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Single Bubbles and Drops, Simple Derivation of Slug Velocity and Formulation of a General Velocity Equation

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ABSTRACT

For single bubbles and drops, equations have been formulated to find :

(a) maximum terminal velocity, eq(29)
(b) minimum terminal velocity, eq(30)
(c) probable terminal velocity in the often-encountered moderately impure systems, eq(26a).

The equations are valid for the whole range of particle Reynolds numbers discussed in the literatue on bubbles and drops, as well as the range of axi-symmetric bounds, from those which prevent motion to those at infinity. A simple derivation defines the shape of the spherical cap bubble, eq(34).

INTRODUCTION

Dispersions of bubbles and drops in liquids are often encountered in industrial operations. Quantitative description is based on knowledge of behaviour of single particles, yet to-date, rational mechanical models that quantify rigorously the behaviour of even single bubbles are limited to asymptotic conditions, e.g. for terminal velocity, it is assumed that either the substantial derivative is zero, or that the shear stress at the interface is zero.

Here, it is intended to provide further explanations and relations for single fluid particles that can improve the modelling of dispersions in general.

From reviews, e.g. Brodkey (1967), Clift et al (1978) and other work cited in context below, the apparent state of published knowledge dealing with single fluid particles is voluminous and not always unanimous. A brief summary can be as follows :

Significant Parameters

Motion of fluid particles may be affected by the following parameters : density of fluid particle phase = ρ_d, density of continuous liquid phase = ρ_c, density difference = $\Delta\rho = |\rho_d - \rho_c|$, viscosity of fluid particle = μ_d, viscosity of continuous liquid phase = μ_c, interfacial tension (surface tension) = σ, "surface viscosity" = b/D, diameter of vessel or duct containing the fluid = d, volume of fluid particle = V, nominal diameter of fluid particle = D = $(6V/\pi)^{1/3}$. The surface viscosity b/D is not known a priori and has value as an explanatory concept only.

The foregoing parameters can be incorporated in dimensionless groups for more concise presentation of equations. The groups are :

$$\text{Particle Reynolds number} = Re_D = \frac{\mu_c\, U_f D}{\mu_c}, \qquad (1)$$

U_f = terminal velocity in free motion

$$\text{Hadamard-Rybczynski factor} = F_{HR} = \frac{3\,\mu_d + 2\,\mu_c}{3\,\mu_d + 3\,\mu_c} \qquad (2)$$

$$\text{Boussinesq factor} = F_B = \frac{3\,\mu_d + 2\,\mu_c + (b/D)}{3\,\mu_d + 3\,\mu_c + (b/D)} \qquad (3)$$

$$\text{Particle Eotvos number} = Eo_D = \frac{\Delta\rho g\, D^2}{\sigma} \qquad (4)$$

$$\text{Vessel or duct Eotvos number} = Eo_d = \frac{\Delta\rho g\, d^2}{\sigma} \qquad (5)$$

Bubble Formation

Bubbles are created by various processes, here, discussion of bubble formation is restricted to the case of gas injection into liquid.

At low frequency of bubble formation, surface tension, gravity and buoyancy provide the only significant forces. Thus with gas orifice diameter d_o,

$$D_o = (6 d_o \sigma\, (\Delta\rho g)^{-1})^{1/3} \qquad (6)$$

With increasing gas flowrate, bubble formation frequency increases until the bubbles form a bubble chain. Further increases of gas flowrte result in larger bubbles which are formed at nearly constant frequency. A simple bubble chain model (Lehrer, 1971) preicts that at constant gas flowrate V_d' from a single orifice,

$$D_o = \left(\frac{6\,V_d'}{\pi}\right)^{2/5} \left(\frac{3\,\rho_c}{4\,\Delta\rho g}\right)^{1/5} C_D^{\;1/5} \qquad (7)$$

where C_D is the drag coefficient of the bubble of size D_o in free motion. Eq.(7) predicts values that are close to those predicted by models based on inertia or on viscosity dominance within their respective range of applicability.

