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Modelling of the Decay of Kinetic Energy and Temperature in Isotropic Turbulence

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ABSTRACT: one studies, with the aid of the stochastic model E.D.Q.N.M. the dynamics of a passive scalar (such as temperature in a slightly heated flow) in three-dimensional isotropic turbulence. Starting initially with energy and temperature spectra concentrated at a low wave number, without any external forcing, one shows numerically that the temperature gradient variance diverges with the enstrophy (mean vorticity variance) at a finite time when the viscosity goes to zero, and if the Prandtl number remains finite. This critical time t_c is such that, in the same conditions, the kinetic energy and the temperature variance are conserved before t_c and dissipated at a finite rate after t_c.

For larger times, the kinetic energy and the temperature variance time decay exponents α_E and α_θ are determined with respect to the infrared behaviour of the initial spectra, and the relative position of the temperature and velocity integral scales. It is found that the temperature integral scale satisfies an analogous Richardson law , and that the temperature decreases the more faster as it is injected in small scales relatively to the energy. It provides an explanation of apparently anomalous experimental decay results.

1 INTRODUCTION

Three-dimensional turbulence is characterized by a finite kinetic energy dissipation, due to viscous forces, even when the molecular viscosity goes to zero. More precisely, it can be shown, on the basis of stochastic models of the E.D.Q.N.M. (Eddy- Dammped Quasi-Normal Markovian) type, that when the velocity fluctuations are confined into the large scales, there exists a critical time t_c before which the kinetic energy is conserved when $\nu \to 0$ (Andre and Lesieur, 1977). At t_c , and still in the limit of zero viscosity, the enstrophy

$$D(t) = (1/2) < (\nabla x \underline{u})^2 > \qquad (1-1)$$

becomes infinite. For times greater than t_c, the kinetic energy is dissipated at a finite rate ϵ . This can be understood when looking at the dissipation relation

$$\epsilon = 2\nu D(t) \qquad (1-2)$$

which indicates that ϵ will go to zero with ν as far as the enstrophy remains finite. The critical time t_c is the time necessary to build up a $k^{-5/3}$ energy cascade, and depends only on the initial enstrophy.

In this paper it will be shown that the same results hold for the passive scalar (called here the temperature), whose mean gradient variance will blow up with the enstrophy. The analysis will parallel that carried out by Andre and Lesieur (1977) with both an E.D.Q.N.M. numerical calculation and an exact analytical result using a simplified model called the Markovian Random Coupling Model (M.R.C.M., Frisch et al., 1974): the dissipation rate of temperature variance

$$\eta = 2\kappa D_\theta(t) \qquad (1-3)$$

will go to zero with κ as far as the temperature enstrophy

$$D_\theta(t) = (1/2) < (\nabla\theta)^2 > \qquad (1-4)$$

remains finite.

For $t > t_c$, the temperature variance will decay at a finite rate. The asymptotic time decay laws of kinetic energy and temperature are quite well understood in that case, where the wave-numbers $k_i(t)$ and $k_\theta(t)$ characterizing respectively the peaks of the kinetic energy and temperature spectra are of the same order: indeed, assuming self similar evolving energy spectra, one has

$$(1/2) < \underline{u}^2 > \sim t^{-\alpha_E} \qquad (1-5),$$

and α_E depends on the $k \to 0$ exponent s of the kinetic energy spectrum, such that

$$E(k,t) = Cs(t)k^s \quad (k \to 0) \qquad (1-6)$$

s, which cannot be larger than 4 , is conserved with time (Comte-Bellot and Corrsin, 1971, Lesieur and Schertzer, 1978). One finally finds for the kinetic energy decay.

$$\alpha_E = 2\frac{s+1-\gamma}{s+3} \qquad (1-7)$$

with

$$Cs(t) \sim t^\gamma \qquad (1-8)$$

γ is zero for $s < 4$. An E.D.Q.N.M. calculation has shown $\gamma = 0.16$ in the case $s = 4$, corresponds to $\alpha_E = 1.38$. It can be shown in the same way that (Larcheveque et al., 1980, Herring et al., 1982) that

$$(1/2) < \theta^2 > \sim t^{-\alpha_\theta} \qquad (1-8)$$

$$\alpha_\theta = (s'+1)\frac{2+\gamma}{s+3} - \gamma' \qquad (1-9)$$

where s' is such that

$$E_\theta(k) \sim C_{s'}k^{s'}, k \to 0 \qquad (1-10)$$

$E_\theta(k)$ being the temperature spectrum. In the same way as for the energy spectrum, s' is conserved with time and cannot be larger than 4. γ' , such that

$$C_{s'} \sim t^{\gamma'} \qquad (1-11)$$

is zero for $s' < 4$, and is equal to 0.06 for $s = s' = 4$. In this case α_θ is equal to 1.48.

