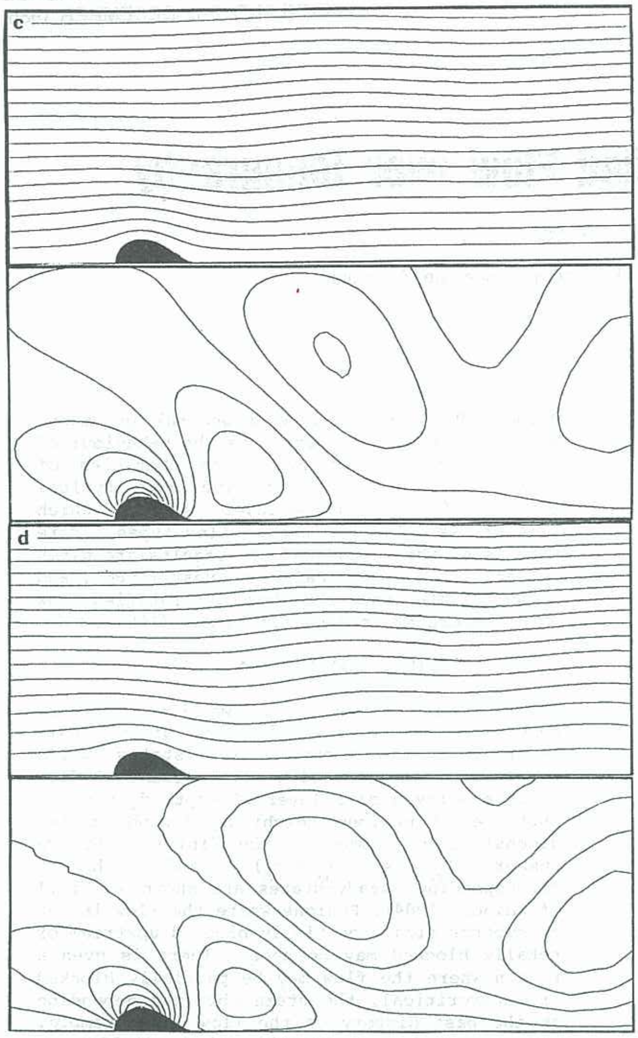


Figure 3. Streamfunction and vorticity plots for flow past a symmetric aerofoil when (a) $\lambda=\alpha=1$ (b) $\lambda=\alpha=2$ (c) $\lambda=\alpha=4$ (d) $\lambda=\alpha=8$.

parameters along the line $\lambda=\alpha$. The contour intervals are identical in each case and from this it is clear that the strengths of both the upstream effect, apparent in the vorticity fields, and the lee waves are increase as λ and α are increased, and that they extend further from the obstacle for larger values of λ. In addition, the wavelength of the lee waves is similar in all cases, as would be expected from the limiting wavenumber $b=(2\alpha/\lambda)^{\frac{1}{2}}$ being constant on the line $\lambda=\alpha$.

BOUNDARY-LAYER EFFECTS

In Page (1982) the flow in a rotating annulus with bottom topography was examined and it was shown that the $E^{\frac{1}{4}}$-layers on the vertical surfaces could separate if inertial effects were sufficiently strong i.e. that λ was large enough. Similar effects can be expected in the flow examined here, particularly for bodies with bluff trailing edges, such as the circular cylinder. In fact, as is shown in Page (1986), the flow past a circular cylinder separates even for $\alpha=0$, provided λ is greater than $\frac{1}{2}$, so that it is likely that separation will effectively invalidate the solutions in that case for all parameter values other than small λ. A necessary condition for separation can be calculated in the same manner as Page (1982) and gives that

$$\lambda \frac{dv_0}{ds} \leqslant -1 \qquad (10)$$

for separation to occur, where v_0 is the velocity around the obstacle surface and s is measured from the forward stagnation point. For a circular cylinder the maximum value of dv_0/ds occurs at the rear stagnation point with separation occurring for λ of $O(1)$ when α is $O(1)$, and λ of at most $O(1/\alpha)$ when α is large. The latter result contradicts the conclusions in Foster (1985); details of the reasons for this are given in Johnson and Page (1987). For an aerofoil, gradients of v_0 are less severe near the trailing edge and separation is delayed to some extent. Separation will, however, occur for large enough values of λ.

CONCLUSION

Numerical solutions were presented for β-plane flow past obstacles, in the presence of Ekman dissipation, and the general features are in accord with previous linear and nonlinear theories which are valid in restricted parameter ranges. Upstream influence and lee waves are shown to extend to within $O(\lambda)$ of the obstacle for steady flow when $\lambda=O(\alpha)$. For $\lambda=0$ the obstacle affects the flow over distances $O(\alpha^{\frac{1}{2}})$ ahead, with a 'western-boundary current', of thickness $O(1/\alpha)$ for $\alpha \gg 1$, behind. The flow in that layer becomes nonlinear for $\lambda=O(1/\alpha)$, leading to difficulties near the rear stagnation point of bluff bodies. Further details of all flows regimes are presented in Johnson and Page (1986,1987).

REFERENCES

Baines, P G (1977): Upstream influence and Long's model in stratified flows. J. Fluid Mech., vol. 82, 147-159.

Boyer, D L and Davies, P A (1982): Flow past a circular cylinder on a β-plane. Phil. Trans. Roy. Soc. Lond. A., vol. 306, 553-556.

Foster, M R (1985): Delayed separation in eastward, rotating flow on a β-plane. J. Fluid Mech., vol. 155, 59-75.

Greenspan, H P (1968): The theory of rotating fluids, Cambridge University Press.

Greenspan, H P (1970): A note on the laboratory simulation of planetary flows. Stud. Appl. Math., vol. 48, 147-152.

Johnson, E R and Page, M A (1986): Flow past an aerofoil on a β-plane. To be submitted to J. Fluid Mech.

Johnson, E R and Page, M A (1987): Flow past a circular cylinder on a β-plane. In preparation.

Miles, J W and Huppert, H E (1968): Lee waves in stratified flow: Part II Semi-circular obstacle. J. Fluid Mech., vol. 33, 803-814.

Page, M A (1982): Flow separation in a rotating annulus with bottom topography. J. Fluid Mech., vol. 156, 205-221.

Page, M A (1986): Separation and free-streamline flows in a rotating fluid at low Rossby number. Submitted to J. Fluid Mech.

Roberts, P H and Soward, A M (1978): Rotating fluids in geophysics, Academic Press.