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Parabolic Quadratic Weir

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ABSTRACT

This paper is devoted to the consideration of the design of a proportional weir called 'parabolic proportional weir' capable of giving a discharge proportional to the square root of the head measured above a datum. It consists of a base in the form of a parabolic weir of height 'a' and top width '2W' over which a designed curve is fitted. The problem is solved by using the generalised theory and the slope-discharge continuity theorem. It is found that the reference plane for this weir is situated at (3/4)a above the crest, so that the discharge is proportional to the square root of the head measured above this plane.

This weir has a very interesting and significant feature in that the equations governing the complimentary weir (above the base) is very rapidly convergent converting it into a proportional orifice even at small depths of flow. It is theoretically proved that the complimentary profile is positive and decreasing throughout $0 \leq x/a \leq \infty$. This characteristic makes it as another example of Notch-Orifice invented by the first author. Experiments conducted on three weirs show excellent agreement with the theory by giving a constant coefficient of discharge of 0.61.

1. INTRODUCTION

Weirs and notches have been the subject of considerable research in hydraulics because of their simplicity and wide use. Rectangular weir, V-notch, trapezoidal weir are amongst the most explored weirs. Their characteristics are well understood. As against this, proportional weirs which can give a known head-discharge relationship have been relatively less known. In 1908, Sutro (1) designed a linear proportional weir, which had a rectangular base weir of depth 'a' over which a designed curve was fitted. He found out that for flows above the base, the discharges proportional to the head measured above a reference plane or datum situated at (1/3)a above the crest. Although this worked out alright, no rational explanation was given for the choice of the reference plane. It was till recently held (2) that the reference plane could be arbitrarily chosen. This was a major lacuna that existed in the theory of proportional weirs. A rational explanation for the choice of the reference plane and an analytical method for the design of the weir was presented by Keshava Murthy and Seshagiri (3,4). Accordingly every weir is uniquely associated with a reference plane which is in accordance with the slope discharge continuity theorem which states that the "rate of change of discharge with respect to head in any physically realisable compound weir, is continuous at all points of discontinuity".

A new classification of weirs was presented by Keshava Murthy and K.G.K. Pillai (5,6). The weirs are classified as Base and Non-base weirs, according to the head-discharge relationship $Q = bH^m$. If $m \geq 3/2$, they are called non-base weirs; the rectangular weir, V-notch are the well known examples. It was further pointed out that the Non-base weirs can also be designed with base weirs. Hence they are further classified as compensating non-base weirs and non-compensating non-base weirs. When $m < 3/2$, they are called base weirs which invariably require a base for their design. Logarithmic weir, quadratic weir are some examples of this case.

Haszpra (7) was the first to tackle the design of the quadratic weir. This arose in the proportional method of flow measurement in a by-pass. The criterion for the applicability of proportional measurement of flow is that the differential head actuating the meter in the by-pass should hold an approximate quadratic relationship with the discharge conveyed in the channel. This is so because the resistance to flow offered by a meter is in quadratic relationship with the discharge. In trying to solve the problem, Haszpra met with expected difficulties and his solution is not even approximate.

An exact solution to the problem was presented by Keshava Murthy (3,4). The problem was solved using the slope discharge continuity theorem. In this paper another interesting solution is provided which gives another example of a Notch-orifice.

2. MATHEMATICAL ANALYSIS

The base for the weir is chosen in the form of a parabolic weir of width '2W' and depth 'a', over which a compensating weir to be designed is fitted (fig.1). Let the reference plane of the weir be located at a depth 'λa', below the origin, where 'λ' is the datum constant. The weir is considered to be sharp crested and symmetrical.