Things are quite different when the scalar is injected initially in much smaller scales than the kinetic energy. The experiments of Warhaft and Lumley (1978) and Sreenevasan et al.(1980) showed then that the temperature variance decayed much quicker. This behaviour will be discussed both phenomenologically and with the aid of numerical EDQNM calculations.

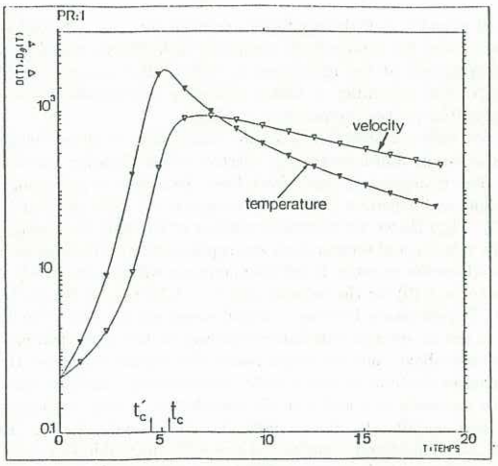

Fig 1: EDQNM calculation of the simultaneous evolution of the velocity and temperature enstrophies, showing the tendancy to a divergence at a time of order 5 initial large-eddy-turnover times $1/v_0k_i(0)$. The Prandtl number of the calculation is one, and the initial Reynolds number $v_0/\nu k_i(0)$ is 40000.

2 BLOW UP OF THE TEMPERATURE VARIANCE

One considers the free evolution of an unforced turbulence of kinetic energy and temperature spectra initially concentrated at a wave-number $k_i(0)$. As recalled in the introduction, the EDQNM theory leads, for the enstrophy

$$D(t) = \int_0^{+\infty} k^2 E(k,t)dk \qquad (2-1),$$

to a divergence (when $\nu \to 0$) at a time $t_c \approx 5/v_0 k_i(0)$, where v_0 is the r.m.s. initial velocity. In these calculations, the initial enstrophy was $1.25[k_i(0)v_0]^2$, which gives $t_c = 5.6D(0)^{-1/2}$. This numerical result has not been yet demonstrated analytically in the general E.D.Q.N.M. case, but can be derived for a simplified model (the M.R.C.M.) proposed by Frisch et al.(1974): it corresponds to a constant triple correlation relaxation time of the E.D.Q.N.M. theory:

$$\Theta_{kpq} = \Theta_0 \qquad (2-2)$$

One then obtains analytically (Andre and Lesieur, 1977) the following evolution equation for the enstrophy

$$\frac{dD(t)}{dt} = \frac{2}{3}\Theta_0 D(t)^2 - 2\nu \int_0^{+\infty} k^4 E(k,t)dk \qquad (2-3)$$

During the initial phase, and as far as the energy spectrum is raplidly decreasing in the large k , the viscous term in the r.h.s. of (2-3) will tend to 0 with ν. Then the enstrophy will blow up at $t = 3/[2\Theta_0 D(0)]$.

Actually eq. (2-3) is not correct, because of the unphysical assumption concerning the time Θ_{kpq}. A more physical enstrophy evolution equation can be obtained directly from the Navier-Stokes equations (Orszag, 1977)

$$\frac{dD(t)}{dt} = \left(\frac{98}{135}\right)^{1/2} s(t)D(t)^{3/2} - 2\nu \int_0^{+\infty} k^4 E(k,t)dk \qquad (2-4)$$

where $[-s(t)]$ is the velocity skewness factor. If one assumes that $s(t)$ is constant with time and equal to the values of order 0.4 usually found in the experiments or in the direct numerical simulations of turbulence, one finds (when $\nu \to 0$) a blow up of the enstrophy at $t_c = (5.9)D(0)^{-1/2}$, close to the above numerical E.D.Q.N.M. value.

