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# The Numerical Modelling of Natural Convection in Vertical Slots

G. D. MALLINSON

Department of Mechanical Engineering, University of Auckland, New Zealand.

ABSTRACT

A two-dimensional numerical model has been developed to predict natural convection in slots. In this paper calculations made by this method are compared with the results of a full three-dimensional analysis and a quasi-two-dimensional method which seeks plane flow solutions but retains the three-dimensionality of the temperature and stream function.

These models have been applied to a slot heated from below and the results compared with those from a Galerkin analysis made by Frick (1983). Very good agreement between all models is found for the case of adiabatic sidewalls. When the walls are highly conducting, the agreement is affected by the influence of the modelling assumptions on the critical Rayleigh number and the buoyancy created by cross-slot temperature gradients. In all cases the two-dimensional method offers considerable savings in computational effort.

## INTRODUCTION

One method used to inhibit natural convection in the air space between a solar collector and its cover plate relies on inserting thin plane partitions which divide the space into a series of vertical slots. Because the partitions are normal to the surface of the collector they have little effect on the heat input by radiation into the collector. However, viscous interaction between the air and the partitions significantly reduces the strength of convection and hence heat loss back through the cover plate.

In the northern hemisphere it is usual to arrange the partitions so that the slots are aligned with the their planes across the direction of slope of the collector. As demonstrated by Symons and Peck (1984) more effective supression may be achieved by aligning the slots up the slope of the collector.

The convection in cross-slope slots consists of a series of rolls with their axes normal to the partitions. In up-slope slots a single roll can exist when the angle of inclination exceeds 24 deg. When multiple rolls occur, the essential mechanisms of the convection can be understood by studying convection in vertical slots between horizontal isothermal surfaces, a situation which corresponds to a solar collector which is not inclined. When the partition spacing is very small, the convective flow is very similar to that in a porous medium heated from below and is amenable to analysis using the Hele Shaw approximation as reported, for example, by Elder (1967), Hartline and Lister (1977) and Frick and Clever (1982).

The work reported here is part of a of a programme of research directed at developing a numerical method which can be applied to the single roll convection in the up-slope slots, and concentrates on the validation of the model described by Mallinson (1984,1986). This model uses assumed forms for the variation of velocity and temperature across the slot to derive a set of two-dimensional governing equations which can be solved sufficiently quickly to permit application to slots having large up-slope aspect ratios.

The data used for validation were published by Frick (1983) who used a Galerkin procedure to predict natural convection by an infinte Prandtl number fluid in vertical slots between horizontal isothermal surfaces. The partitions forming the slots were either adiabatic or perfectly conducting. Frick's data were produced by a fully three-dimensional analysis and an approximate plane flow analysis. The plane flow model described here replicates this approximation.

Finite difference methods are used in the present study. Three different models for the convection in the slots are used. The first is a fully three dimensional model similar to that described by Mallinson and de Vahl Davis (1973). The second model is based on the assumptions made by Frick (1983) to produce plane flow solutions. The third model is the two-dimensional model described by Mallinson (1984).

## NUMERICAL MODELS FOR SLOT CONVECTION

### Three Dimensional Model

The slot, shown in Fig. 1, has dimensions $L_x$, $L_y$ and $L_z$ such that $L_y$ is much smaller than either $L_x$ or $L_z$. The x boundaries of the slot are isothermal with the temperature of the boundary at x=0 being the greater, (i.e. $T_0 > T_1$). Using $L_x$, $\kappa_f/L_x$ and $\rho_0\kappa_f^2/L_x^2$ as the scale factors for distance, velocity and pressure, and defining $\theta = (T-T_1)/(T_0-T_1)$, the equations governing steady Boussinesq convection are;

$$\underset{\sim}{v}\cdot\nabla\underset{\sim}{v} = -\nabla P + RaPr\theta\underset{\sim}{\hat{i}} + Pr\nabla^2\underset{\sim}{v}, \tag{1}$$

$$\nabla\cdot\underset{\sim}{v} = 0 , \tag{2}$$

$$\underset{\sim}{v}\cdot\nabla\theta = \nabla^2\theta . \tag{3}$$

Ra and Pr are the Rayleigh and Prandtl numbers respectively.

