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The Fourth Order Evolution Equation for Deep Water Waves

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ABSTRACT

The stability of a train of nonlinear gravity-capillary waves on the surface of an ideal fluid of infinite depth is considered. An evolution equation for the wave envelope is derived from the Zakharov equation. The main difference from the third-order evolution equation is, as far as stability is concerned, the introduction of a mean flow response. In general the mean flow effects for pure capillary waves are of opposite sign to those of pure gravity waves. The second-order corrections to first-order stability properties are shown to depend on the interaction between the mean flow and the envelope frequency-dispersion term. The results are shown to be in agreement with some recent computations of the full problem.

INTRODUCTION

The problem of the stability of a wave train on water has attracted a great deal of interest ever since a uniform train of gravity waves was shown to be unstable in the inevitable presence of side bands (Benjamin and Feir 1967). Subsequently for small amplitudes the cubic Schrödinger equation was derived to describe the evolution of a wave train. Later Dysthe (1979) derived the fourth order evolution equation for pure gravity waves.

In this paper we go a stage further by including the effects of surface tension and adopting the approach based on Zakharov's (1968) equation. This leads us to the general result that dispersion effects, when combined with the mean flow term, provide the second order corrections to first order stability properties. We are also able to assign roles to all the other terms in the evolution equation, except one.

FOURTH-ORDER EVOLUTION EQUATION

We consider a spectrum of gravity-capillary waves on the surface of an ideal fluid of infinite depth. The free surface is given by $z = \eta(\underline{x},t)$ where $\underline{x} = (x,y)$ is the horizontal space vector, z is the vertical coordinate and t is the time. The frequency ω is related to the wavenumber $\underline{k}$ by the dispersion relation

$$\omega^2(\underline{k}) = g|\underline{k}| + S|k|^3 \qquad (1)$$

where g is the acceleration due to gravity and S is the surface tension coefficient divided by the density.

If the free surface $\eta(\underline{x},t)$ is written as

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|\underline{k}|}{2\omega(\underline{k})}\right)^{\frac{1}{2}}\{B(\underline{k},t)\exp[i(k\cdot x-\omega(k)t)]+c.c.\}dk \qquad (2)$$

where c.c. denotes the complex conjugate, then the Zakharov integral equation is given by

$$i\frac{\partial B}{\partial t}(\underline{k},t) = \int_{-\infty}^{\infty}\iint T(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)B^*(\underline{k}_1,t)B(\underline{k}_2,t)B(\underline{k}_3,t)$$
$$\delta(\underline{k}+\underline{k}_1-\underline{k}_2-\underline{k}_3)\times\exp\{i[\omega(\underline{k})+\omega(\underline{k}_1)-\omega(\underline{k}_2)-\omega(\underline{k}_3)]t\}d\underline{k}_1d\underline{k}_2d\underline{k}_3 \qquad (3)$$

The real function $T(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)$ was first given for free surface gravity-capillary waves by Zakharov (1968).

Equation (3) has been used to analyse the stability of gravity waves (Crawford et al. 1981, Stiassnie and Shemer 1984). It is also valid for gravity-capillary waves if the wave packet is sufficiently narrow, that is $|\underline{k}-\underline{k}_0|<<|\underline{k}_0|$ for some $|\underline{k}_0|$. A broad spectrum may lead to triad interactions of the sort

$$\left.\begin{aligned}\omega(\underline{k}) &= \omega(\underline{k}_1) + \omega(\underline{k}_2)\\ \underline{k} &= \underline{k}_1 + \underline{k}_2\end{aligned}\right\} \qquad (4)$$

The set of equations (4) does have a solution for gravity-capillary waves, unlike pure gravity waves. This is precisely the condition that will give a zero denominator in one of the terms of T. But if a narrow spectrum of waves is considered then the triad resonance conditions (4) can not be satisfied.