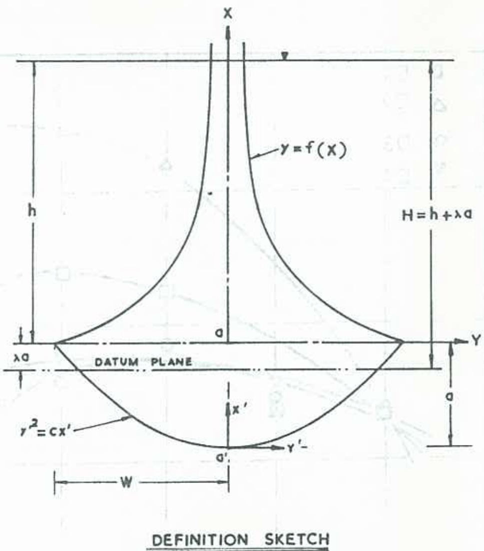


Fig. 1

For convenience let O' be the origin of co-ordinates x' and y' for the parabolic base.

Let the equations of the parabola be given by

$$y'^2 = cx' \tag{1}$$

Referring to fig.1, the discharge passing through the parabolic portions (acting as an orifice) when the head is 'h' above the origin

$$q_1 = 2\,C_d\sqrt{2g}\int_0^a \sqrt{h+a-x'}\; y'(x')\,dx' \qquad (2)$$

where C_d = coefficient of discharge which is assumed to be the same for both notch and orifice portions.

g = acceleration due to gravity
h = head of flow above the parabolic base.

On substituting for $y'(x')$ in (1), and integrating, we get

$$q_1 = 2C_d\sqrt{2g}\int_0^a \sqrt{h+a-x'}\;(cx')^{1/2}\,dx'$$

$$= KC^{1/2}\left[\frac{a-h}{4}\sqrt{ah} + \frac{(h+a)^2}{8}\left\{\sin^{-1}\frac{a-h}{a+h} + \frac{\pi}{2}\right\}\right] \qquad (3)$$

where $K = 2C_d\sqrt{2g}$.

The discharge through the curved portion above the base is given by

$$q_2 = 2C_d\sqrt{2g}\int_0^h \sqrt{h-x}\; f(x)\,dx$$

$$= K\int_0^h \sqrt{h-x}\; f(x)\,dx \qquad (4)$$

where $f(x)$ is the function defining the complimentary weir to be determined.

The total discharge passing through the weir when the depth of flow is 'h' above the base weir, is,

$$Q = q_1 + q_2$$

i.e.
$$Q = KC^{1/2}\left[\frac{a-h}{4}\sqrt{ah} + \frac{(h+a)^2}{8}\left\{\sin^{-1}\frac{a-h}{a+h} + \frac{\pi}{2}\right\}\right] + K\int_0^h \sqrt{h-x}\; f(x)\,dx \quad . \qquad (5)$$

We want this discharge to be proportional to the square root of the head measured above a reference plane. Hence we can write

$$Q = KC^{1/2}\left[\frac{a-h}{4}\sqrt{ah} + \frac{(h+a)^2}{8}\left\{\sin^{-1}\frac{a-h}{a+h} + \frac{\pi}{2}\right\}\right] + K\int_0^h \sqrt{h-x}\; f(x)\,dx$$

$$= b\sqrt{h+\lambda a} \quad \text{for } h \geq 0 \quad . \qquad (6)$$

Here, 'b' is the constant of proportionality and 'λ' is the datum constant to be determined.

Equation (6) has to satisfy two conditions. The first condition is that there is no flow through the curved portion when h=0.

i.e., $b\sqrt{\lambda a} = KC^{1/2}\dfrac{\pi a^2}{8}$

or $$b = \frac{K\pi}{8\sqrt{\lambda}}\,C^{1/2}\,a^{3/2} \qquad (7)$$

Differentiating (6) using Leibnitz's rule and rearranging

$$\int_0^h \frac{f(x)}{\sqrt{h-a}}\,dx = \frac{b}{K}\,\frac{1}{\sqrt{h+\lambda a}} + C^{1/2}\sqrt{ah} - \frac{C^{1/2}(a+h)}{2}\left[\frac{\pi}{2} + \sin^{-1}\frac{a-h}{a+h}\right] \qquad (8)$$

$$= \phi(h) \text{ say.}$$

The second condition arises out of the compliance with the slope discharge continuity theorem (i.e., the rate of change of discharge is continuous at h=0). This means that the R.H.S. of eq.(8) should be zero for h=0.