Let us consider now the "temperature enstrophy"

$$D_\theta(t) = \int_0^{+\infty} k^2 E_\theta(k,t)dk = \frac{1}{2} < (\nabla\theta)^2 > \qquad (2-5)$$

The E.D.Q.N.M. theory leads then to the temperature enstrophy evolution equation

$$\frac{dD_\theta(t)}{dt} = \frac{4}{3}\int_0^{+\infty}\int_0^{+\infty} \Theta_{00pq} p^2 q^2 E(q) E_\theta(p) dp dq - 2\kappa \int_0^{+\infty} k^4 E_\theta(k,t)dk \qquad (2-6)$$

which can be written as, using

$$\Theta_{00q} \approx \left[\int_0^q \alpha^2 E(\alpha,t)\right]^{-1/2} \qquad (2-7)$$

$$\frac{dD_\theta(t)}{dt} = \frac{8}{3} D_\theta(t) D(t)^{1/2} - 2\kappa \int_0^{+\infty} k^4 E_\theta(k,t)dk \qquad (2-8)$$

If ν and $\kappa \to 0$ the conductive term in the r.h.s. of (2-8) will tend to zero, and the temperature enstrophy will diverge together with the enstrophy and at the same time t_c. Physically, one can say that the catastrophic stretching of vortex filaments by turbulence will lead to singularities for the temperature gradients.

We have performed E.D.Q.N.M. calculations of the evolution of the kinetic energy and temperature spectra at low viscosity and conductivity. The numerical methods have been described in preceeding papers (Lesieur and Schertzer 1978, Larcheveque et al. 1980 and Herring et al. 1982). Fig 1 shows, for a Prandtl number ν/κ of one, the strong increase of the velocity and temperature enstrophies at times (defined here as the time where the quantity is half its maximum value) respectively equal to $t_c = 5.4/v_0 k_i(0)$ for the velocity and $t'_c = 4.35/v_0 k_i(0)$ for the temperature. The difference between t_c and t'_c seems simply due to the fact that this is not an inviscid calculation contrary to the analytical predictions presented above, and one might reasonably expect that both times will collapse in the limit of zero ν and κ. conductivity. Another consequence of the finiteness of ϵ and η in the calculation, is that both enstrophies will decrease after t_c, due to molecular effects: indeed, since $D(t)$ and $D_\theta(t)$ are respectively equal to $\epsilon/2$ and $\eta/2$, one will have $\epsilon \sim t^{-n_E-1}/\nu$ and $\eta \sim t^{-n_\theta-1}/\kappa$ for $t > t_c$. Their maximum will then go to infinity as respectively ν^{-1} and κ^{-1}. For $t > t_c \approx t'_c$, the kinetic energy and temperature variance will then be dissipated at finite rates ϵ and η, as shows the calculation presented in Fig 2. We have checked that the results are not significantly different at small and large Prandtl numbers.

A third event will happen at t_c, that is the simultaneous appearance of a $k^{-5/3}$ inertial kinetic energy spectrum and of a $k^{-5/3}$ inertial convective temperature spectrum, which is shown on Fig 3 (for a Prandtl number of 1): before t_c, the temperature spectrum spreads out towards larger and larger wave-numbers; at $t = 4.6/v_0 k_i(0)$ the inertial convective subrange starts to show off, and is completely established at $t = 5.1/v_0 k_i(0)$. For times larger than t_c , and since the conductive scales have been reached, the temperature spectrum will decay self- similarly (see the next section). Fig 4 shows the temperature spectrum for a Prandtl number Pr=0.1 and Pr=10. One can remark in particular the formation of the k^{-1} Batchelor's viscous convective range in the case Pr=10.