Under the assumption of a narrow band of waves centred on $\underline{k} = \underline{k}_0 = (k_0,0)$, we can write

$$\eta(\underline{x},t) = \mathrm{Re}\{a(\underline{x},t)e^{i(k_0x-\omega(k_0)t)}\} \qquad (5)$$

We then expand the coefficient T in equation (3) in terms of the spectral width, retaining terms up to the third order. In this way an evolution equation is derived correct to fourth order in the amplitude $a(\underline{x},t)$ (Hogan 1985). It is

$$2i(a_t+c_ga_x) + pa_{xx} + qa_{yy} - \gamma|a|^2a$$
$$= -isa_{xyy} - ira_{xxx} - iua^2a^*_x + iv|a|^2a_x + a\bar{\phi}_x \qquad (6)$$

The scaling transformation $t = t'/\omega_0$, $(x,y)=(x',y')/k_0$, $a = a'/k_0$, $\bar{\phi} = \omega_0\bar{\phi}'/2k_0^2$ have been made and the primes dropped. $\bar{\phi}$ represents an induced mean flow which satisfies

$$\nabla^2\bar{\phi} = 0 \qquad z \leqslant 0 \qquad (7)$$

and

$$\bar{\phi}_z = (|a|^2)_x \quad \text{on} \quad z = 0, \quad \bar{\phi}_z \to 0 \text{ as } z \to -\infty \qquad (8)$$

In equation (6), c_g is the group velocity given by

$$c_g = \frac{1+3\kappa}{2(1+\kappa)} \qquad (9)$$

and the other constants (all real) are given in Hogan (1985).

The parameter κ is given by

$$\kappa = Sk_o^2/g \tag{10}$$

The constants γ, u and v are singular at $\kappa = \frac{1}{2}$. This corresponds to a special case of triad resonance with $\underline{k}_1 = \underline{k}_2 = \frac{1}{2}\underline{k}$, known as subharmonic resonance. In this case an evolution equation is required for the amplitude of the wave and its first harmonic. We have excluded this case from our analysis. (see also McGoldrick 1970).

The first two terms on the left hand side of equation (6) show that to second order, the wave envelope moves with the group velocity. The whole of the left hand side is the well-known cubic Schrödinger equation. From this equation we know that the stability of a nonlinear wavetrain is governed by the terms pa_{xx} and $\gamma|a|^2a$.

This balance of dispersion and nonlinearity maintains a resonant quartet of a wave and its sidebands and allows energy to leak away.

In the next section we consider the effect of the higher order terms on the stability of a uniform wave train.

STABILITY OF A NONLINEAR WAVETRAIN

It is straightforward to show that one solution to equations (6)-(8) is the nonlinear uniform wavetrain given by

$$\bar{\phi} = 0 \quad , \quad a = a_o \exp(-\tfrac{1}{2}ia_o^2\gamma t) \tag{11}$$

where a_o is the wave steepness. Following Dysthe (1979) we examine the stability of this solution by taking

$$\begin{aligned} \bar{\phi} &= \bar{\phi}' \\ a &= a_o(1+a')\exp\{i(\theta'-\tfrac{1}{2}a_o^2\gamma t)\} \end{aligned} \tag{12}$$

We seek solutions for $\bar{\phi}'$ of the form $\exp\{Kz+i(\lambda x+\mu y-\Omega t)\}$ where $K^2 = \lambda^2 + \mu^2$ and for a' and θ' of the form $\exp\{i(\lambda x+\mu y-\Omega t)\}$.

On substituting equations (12) into equations (6)-(8) we find the dispersion relationship for the perturbation is

$$\Omega = (c_g-\tfrac{1}{2}va_o^2)\lambda - \tfrac{1}{2}r\lambda^3 - \tfrac{1}{2}s\mu^2\lambda$$

$$\pm \tfrac{1}{2}\left\{(p\lambda^2+q\mu^2)\left[p\lambda^2+q\mu^2+2a_o^2(\gamma-\frac{\lambda^2}{K})\right]\right\}^{\frac{1}{2}} \tag{13}$$

The constant u does not appear in this expression and the only contribution of the fourth order terms to the expression inside the square root sign is λ^2/K. This comes from the mean flow term. This is identical to the conclusion of Dysthe (1979) in his work on pure gravity waves.