Hence

$$0 = \frac{b}{K\sqrt{\lambda a}} - C^{1/2}\frac{\pi a}{2} \qquad (9)$$

Solving eqs. (7) and (9), for 'b' and 'λ', we have

$$b = c^{1/2}\,a^{3/2}\,\frac{\pi K}{4} \;,\quad \lambda = 1/4 \qquad (10)$$

If we put $Ca^{1/2}$ = W (half width of the weir)

$$b = \frac{Wa\pi K}{4}$$

Equation (8) is in the Abel's form of the Volterra's integral equation (RE)F and can be solved as

$$f(x) = \frac{1}{\pi}\int_0^x \frac{\phi'(h)}{\sqrt{x-h}}\,dh$$

Substituting for $\phi(h)$ and integrating and simplifying, we have

$$f(x) = W\left[\sqrt{1+x/a} - \sqrt{x/a} - \frac{2\sqrt{x/a}}{1+4x/a}\right] \qquad (11)$$

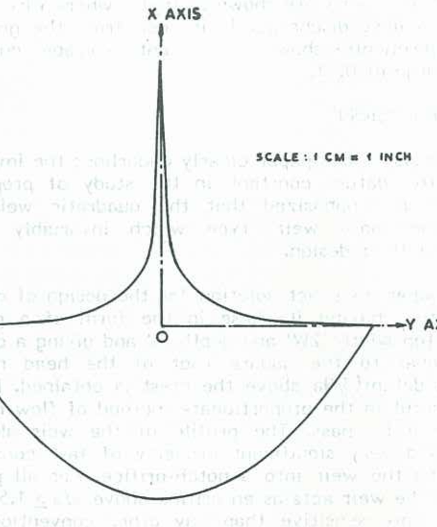
The coordinates of the points of the weir are given in table 1. A weir plotted to scale is shown in fig.(2).

Table 1

Co-ordinates of points on f(x) given by eqn.(11)

$\frac{x}{a}$	$\frac{y}{W}$	$\frac{x}{a}$	$\frac{y}{W}$	$\frac{x}{a}$	$\frac{y}{W}$
0.00	1.0000	1.20	0.0101	2.40	0.0024
0.10	0.2808	1.30	0.0086	2.50	0.0022
0.20	0.1513	1.40	0.0074	2.60	0.0020
0.30	0.0945	1.50	0.0065	2.70	0.0019
0.40	0.0643	1.60	0.0057	2.80	0.0017
0.50	0.0462	1.70	0.0050	2.90	0.0016
0.60	0.0347	1.80	0.0045	3.00	0.0015
0.70	0.0268	1.90	0.0040	3.50	0.0011
0.80	0.0213	2.00	0.0036	4.00	0.0008
0.90	0.0173	2.10	0.0032	4.50	0.0006
1.00	0.0142	2.20	0.0030	5.00	0.0005
1.10	0.0119	2.30	0.0027		

For further values of $x/a > 5.00$, y/W can be practically taken as zero.



QUADRATIC WEIR WITH PARABOLIC BOTTOM
a=5 W=6
(12.7 cm) (15.24 cm)

Fig.2.

3. EXAMPLE OF NOTCH-ORIFICE

From table 1 we observe a very interesting and significant fact that f(x) given by (11) is rapidly convergent and almost tends to zero for $x/a \geq 1.7$. For even as small a value of $x/a=5$ the value of y/W becomes zero correct to three significant digits after the decimal. Beyond this we can say that the weir for all practical purposes acts as a weir of zero width or as an orifice. However, before we may conclude this, it is necessary to show that f(x) never crosses the x-axis, or in other words, it always remains positive for positive values of x, as otherwise it would lead to physically absurd results. We have from equation (11) when $x=0$, $y=W$; and when $x \to \infty$, $y \to 0$. Differentiating equation (11) w.r.t. x, we have