Particle Shape

In a given system, small bubbles and drops are nearly spherical, those of intermediate size are labile but appear spheroidal to the unaided human eye. Large bubbles moving freely into quiescent liquid tend to be cap-like with a frontal surface of constant radius R near the pole (Fig.1). The angle θ which defines the shape of the cap has an average value of nearly 50 deg when Re_D>200. The angle θ increases sharply with decreasing values of Re_D below 200. In turbulent liquids, large bubbles become irregular in shape and break up, often with recoalescence.

When confined to move vertically in axisymmetric tubes, bubbles whose D-value is of the same order of magnitude or larger than the tube diameter d have a convex domed front which merges into an almost cylindrical frustum.

Particle Motion

The force balance for a single particle in free steady motion is

$$\Delta\rho\; g\; V = C_D \pi D^2 \rho_c\; U_f^2/8 = \Delta\rho\; g\pi D^3/6 \qquad (8)$$

therefore

$$U_f = \left(\frac{4\;\Delta\rho\; g\; D}{3\;\rho_c\; C_D}\right)^{1/2} \qquad (9)$$

It is not certain that all experimental determinations of terminal velocity report truly free motion.

For motion of fluid spheres at very low Re_D values, successive arguments have resulted in, using eq's(1) and (3),

$$C_D = \frac{24}{Re_{D,\text{rigid sphere}}} F_B^2 \cdot Re_D<1,\text{ say} \quad (10)$$

Eq(10) is based on creeping flow around a sphere, and allowing for full circulation within a fluid sphere;- hindrance to such circulation due to surface effects and tendency to rigid sphere velocity with decreasing size D, is discussed in texts, e.g. Happel and Brenner (1965). In a given fluid/fluid system, the influence of viscosity on particle motion decreases with increasing particle size. There is a critical particle size D_{crit} above which description of particle motion no longer requires consideration of viscosities. This diameter at transition from viscosity-dependent to viscosity-independent motion can be estimated for both bubbles and drops from (Lehrer, 1980)

$$D_{\text{crit},\,(\text{min})} = \left\{\frac{3888\ \sigma}{\rho_c}\left(\frac{u_c + u_d}{\Delta\rho\ g}\right)^2\right\}^{1/5}, \quad Eo_{D_{crit}}<6 \quad (11)$$

When there are impurities in the liquid, the transition point is less sharply defined on the U_f vs D plot. Transition is not clearly seen when $Eö_{D_{crit}}>6$.

For the viscosity-independent regime of bubble motion, Mendelson (1967) set velocity U_f in eq (9) equal to that of a surface wave, the concomitant drag coefficient is

$$C_D = \frac{8\ Eo_D}{3\ (4 + Eo_D)}, \quad D>D_{crit} \quad (12)$$

Based on conversion of potential to kinetic energy, a terminal velocity and hence a drag coefficient were derived for bubbles and drops in the viscosity-independent regime (Lehrer, 1976). The drag coefficient is

$$C_D = \frac{8}{3}\ \frac{Eo_D}{(6 + Eo_D)}, \quad D>D_{crit} \quad (13)$$

Using eq's (9) and (13), a minimum terminal velocity occurs in free motion when (gravity + buoyancy force) = (surface tension force) = (drag force), i.e.

$$\frac{\pi\Delta\rho g D^3}{6} = \pi D\sigma = C_D \pi D^2 \rho_c U_f^2/8, \quad Eo_D = 6,$$

$$D = (6\sigma(\Delta\rho g)^{-1})^{1/2} \quad (14)$$

For analysing the rise of large bubbles into quiescent liquid, Davies and Taylor (1950) postulated a spherical surface near the apex of the cap-shaped bubble (Fig. 1), potential flow over this spherical shape and applicability of the Bernouilli equation to this tangential flow. Thus with cap radius R,

$$(9/4)\ U_f^2 \sin^2\theta = 2\ g\ R\ (1 - \cos\theta) \quad (15)$$

$$U_f = (2/3)\ (gR)^{1/2}, \quad \theta \to 0 \quad (16)$$

Because the bubble has constant shape, the velocity of the apex is the velocity of the bubble, therefore eq (16) states the terminal velocity.