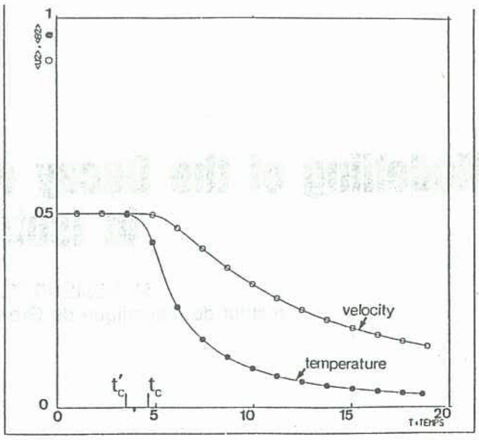

Fig 2: Time evolution of kinetic energy and temperature half variances, in the same calculation as in Fig 1

3 PHENOMENOLOGY OF THE TEMPERATURE DECAY

The following analysis holds for $t > t_c$, when the kinetic energy and temperature spectra decay self-similarly, and whatever the value (greater than 1) of $k_\theta(t)/k_i(t)$. We assume that the kinetic energy spectrum decays as described in the section 2, with a non zero value for γ in the only case $s = 4$. The temperature spectrum will be assumed to peak at a wave-number $k_\theta(t)$ located in the kinetic energy spectrum $k^{-5/3}$ inertial range. The particular case of k_θ of the order of k_I will allow to recover the temperature variance decay laws of section 1. We assume that $E_\theta(k,t)$ is given by

$$E_\theta(k,t) = A\,\frac{\eta}{\epsilon}\,E(k,t) \ \text{ for } \ k > k_\theta(t) \qquad (3-1)$$

$$E_\theta(k,t) = C_{s'}(t)\,k^{s'} \ \text{ for } \ k < k_\theta(t) \qquad (3-2)$$

The justification of (3-1) comes from an inertial-convective range assumption for the temperature spectrum. We still have

$$\frac{dC_{s'}(t)}{dt} = 0 \ \text{ for } \ s' < 4 \qquad (3-3)$$

and for $s' = 4$

$$dC_{s'}(t)/dt \approx \int_{k_\theta}^{+\infty} [q^3 E(q)]^{-1/2} E(q) \frac{E_\theta(q)}{q^2} dq \qquad (3-4)$$

which come from a "non local" expansion of the EDQNM temperature transfer term, following techniques described in Lesieur and Schertzer (1978).

Let us first consider the temperature dissipation rate η of the order of $(1/2) < \theta^2 >$ divided by a characteristic dynamical time at scales of

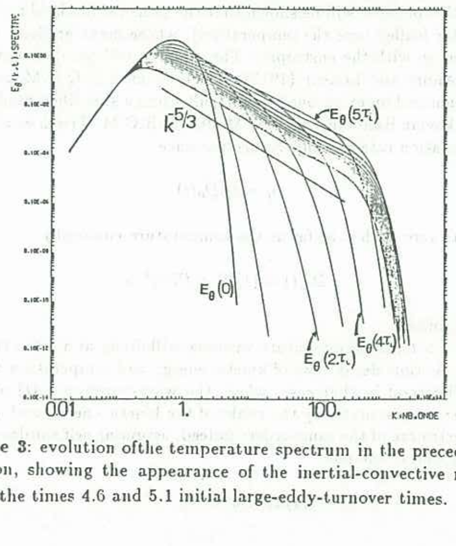

Figure 3: evolution ofthe temperature spectrum in the preceeding calculation, showing the appearance of the inertial-convective range between the times 4.6 and 5.1 initial large-eddy-turnover times.

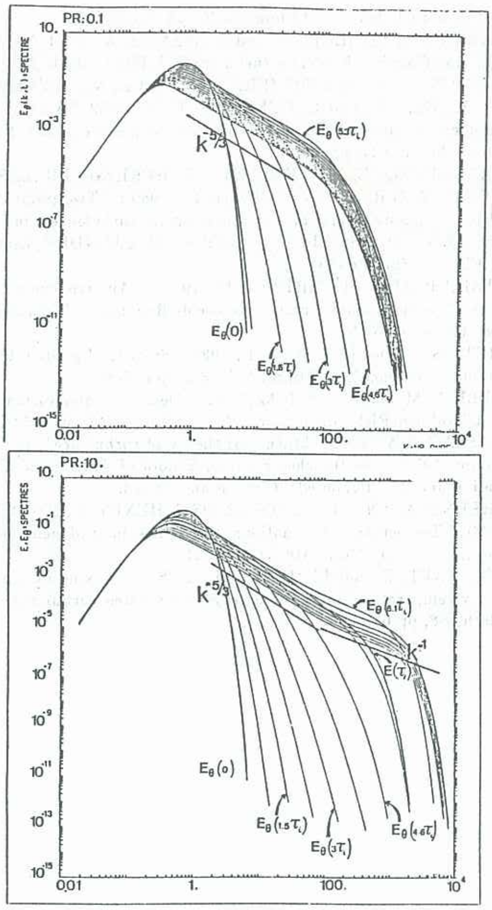

Fig 4: Evolution in time of the temperature spectrum for Prandlt numbers of 0.1 (Fig 4 a) and 10 (Fig 4 b).