Let us consider perturbations in the propagation, that is x, direction only. Thus we set $\mu = 0$. From (13) we find that we have instability when

$$\lambda < \left(-\frac{2\gamma}{p}\right)^{\frac{1}{2}} a_o + \frac{1}{p}a_o^2 \tag{14}$$

and marginal stability occurs at

$$\Omega = c_g\left[\left(-\frac{2\gamma}{p}\right)^{\frac{1}{2}} a_o + \frac{1}{p}a_o^2\right] \tag{15}$$

There is a region of stability given by $\frac{2}{\sqrt{3}} - 1<\kappa<\frac{1}{2}$ (see Hogan 1985). The maximum growth rate is given by

$$\delta_m = \frac{|\gamma|}{2}\left[a_o^2 - \frac{1}{\gamma}\left(-\frac{\gamma}{p}\right)^{\frac{1}{2}} a_o^3\right] \tag{16}$$

and this occurs at a wavenumber

$$\lambda_m = \left(-\frac{\gamma}{p}\right)^{\frac{1}{2}} a_o + \frac{3a_o^2}{4p} \tag{17}$$

when the real part of Ω has the value

$$\mathrm{Re}\,\Omega_m = c_g\left[\left(-\frac{\gamma}{p}\right)^{\frac{1}{2}} a_o + \frac{3a_o^2}{4p}\right] \tag{18}$$

In each expression (14), (15), (17) and (18) the key element at $O(a_o^2)$ is the term 1/p. This represents a balance between dispersion and the mean flow response.

We recover Dysthe's (1979) results for pure gravity waves by setting $\kappa = 0$ and obtain results for pure capillary waves by setting κ infinite (see Hogan 1985 for full details). We have an independent check on our results for the case $\mu = 0$ and very long perturbations, using the averaged Lagrangian method as shown by Lighthill (1965, 1967)); see Hogan 1985 for details. We can also consider general perturbations in the horizontal plane by taking $\mu \neq 0$.

These results can be used as an important check on full-scale computations of gravity-capillary wave instability. The work of Zhang & Melville (1985) supports our conclusions, see Table 1.

TABLE 1. VALUES OF PERTURBATION WAVENUMBER λ CORRESPONDING TO ONSET OF STABILITY OF GRAVITY-CAPILLARY WAVES OF STEEPNESS $a_o = 0{\cdot}05$

source	$\kappa = 3$	$\kappa = 7$
equation (14) (to $O(a_o)$)	0.036	0.030
equation (14) (to $O(a_o^2)$)	0.040	0.034
full calculation (Zhang & Melville 1985)	0.042	0.037

We note that the $O(a_o^2)$ correction is negative for $\kappa < \frac{1}{2}$ but positive for $\kappa > \frac{1}{2}$, as reflected in the values of Table 1. The work of Chen and Saffman (1985) appears to place an upper limit on a_o and hence on the range of validity of this approach.

Finally we note that we can derive the fourth order evolution equation in terms of the complex velocity potential $\psi(\underline{x},t)$ given by

$$\psi = -ia + \frac{(1-\kappa)}{2(1+\kappa)}a_x \tag{19}$$

The resulting equation is similar to equation (6) except that the unassigned term $-iua^2a_x^*$ becomes $+iu\psi^2\psi^*_x$. The reason for such close similarity remains unresolved (see Hogan (1986) for details).

CONCLUSIONS

(i) We have incorporated surface tension in the fourth order evolution equation of deep water waves.

(ii) The $O(a_o^2)$ corrections to $O(a_o)$ stability properties are caused by dispersion (pa_{xx}) balancing the mean flow ($a\bar{\phi}_x$) to maintain a resonant quartet and hence keep energy flowing from the main wave into sidebands.

(iii) For gravity waves the mean flow detunes the nonlinearity to $a(|a|^2 + \bar{\phi}_x)$ to balance the dispersion $-\frac{1}{4}a_{xx}$. For capillary waves we must balance $a(\frac{1}{8}|a|^2-\bar{\phi}_x)$ $-\frac{3}{4}a_{xx}$.

(iv) Agreement has been found with full computations.

(v) The term $iv|a|^2a_x$ provides the real $O(a_o^2)$ correction to the frequency of very long plane perturbations to the waveform.

(vi) The term $iua^2a^*_x$ appears to have no role in the stability calculations.

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