In bounded motion of fluid particles, limiting conditions exist when the equivalent diameter of the bounding vessel is the only length which is significant in bubble motion.

In a round tube of diameter d, bubble velocity tends to zero when (Gibson, 1913)

$$\rho_c g d = 4\ \sigma\ d^{-1} \quad (17)$$

i.e. $U \to 0$ when $\rho_c g\ d^2 \sigma^{-1} = Eo_d<4$ (18)

The frontal shape and the motion of "infinitely long" gas bubbles in round vertical tubes of diameter d were investigated by Dumitrescu (1943). The analysis assumed insignificance of viscosity and surface tension effects. The resulting value of rise velocity U was confirmed by experiment and is

$$U = 0.35\ (g\ d)^{1/2} \quad (19)$$

Maneri and Mendelson (1968) proposed that for single bubbles in round vertical tubes at large Eo_d rise velocity U is

$$U = U_f\ (\tanh\ (0.25\ d\ D^{-1})\ , \quad Eo_d \to \infty \quad (20)$$

At large values of d/D, the motion approaches free motion and $U = U_f$. For small values of Eo_d a more complex equation was proposed.

DISCUSSION

Drag Coefficient in Free Motion.

Figure 2 shows typical relations between C_D and Re_D. There are four regimes of motion. Also shown are the physical properties that are significant within each of the regimes. Not quantifiable in their effect on motion are contaminants and particle deformation. Both of these are significant in regime II, causing a noticeable increase of C_D above the values provided by eq. (10). Within regimes III and IV, eq. (13) holds for pure fluids; it holds for all fluids in regime IV. For regime II, a number of empirical equations that require testing for Re_D have been proposed. Regime I, i.e. $Re_D<1$, has been investigated with some highly viscous liquids and there are few data.

One may consider the following : The lowest values of eq. (10), i.e. with $F_B = F_{HR}$, vide eqs. (2) and (3), do not appear as experimental data.

The usually higher C_D values are generally ascribed to surface impurities. An additional argument which may also help to explain the often-observed spiralling or zig-zag motion assumes that internal circulation may be periodic and/or may exist in only part of the sphere. External drag causes internal circulation which reduces drag, but the concomitant increased particle velocity results in higher rate of energy dissipation. This is an unstable condition. Hence a model here is based on periodicity, C_D cycling between values for a rigid sphere and for a fully circulating sphere. In view of the above argument,

$$\frac{d\ C_D}{dt} = -\ B\ C_D\ ;\ \log\ C_D = -\ Bt + \text{constant}\ ;$$
$$B = \text{constant} \quad (21)$$

At t = 0, $C_D = C_{D,\text{rigid sphere}}$; at $t = t_{\text{half-period}}$,

$C_D = C_{D,\text{ciruclating sphere}}$,

thus $(C_D/C_{D,\text{rigid sphere}}) = \exp(-\ Bt)$. (22)

A mean value of C_D is that at $t = t_{\text{half-period}}/2$; the drag coefficient is then

$$C_D = (C_{D,\text{rigid sphere}}\ C_{D,\text{circulating sphere}})^{\frac{1}{2}} \quad (23)$$

Eq. (23) pertains to the viscosity-dependent regime. Period is of order (D/U), which from eq's (9) and (10) is proportional to D^{-1}, i.e. rigidity increases with decreasing D. The detailed equation for the minimum drag coefficient is, from eqs (2),(10),(12) and (23)

$$C_{D,\text{minimum}} = \frac{24\ F_{HR}}{Re_{D,\text{rigid sphere}}} + \frac{8}{3}\ \frac{Eo_D}{(6 + Eo_d)} \quad (24)$$