order l_θ . The latter scale is the temperature integral scale, of the order of k_θ^{-1} , but which we are going to define more precisely. The local time at l_θ is l_θ/v_θ, where v_θ is a velocity characteristic of eddies of size l_θ, that is $(\epsilon l_\theta)^{1/3}$. Hence one obtains

$$\eta = < \theta^2 > \epsilon^{1/3} l_\theta^{-2/3} \qquad (3-5)$$

In the same way, the well known relationship $\epsilon = < \underline{u}^2 >^{3/2} /l$ can be written as

$$\epsilon = < \underline{u}^2 > \epsilon^{1/3} l^{-2/3} \qquad (3-6)$$

Let us define instantaneous kinetic energy and temperature decay exponents $\alpha'_E(t)$ and $\alpha'_\theta(t)$ by

$$\alpha'_E(t) = -\frac{1}{2}\frac{d < \underline{u}^2 >}{dLogt} = 2t\frac{\epsilon}{< \underline{u}^2 >} \qquad (3-7)$$

$$\alpha'_\theta(t) = -\frac{1}{2}\frac{d < \theta^2 >}{dLogt} = 2t\frac{\eta}{< \theta^2 >} \qquad (3-8)$$

which yields

$$\frac{\alpha'_\theta(t)}{\alpha'_E(t)} = [l(t)/l_\theta(t)]^{2/3} \qquad (3-9)$$

which implies that the temperature decays the more faster as the relative ratio of velocity to temperature integral scales is large. This is in good agreement with the observations of Warhaft and Lumley (1978). (3-9) is also valid for the asymptotic exponents α_θ and α_E given in section 1. Since the latter temperature decay law in power of time $t^{-\alpha_\theta}$ is only possible with one value of α_θ given by (1-9), which depends itself on the infrared energy and temperature spectral exponents s and s' , there can be only one ratio l/l_θ corresponding to such a decay: this ratio is thus equal to $[(s'+1)/(s+1)]^{3/2}$ when $[s, s' < 4]$, and to 1.11 when $s = s' = 4$. This is in any case very close to one, and fixes what is meant by the statement that the asymptotic values (1-8) and (1-9) imply that l and l_θ are of the same order. Now, if l_θ is smaller than l , there cannot be a power law decay for the temperature: following (3-9), the temperature decays faster than the kinetic energy, while $l_\theta(t)$ catches up with $l(t)$, as has been shown in Lesieur et al.(1986): eventually, the temperature will decay following (1-8) and (1-9) when l/l_θ will have the particular ratio of order one determined above.

We skip the details of the calculation which is given in Lesieur et al.(1986). The analytical results are that l_θ grows whith time following a Richardson-type law :

$$\frac{1}{2}\frac{dl_\theta^2}{dt} \sim \epsilon^{1/3} l_\theta^{4/3} \qquad (3-10)$$

This result is valid whatever the ratio (greater than one) of l/l_θ , for both situations of a stationary (that is artificially maintained by external forces injecting energy at k_i) or decaying kinetic energy spectrum. ϵ is then either a constant or a decreasing function of time, but the temperature is always decreasing. Such a Richardson law was, in the particular case $s' = 2$, employed by Nelkin and Kerr (1981) to look at the same problem of temperature decay. Actually our study generalizes their study to an arbitrary value of s'.