These equations can be expressed in terms of the vorticity, $\underset{\sim}{\zeta}$, and a vector potential, $\underset{\sim}{\psi}$, defined by;

$$\underset{\sim}{v} = \nabla\times\underset{\sim}{\psi} \quad ; \quad \nabla\cdot\underset{\sim}{\psi} = 0. \tag{4}$$

The curl of equation (1) yields,

$$\nabla\times(\underset{\sim}{\zeta}\times\underset{\sim}{v}) = -RaPr(\nabla\times\theta\underset{\sim}{\hat{i}}) + Pr\nabla^2\underset{\sim}{\zeta} \tag{5}$$

and the equation relating vorticity and the vector potential is

$$\nabla^2\underset{\sim}{\psi} = -\underset{\sim}{\zeta} . \tag{6}$$

Boundary conditions for $\underset{\sim}{\psi}$ and $\underset{\sim}{\zeta}$ can be derived from the relevant boundary conditions for velocity. For the plane boundary at z = 0, say,

$$\psi_1 = \psi_2 = \partial\psi_3/\partial z = 0. \tag{7}$$

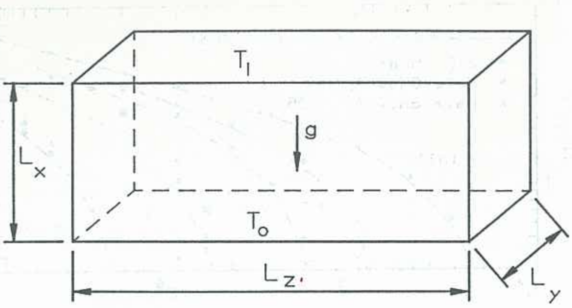


Fig. 1. Cavity dimensions and orientation.

The boundary conditions for $\zeta$ depend on whether the boundary is rigid (non slip) or free (perfect slip). The conditions at $z = 0$ when that boundary is rigid are,

$$\zeta_1 = -\partial^2\psi_1/\partial z^2\ ,\quad \zeta_2 = -\partial^2\psi_2/\partial z^2,\quad \zeta_3 = 0. \tag{8}$$

When the same boundary is free, the boundary conditions are,

$$\zeta_1 = \zeta_2 = 0\ ,\quad \partial\zeta_3/\partial z = 0. \tag{9}$$

The boundary conditions for $\theta$ at x=0 and x=1 are,

$$\theta = 1 \text{ and } \theta = 0, \text{ respectively.} \tag{10}$$

If the y boundaries are adiabatic,

$$\partial\theta/\partial y = 0, \text{ at } y = 0 \text{ and } y = A_y. \tag{11}$$

The conditions for perfectly conducting y boundaries are,

$$\theta = 1-x \text{ at } y = 0 \text{ and } y = A_y. \tag{12}$$

For this study, the z boundaries are adiabatic so that

$$\partial\theta/\partial z = 0, \text{ at } z = 0 \text{ and } z = A_z. \tag{13}$$

The above equations can be solved numerically using the finite difference method described by Mallinson and de Vahl Davis (1973) in which second order finite difference approximations are used throughout and steady state solutions are generated by an alternating direction implicit iterative procedure.

The full three-dimensional analysis can, in principle, be applied to any rectangular cavity. However, as discussed by Mallinson (1986) time step restrictions resulting from small mesh intervals in the y direction render the analysis to be computationally expensive. This difficulty provided the motivation for developing the approximate methods described below.