The upper bound of drag coefficient values for fluid particles is, from inspection of experimental data and eq. (24)

$$C_{D,\mathrm{maximum}} = \frac{24}{Re_{D,\mathrm{rigid\ sphere}}} + \frac{8}{3} \qquad (25)$$

Figures 3 and 4 show typical observed results. There is considerable scatter in regime II and at the low Re_D end of regime III. Considering the difficulty of formulating an accurate, quantitative, analytical description, it is proposed that for the often-met systems with moderate impurities,

$$C_D = \frac{8}{3}\left[\frac{24\ F_{HR}}{Re_{D,\mathrm{rigid\ sphere}}} + \frac{Eo_D}{6 + Eo_D}\right] \qquad (26)$$

The equation applies to bubbles and drops.
In regime III, eq(26) is close to the correlation for this regime based on data for pure liquids(Thorsen etal) The velocities based on eqs (24), (25) and (26) respectively are shown on Figure 3 for bubbles in water

Bubble Motion and Drop Motion in Round Vertical Tubes

In free motion, terminal velocity U_f is estimated from physical properties and the bubble diameter D, using eqn. (9) and an appropriate value of C_D. Eq(19) estimates the rise velocity of a bubble in a vertical, round tube when bubble size D is near to, or larger than tube diameter d and when viscosity and interfacial tension have negligible effect. Eq. (19) is independent of bubble size D and it should be used only within it's range of validity, which is within the slugflow regime. Published flow maps indicate the conditions for slugflow, but there are discrepancies between flowmaps (Clark and Flemmer, 1985). Models that are valid for the whole range of flow conditions are preferable. Maneri and Mendelson (1968) proposed eq. (20) for large Eo_d, for small Eo_d, they proposed a more complicated equation.

Here, an equation is proposed that is simple, satisfies eq's (17) and (18) and reduces to eq. (19) at the appropriate conditions. The equation is :

$$U = U_f\ \left(\tanh\left(\frac{d}{D}\left(\frac{1}{4} - \frac{1}{Eo_d}\right)\right)\right)^{\frac{1}{2}}, \quad Eo_d = \Delta\rho\ g\ d^2\sigma^{-1} > 4 \qquad (27)$$

Then, the drag coefficient in eq(5) can be revised to provide a comprehensive equation for terminal velocity of a single particle, thus

$$U^2 = \frac{4\ \Delta\rho\ g\ D}{3\ \rho_c\ C_{D,d}}, \quad d > \left(\frac{4\ \sigma}{\Delta\ \rho\ g}\right)^{\frac{1}{2}}, \quad \text{i.e. } Eo_d > 4 \qquad (28)$$

and the comprehensive drag coefficient $C_{D,d}$ is, using eq. (24), and detailing all parameters in eq's(29)and (30),

$$C_{D,d,\mathrm{minimum}} = \left[\frac{432\ \mu_c^2}{\rho_c\ \Delta\rho g\ D^3}\left(\frac{3\mu_d + 2\mu_c}{3\mu_d + 3\mu_c}\right) + \frac{8\ \Delta\rho g\ D^2\sigma^{-1}}{3(6 + \Delta\rho g\ D^2\sigma^{-1})}\right] \times \left[\tanh\left(\frac{d}{D}\left(\frac{1}{4} - \frac{\sigma}{\Delta\rho g\ d^2}\right)\right)\right]^{-1}, \qquad (29)$$

Similarly, using eq. (25),

$$C_{D,d,\mathrm{maximum}} = \left(\frac{432\ \mu_c^2}{\rho_c\Delta\rho g\ D^3} + \frac{8}{3}\right)\left[\tanh\left(\frac{d}{D}\left(\frac{1}{4} - \frac{\sigma}{\Delta\rho g\ d^2}\right)\right)\right]^{-1}, \qquad (30)$$

and (26),

$$C_{D,d\ \mathrm{probable}} = \frac{8}{3}\left[\frac{\dfrac{24\ F_{HR}}{Re_{D,\mathrm{rigid\ sphere}}} + \dfrac{Eo_D}{6 + Eo_D}}{\tanh\left(\frac{d}{D}\left(\frac{1}{4} - \frac{1}{Eo_d}\right)\right)}\right] \qquad (26a)$$

Figure 4 illustrates the accuracy of eq. (26a).