It has to be stressed that in the freely-decaying kinetic energy case, the velocity integral scale l follows also a Richardson-type law

$$\frac{1}{2}\frac{dl^2}{dt} \sim \epsilon^{1/3} l^{4/3} \qquad (3-11)$$

Finally one obtains

$$l_\theta(t) = [\frac{\alpha_E}{\alpha_\theta}]^{3/2} l(t)[1 + Bt^{-(2/3)\alpha_E}]^{3/2} \qquad (3-12)$$

and

$$< \theta^2 > \sim [B + t^{(2/3)\alpha_E}]^{-3\alpha_\theta/(2-\alpha_E)} \qquad (3-12)$$

This result is equivalent to

$$< \theta^2 > \sim l_\theta^{-\alpha_\theta/\alpha_E} \qquad (3-13)$$

and expresses the fact that the temperature variance decays much faster than the kinetic energy if the temperature integral scale increases rapidly. We have performed an EDQNM calculation in that case, where the temperature is initially introduced in the kinetic energy inertial range, with $l(t_0)/l_\theta(t_0) = 362$. Figure 5 shows the self-similar decay of the spectra obtained in this calculation for $s = s' = 4$. The Richardson law is well verified for a Prandtl number of 1, whith a numerical constant (arising in the r.h.s. of (3-10)) equal to 0.24. This value is not far from the value 0.22 found by Herring et al.(1982). We have also evaluated this constant for a Prandtl number of 0.1 and 10 (and for $s' = 4$) and found an average value of respectively 0.21 and 0.18.

Fig 6 shows the evolution with time of the "instantaneous" temperature decay exponent $\alpha'_\theta(t)$, evaluated both in the preceding EDQNM calculation and using the above analytical result. The agreement is fairly good. The final value of α'_θ obtained in this calculation is of 3.8 at $t = 3t_0$, and it seems difficult to push the numerical calculation further, for the temperature variance has been nearly completely dissipated by molecular conductivity. However, and if the calculation had been performed with infinite inertial and inertial-convective spectral ranges, $\alpha'_\theta(t)$ would eventually tend towards α_E , so that the values displayed in Fig 6 represent only a transient evolution of the temperature, and have no universal character. Such a result is reminiscent of the conclusions obtained for the statistical predictability problem (Metais and Lesieur, 1986).

4 EDQNM CALCULATIONS AND COMPARISON WITH EXPERIMENTS

Experiments in the air done by Warhaft and Lumley (1978) and Sreenevasan et al.(1980) are a good candidate for comparisons with the above theoretical and numerical predictions: a turbulence which is approximately homogeneous and isotropic is produced in a wind tunnel downstream of a grid (of mesh size M), and temperature fluctuations are introduced with the aid of a thin heated array of parallel wires whose mesh size is $M\theta$.

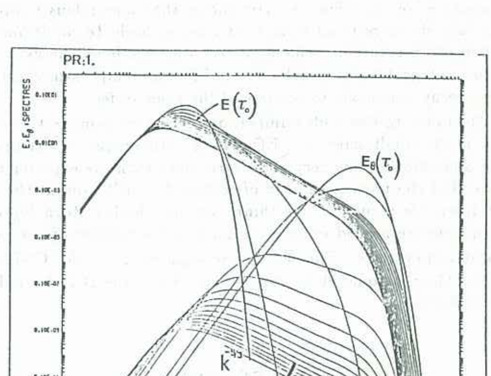

Fig 5: EDQNM calculation of the kinetic energy and temperature spectra, when the temperature is injected at a time t_0 in scales much smaller than $l(t_0)$.

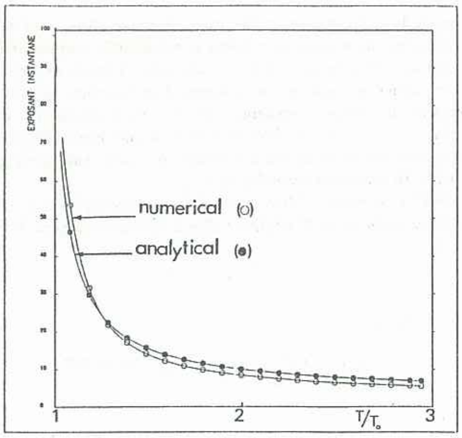

Fig 6: instantaneous temperature decay exponent

We have carried out an EDQNM calculation in conditions close to these experiments (and with a Prandtl number of one). The values of the decay exponent $< \alpha_\theta >$ given here is the slope of the experimental and numerical curves averaged during the period of evolution. At this moderate Reynolds number our calculation show that the kinetic energy decays as $t^{-1.32}$, in good agreement with the $t^{-1.34}$ experimental result of Warhaft and Lumley (1978). The temperature, introduced at t_θ, decreases along an approximate slope of -3.5 , close to the experimental results. It is encouraging to see that the EDQNM spectral equations are able to handle this experimental situation.