## Plane Flow Model

Frick (1983) used a Galerkin method to solve governing equations which had been expressed in terms of two scalar potentials for velocity. Approximate solutions were obtained by ignoring one of those potentials. This approximation is equivalent to assuming that v=0 so that the flow is plane in the x-z directions and allows the velocity field to be described by a single scalar potential, $\psi$ say, where,

$$u = -\partial\psi/\partial z \text{ and } w = \partial\psi/\partial x. \tag{14}$$

The vorticity equation (Eq. (5)) then reduces to a single equation for the y component of vorticity, $\zeta_2$, i.e.,

$$\frac{\partial}{\partial x}(u\zeta_2) + \frac{\partial}{\partial z}(w\zeta_2) = \mathrm{RaPr}\frac{\partial\theta}{\partial z} + \nabla^2\zeta_2 \tag{15}$$

The stream function is related to this vorticity component by,

$$\nabla_2^2\psi = -\zeta_2, \tag{16}$$

and the energy equation reduces to

$$\frac{\partial}{\partial x}(u\theta) + \frac{\partial}{\partial z}(w\theta) = \nabla^2\theta. \tag{17}$$

Note that the functions u, v, $\psi$, $\zeta_2$, and $\theta$ are dependent on all three space variables but the Laplacian in Equ. (16) is two-dimensional.

The appropriate boundary conditions for $\psi$ are that $\psi$=0 on all impermeable boundaries. Boundary conditions for $\zeta_2$ at the x and y boundaries are,

$$\zeta_2 = -\frac{\partial^2\psi}{\partial x^2} \text{ at } x = 0 \text{ and } x = 1; \tag{18}$$

$$\zeta_2 = 0 \text{ at } y = 0 \text{ and } y = A_y\ . \tag{19}$$

For non-slip z boundaries,

$$\zeta_2 = -\frac{\partial^2\psi}{\partial z^2} \text{ at } z = 0 \text{ and } z = A_z\ , \tag{20}$$

whereas for free z boundaries,

$$\zeta_2 = 0 \text{ at } z = 0 \text{ and } z = A_z\ . \tag{21}$$

The assumption of plane flow reduces the number of solution variables from 7 to 3, ($\psi$, $\zeta_2$ and $\theta$), which results in a significant reduction in computational effort. The three-dimensionality of the Laplacian operators in Equs. (15) and (17) does, however, lead to severely restrictive time steps for small $A_y$.

## Two-dimensional Model

The two-dimensional model relies on the assumption that the flow in a narrow slot closely resembles laminar flow between parallel planes. Accordingly the velocity field can be assumed to be approximated by

$$u = f(y)\bar{u}(x,z),\quad v = 0,\quad w = f(y)\bar{w}(x,z) \tag{22}$$

where

$$f(y) = 6y(A_y-y)/A_y^2\ . \tag{23}$$

Making these substitutions, it follows that $\bar{u}$ and $\bar{w}$ can be generated from a stream function $\bar{\psi}$ such that

$$\bar{u} = -\partial\bar{\psi}/\partial z \text{ and } \bar{w} = \partial\bar{\psi}/\partial x. \tag{24}$$

The y component of vorticity is given by

$$\zeta_2 = (\partial\bar{u}/\partial z - \partial\bar{w}/\partial x)f(y) = \bar{\zeta}_2 f(y) \tag{25}$$

and is related to the stream function by,

$$\nabla^2\bar{\psi} = -\bar{\zeta}_2\ . \tag{26}$$

After substitution and integration over y, the vorticity transport equation becomes,

$$\frac{1.2}{Pr}\left[\frac{\partial}{\partial x}(\bar{\zeta}_2\bar{u}) + \frac{\partial}{\partial z}(\bar{\zeta}_2\bar{w})\right] = \mathrm{Ra}\frac{\partial\theta_f^*}{\partial z} + \nabla_2^2\bar{\zeta}_2 - 12\bar{\zeta}_2/A_y^2\ . \tag{27}$$

where the y-averaged temperature field is denoted by $\theta_f^*$.