The Shape of Spherical Cap Bubbles

The motion of spherical cap bubbles has been described successfully by using the equations for ideal flow around a sphere and the relation

$$\rho\ v^2/2 + \rho gz = \text{constant} \qquad (31)$$

(Davies and Taylor, 1950)

Conditions included axisymmetric flow of an inviscid liquid along streamlines in the θ-direction only when θ>0, uniform pressure in the bubble and therefore at the gas/liquid interface as implied by eq. (30) and a spherical surface with radius R in the vicinity of θ = 0. If it assumed thatthe constant radius R exists over the whole frontal surface (Fig. 1), the equations of motion (e.g. Bird et al, 1962)are along the interface at R -

$$-\frac{v_\theta^2}{R} = -\ g\cos\theta \qquad (32)$$

$$\frac{v_\theta}{R}\frac{\partial\ v_\theta}{\partial\ \theta} = g\sin\theta \qquad (33)$$

$v_\theta = 0$ at $\theta = 0$, then from eq's (32) and (33),

$$v_\theta^2 = 2gR(1 - \cos\theta) = g\ R\cos\theta \qquad (34)$$

$$\cos\theta = 2/3, \quad \theta = 48.19^\circ \qquad (35)$$

In a review, Wegener and Parlange (1973) report mean values of $\theta = 48^\circ$ for values of Re_D near 20,000, and $\theta \cong 50^\circ$ for values of Re_D above 200. For $\theta = 48.19^\circ$, the ratio D/R = 0.84. Terminal velocity predictions are also in very good agreement with reported observed values.

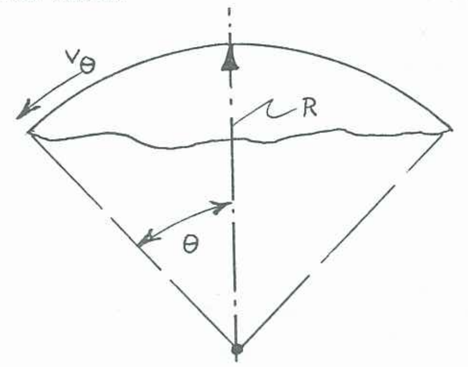

Fig. 1. Spherical cap bubble.

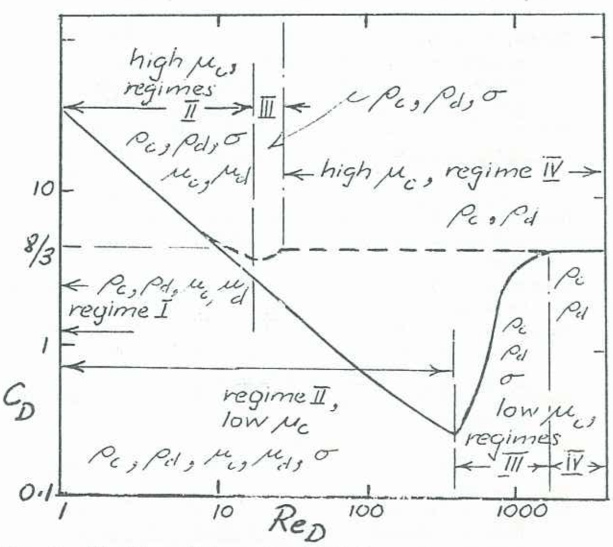

Fig. 2: Fluid particles, typical C_D vs Re_D relations.