5 CONCLUSION

This paper has dealt only with a passive scalar (the temperature for instance) transported by a three-dimensional isotropic turbulence. When the kinetic energy and the temperature are injected in the large scales, we have shown, using the EDQNM approximation or a simplified version of it, the MRCM, that the temperature enstrophy (characteristic of the temperature gradients) would blow up with the enstrophy at a finite time t_c when the viscosity and the diffusivity go to zero (the Prandtl number remaining finite). At t_c appear the inertial and inertial-convective spectra. After t_c both spectra evolve self-similarly, and the kinetic energy and temperature integral scales l and l_θ adjust into a constant ratio close to one.

In the case where the temperature is introduced in smaller scales than the velocity, l_θ follows, like l , a Richardson-type equation, but catches up with l due to the particular initial conditions. A very simple phenomenological equation relating the instantaneous temperature decay exponent to l/l_θ, as well as a non local spectral expansions using the EDQNM theory, has allowed us to find the general law governing the temperature decay. This analysis shows that anomalous temperature decay laws do correspond in fact, at least at high Reynolds and Peclet numbers, to a transient behaviour necessary for both energy and temperature integral scales to collapse, and for the temperature and kinetic energy decay exponents to become of the same order.

At moderate Reynolds number, and when the temperature is introduced in the small scales, the EDQNM kinetic energy and temperature decay predictions are in very good agreement with the experiments. We believe that the phenomenology of section 3 is still valid at these moderate Reynolds numbers. We think also that higher Reynolds number experiments are needed in order to confirm the validity of our analysis.
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REFERENCES

ANDRE, J.C. and LESIEUR, M., 1977, "Influence of helicity on high Reynolds number isotropic turbulence." J. Fluid. Mech., **81**, pp 187-207
COMTE-BELLOT G. and CORRSIN S., 1971, "Simple Eulerian time correlation of full and narrow band velocity signals in grid generated isotropic turbulence" . J.Fluid.Mech., **48** ,pp 273-337
FRISCH, U., LESIEUR, M. and BRISSAUD, A., 1974, "A Markovian Random Coupling Model for turbulence" J. Fluid. Mech.,**65**, pp 145-152
HERRING, J.R., SCHERTZER, D., LESIEUR, M., NEWMAN, G. R., CHOLLET, J.P., and LARCHEVEQUE, M., 1982, "A comparative assessment of spectral closures as applied to passive scalar diffusion" J. Fluid. Mech.,**124**, pp 411-420
LARCHEVEQUE, M., CHOLLET J.P., HERRING J.R.,LESIEUR M., NEWMAN G.R. and SCHERTZER D., 1980 : "Two-point closure applied to a passive scalar in decaying isotropic turbulence", in "Turbulent Shear flows II", edited by F. DURST and B. LAUNDER, Lecture Notes in Physics, Springer-Verlag.
LESIEUR, M. et SCHERTZER, D., 1978 : "Amortissement auto similaire d'une turbulence a grand nombre de Reynolds", Journal de Mecanique,**17**, pp 609-635.
METAIS, O. and LESIEUR, M., 1986, "Statistical predictability of decaying turbulence", J. Atmos. Sci., **43**, pp 857-870
NELKIN, M. and KERR, R.M., 1981, " Decay of scalar variance in terms of a Modified Richardson Law." Phys. Fluids ., **24**, pp 1754-1757
ORSZAG, S.A., 1977, "Statistical theory of turbulence", in "Fluid Dynamics 1973", Les Houches Summer School of Theoretical Physics, R. Balian and J.L. Peube eds, Gordon and Breach
SREENEVASAN K.R.,TAVOULARIS S.,HENRY R.,CORRSIN S., 1980, "Temperature fluctuations and scales in grid generated turbulence." J. Fluid. Mech.,**100**,pp 597-621
WARHAFT, Z. and LUMLEY, J.L. 1978, "An experimental study of decay temperature fluctuations in grid generated turbulence." J. Fluid. Mech, **88**, pp 659-684