In general, the temperature in the fluid can be approximated by,

$$\theta_f = \bar{\theta}_f(x,z)h(y) + \theta_b(x,z) \tag{28}$$

where $\theta_b$ is the temperature at the wall/fluid interface and $\bar{\theta}_f$ is the mean deviation of $\theta_f$ from $\theta_b$.

If the side walls (y boundaries) are adiabatic, $\bar{\theta}_f = 0$ and $\theta_b$ is given by

$$\frac{\partial}{\partial x}(\bar{u}\theta_b) + \frac{\partial}{\partial z}(\bar{w}\theta_b) = \nabla_2^2\theta_b. \tag{29}$$

If the side walls are perfectly conducting, $\theta_h = 1-x$ and $h(y)$ is given by,

$$h(y) = 5(y^4 + A_y^3 y - 2A_y y^3)/A_y^4 \; . \tag{30}$$

Substitution into Eq. (3) and integration over the cavity width yields,

$$(51/42)\left[\frac{\partial}{\partial x}(\bar{u}\bar{\theta}_f) + \frac{\partial}{\partial z}(\bar{w}\bar{\theta}_f)\right] = \nabla^2\bar{\theta}_f - 10\bar{\theta}_f/A_y^2 - \bar{u}. \tag{31}$$

Heat Transfer

The heat transferred by the convecting fluid can be calculated by evaluating a Nusselt number which is estimated by using a 3 point forward difference approximation for the temperature gradient and Simpson's rule to calculate the average heat flux through the x=0 boundary.

## RESULTS AND DISCUSSION

Frick (1983) published solutions for vertical slots heated from below and filled with a high Prandtl number fluid. The Galerkin procedure used by Frick produces estimates of the critical Rayleigh number and wavenumber for the motion following the onset of instability.

The finite difference method used here does not produce direct estimates of the critical Rayleigh number and wavenumber. The critical Rayleigh number can be estimated indirectly by subjecting a "zero flow" solution to a suitable disturbance: growth to form a steady flow implies Ra is greater than the critical value. The wavenumber can be estimated by obtaining solutions for a range of values of roll aspect ratio and using a suitable indicator, such as maximum in the rate of heat transfer, to determine the preferred wavenumber. Both methods can involve prohibitive amounts of computer time, especially when using a three-dimensional solution method.

Accordingly, the technique adopted in this study was to obtain solutions for single roll having an apsect ratio corresponding to the wavemumber predicted by Frick (1983). The z boundaries representing the interface between rolls are free and adiabatic. Following Frick's example, the critical wavemumber was used for all super-critical Rayleigh number solutions. No attempt was made to predict the critical Rayleigh number, the basis of comparison with Frick's data being the predicted Nusselt numbers. All numerical solutions obtained here are for $Pr=10^4$.

All three-dimensional and plane flow solutions used a 21x21x21 mesh. Experiments confirmed that reducing the number of mesh points in the y direction, even for small A was not possible since the truncation errors induced were invariably greater than the errors associated with the assumptions leading to the two-dimensional model.

The two-dimensional solutions used for comparison with the three-dimensional solutions use a 21X21 mesh. Some solutions were, however, obtained with a 41x41 mesh to investigate the effects of mesh refinement in the x-z plane.

Adiabatic Sidewalls

For adiabatic sidewalls, Frick's analysis predicted a critical wavenumber corresponding to $A_z=1$. Data from the two-dimensional model with a 41x41 mesh for the adiabatic sidewalls case are summarised in Fig. 2, in which the broken lines denote data taken from Frick's (1983) Fig. 2 for $A_y = 0.05$, 0.1, 0.2, 0.33 and infinity (i.e. no sidewalls). The Nusselt numbers predicted by the two-dimensional model are approximately 2% higher than the Nusselt numbers obtained by Frick.