NOMENCLATURE

(Any consistent system of units)

b	=	contaminant flow, eq(2),	Mt^{-1}
C_D	=	drag coefficient of particle in free motion	
$C_{D,d}$	=	drag coefficient of fluid particle	
d	=	vessel diameter,	L
D	=	particle diameter, eq(6),	L
Eo_d	=	vessel Eotvos number, eq(5),	L
Eo_D	=	particle Eotvos number, eq(4)	
F_B	=	Boussinesq factor, eq(3)	
F_{HR}	=	Hadamard-Rybczynski factor, eq(2)	
g	=	gravity acceleration,	Lt^{-2}
R	=	radius of spherical cap	R
Re_D	=	particle Reynolds number	
U	=	terminal velocity of fluid particle,	Lt^{-1}
U_f	=	terminal velocity of particle in free motion	Lt^{-1}
v	=	velocity,	Lt^{-1}
V	=	volume of particle	L^3
V_d'	=	volume flowrate of dispersed phase	L^3t^{-1}
z	=	vertical distance,	L
Δ	=	difference	
θ	=	angle	
μ	=	viscosity,	$ML^{-1}t^{-1}$
ρ	=	density,	ML^{-3}
σ	=	surface tension,	Mt^{-2}

Subscripts

c	=	continuous phase
d	=	dispersed phase, - except Eo_d
o	=	at orifice
θ	=	in θ-direction

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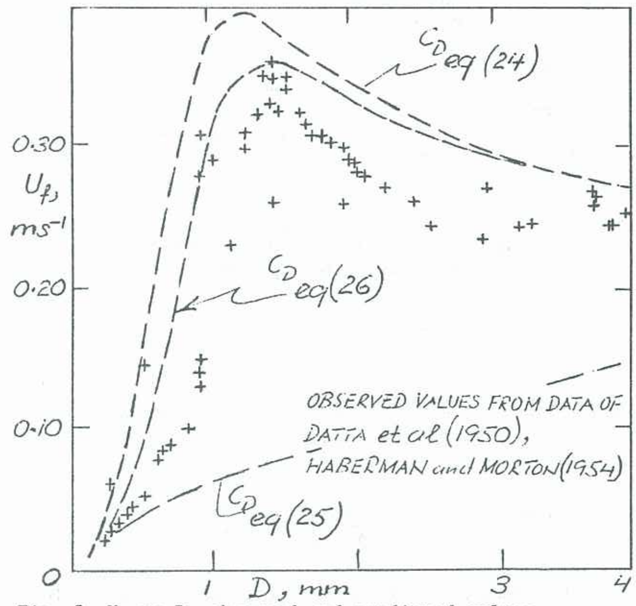

Fig. 3. U_f vs D, observed and predicted values.

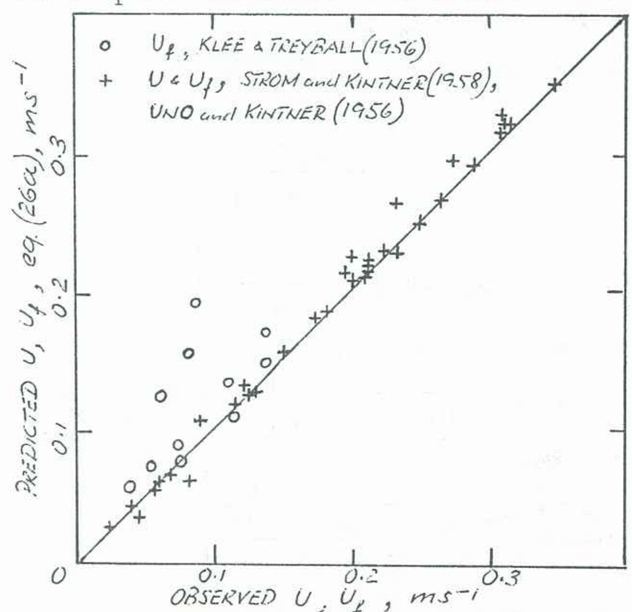

Fig. 4. U and U_f, observed vs predicted values.