$$\frac{dy}{dx} = \frac{W}{x/a} \frac{1}{\left[\sqrt{1+\frac{x}{a}}+\frac{x}{a}\right]^3 \left[1+\frac{4x}{a}\right]} \geq 0 \ ,$$

which is always positive, for all positive values of x and as the curve starts from $y=W$ at $x=0$, and does not change its sign of its slope in $0 \leq x \leq \infty$, it is concluded that f(x) is positive and decreasing for all $0 \leq x \leq \infty$. Practically, we can say for $x \geq 1.7a$, it acts as a "notch of zero width", i.e., a proportional orifice. As the weir can pass a discharge proportional to the square root of the head above a datum, both while acting as a notch and as an orifice it is another example of 'Notch Orifice'.

4. EXPERIMENTS

Experiments to check the theory was conducted on three weirs having the dimensions (i) a=10 cm, W=15 cm; (ii) a=10 cm, W=25 cm; (iii) a=12.5 cm, W=15 cm.

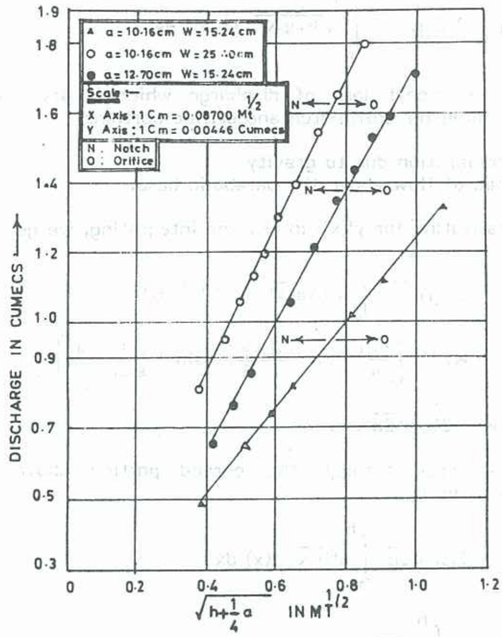
The above three weirs of which weir (i) is shown in fig.(2) were experimented. The weirs were cut in 1/4" (6.25 mm) mild steel plate. The boundaries of the weirs were carefully marked on the plate by a smooth curve. The opening was then cut roughly by gas cutting and then accurately filed to the required shape. The weirs had a sharp edge of 1/16" (1.56 mm) with a 45° chamfer. The weir was fixed at the end of a rectangular channel 40 ft (14 m) long and 4 ft (1.2 m) wide and 3 ft (1 m) deep.

The head over the weir was measured by an electric hook gauge capable of reading 0.003 mm and the discharges were recorded in a tank measuring 3m x 3m x 1.4m. Each experiment was repeated thrice to ensure accuracy. Experiments were conducted for ranges of head constituting both notch and orifice. The experimental results for the three weirs are shown in fig.(3) where $\sqrt{h+(1/4)a}$ is plotted against discharge. It is seen from the graph that the experiments show a constant average coefficient of discharge of 0.61.

5. CONCLUSION

The analysis of the paper clearly underlines the importance of 'λ', the datum constant in the study of proportional weirs. It is emphasized that the quadratic weir comes under the 'base weir' type which invariably requires a base for their design.

In this paper an exact solution for the design of quadratic weir notch, having its base in the form of a parabolic weir of top width '2W' and depth 'a' and giving a discharge proportional to the square root of the head measured above a datum (3/4)a above the crest is obtained. It should prove useful in the proportionate method of flow measurement in a by pass. The profile of the weir above the base has a very significant property of fast convergence converting the weir into a notch-orifice. For all practical purposes the weir acts as an orifice above $x/a \geq 1.5$. This weir is more sensitive than any other conventional weir in that an error of 1% in the head measurement causes only 0.5% error in discharge as against 1.5% in a rectangular weir and 2.5% in a V-notch. Experiments with three typical weirs confirms the theory by giving a constant average coefficient of discharge of 0.62.



GRAPH OF Q VS. $\sqrt{h+\frac{1}{4}a}$

Fig.3.

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