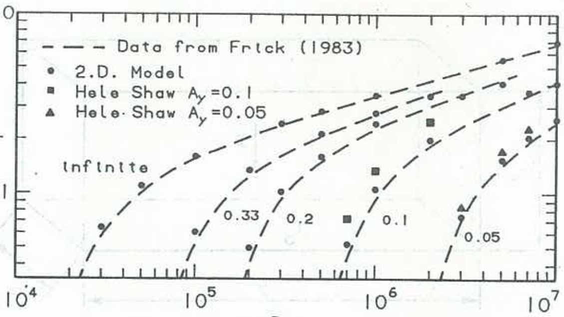


Fig. 2. Adiabatic sidewalls: dependence of $Nu_f$ on Ra. $A_y$ values as marked.

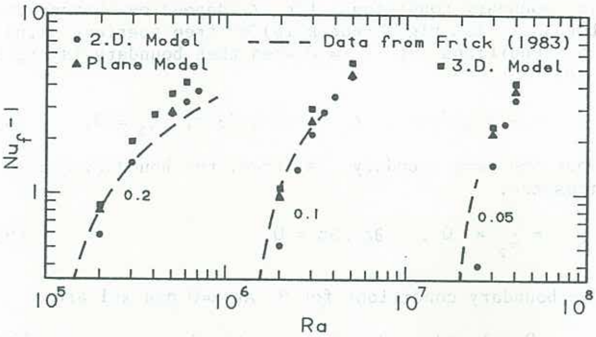


Fig. 3. Perfectly conducting sidewalls: dependence of $Nu_f$ on Ra. $A_y$ values as marked.

| | | $A_y$=0.1 | | $A_y$=0.2 | |
|---|---|---|---|---|---|
| Model | Mesh | $Ra=10^5$ | $Ra=2x10^5$ | $Ra=3x10^4$ | $Ra=5x10^4$ |
| 2D | 21x21<br>41x41 | 2.09<br>2.06 | 3.07<br>2.99 | 2.04<br>2.02 | 2.64<br>2.60 |
| Plane | 21x11x21<br>21x21x21 | 1.90<br>2.07 | 2.81<br>3.00 | 1.84<br>1.98 | 2.39<br>2.58 |
| 3D | 21x11x21<br>21x21x21 | 1.89<br>2.02 | 2.85<br>3.06 | 1.82<br>2.01 | 2.40<br>2.58 |

Table 1. Adiabatic sidewalls: sample values of Nu as predicted by the different models.

Also shown are data for $A_y$=0.05 and 0.1 generated by omitting the advection tems and the Laplacian operator portion of the diffusion term in Equ. (27). These omissions correspond the the assumptions of the conventional Hele Shaw model for convection in narrow slots. These data demonstrate that the Hele Shaw model over-predicts heat transfer for $A_y$>0.05, an observation which is also supported by calculations presented by Mallinson (1986) for air filled slots heated from the side.

Table 1 summarises the results for selected parameter values using the plane flow and three-dimensional models. Generally, the two-dimesnional solutions for a 21x21 mesh produce 2-3% increases in Nusselt numbers referenced to the 41x41 values. The 21x21 two-dimensional Nusselt numbers are within 1.5% of the three-dimensional results which use the same cross sectional mesh and 21 y planes. The data for 11 y planes confirm the assertion made above that at least 21 y planes are requred to ensure adequate representation of the flow field by the three-dimensional mesh. This behaviour is also exhibited by the plane flow solutions.

These results for adiabatic side walls indicate that the two-dimensional model can represent the convection in slot with an accuracy which is commensurate with that of a three-dimensional model having the same cross sectional mesh. The differences between the two-dimensional and three-dimensional results are less than the errors resulting from cross sectional truncation errors.

Conducting Sidewalls

For this case Frick's (1983) formula for the critical wavenumber leads to $A_z=A_y^{\frac{1}{2}}$. Results for $A_y$=0.05, 0.1 and 0.2 (with $A_z$=0.22, 0.33 and 0.45 respectively) are presented in Fig. 3. The influence of $A_y$ is more pronounced in this case that it was for adiabatic side walls and there is a greater sensitivity to the differences between the various models. The two dimensional model under-predicts heat transfer when compared to the three-dimensional model as does, generally to a lesser degree, the plane flow model.

As Ra increases beyond the critical value, the rate of increase of Nu is greater than in the case of adiabatic walls and is greatest for $A_y$=0.05. The results for $A_y$=0.05 and $A_y$=0.1 show that the differences between the models increase with decreasing Ra. This suggests that sensitivity of the critical Rayleigh number to the modelling assumptions could account for the differences between the Nusselt numbers predicted by the three models. The results for $A_y$=0.2 show an increasing difference between the three-dimensional model on the one hand and the two-dimensional and plane flow models on the other. Buoyancy generated by gradients of temperature in the y direction could be responsible for this effect. It would be difficult to isolate the causes of the observed differences any further without more precise estimates of the critical Rayleigh number and wavenumber.

CONCLUSIONS

Generally, there is favourable agreement between the two-dimensional model and the more complex models. In the case of adiabatic sidewalls, the agreement is commensurate with the accuarcy of the representation of the cross sectional flow.

In the case of isothermal sidewalls, discrepancies between the models for small Ay could be caused by shifts in the critical Rayleigh number: for larger $A_y$, buoyancy effects arising from temperature gradients in the y direction may also contribute.

As a final comment, the computer times for the two dimensional solutions were measured in minutes, whereas both the plane flow and three-dimensional solution times were measured in hours.

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NOMENCLATURE

| | |
|---|---|
| $A_y$ | = $L_y/L_x$, cavity aspect ratio in y versus x. |
| $A_z$ | = $L_z/L_x$, cavity aspect ratio in z versus x. |
| f(y) | = y variation for velocity, vorticity and stream function |
| g | = gravitational vector |
| $\bar{h}(y)$ | = y variation for fluid temperature |
| k | = thermal conductivity of the fluid |
| $L_x, L_y, L_z$, | = dimensions of cavity in x, y and z directions |
| $Nu_f$ | = Nusselt number for fluid |
| Pr | = $\nu/\kappa_f$, Prandtl number |
| Ra | = $g\beta_f L_x^3(T_0-T_1)/(\nu\kappa_f)$, Rayleigh number |
| $Ra_{HS}$ | = $RaA_y^2/12$, Hele-Shaw Rayleigh number |
| T | = temperature |
| $T_0$ | = temperature of hot boundary at x = 0 |
| $T_1$ | = temperature of cold boundary at x = 1 |
| u,v,w | = velocity components in x, y and z directions |
| $\bar{u}$ | = y averaged velocity in x direction |
| $\underline{v}$ | = velocity vector |
| $\bar{w}$ | = y averaged velocity in z direction |
| x,y,z | = nondimensional Cartesian coordinates |

Greek Symbols

| | |
|---|---|
| $\beta_f$ | = fluid coefficient of thermal expansion |
| $\theta$ | = $(T-T_1)/(T_0-T_1)$, nondimensional temperature |
| $\theta_b$ | = value of at fluid/wall interface |
| $\theta_f$ | = nondimensional fluid temperature |
| $\bar{\theta}_f$ | = y-averaged deviation of $\theta_f$ from $\theta_b$ |
| $\theta_f^*$ | = y-averaged value of $\theta_f$ |
| $\kappa_f$ | = thermal diffusivity for the fluid |
| $\nu$ | = kinematic viscosity |
| $\rho_0$ | = reference density |
| $\psi_1,\psi_2,\psi_3$ | = x, y and z components of |
| $\underline{\psi}$ | = vector potential for velocity |
| $\bar{\psi}$ | = y-averaged stream function |
| $\underline{\zeta}$ | = vorticity vector |
| $\zeta_1,\zeta_2,\zeta_3$ | = x, y and z components of vorticity |
| $\bar{\zeta}_2$ | = y-averaged value of y vorticity |
| $\nabla_2^2$ | = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ |