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A Comparison of Numerical Schemes used in River Morphology Problems

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SUMMARY

The one-dimensional formulation used to mathematically model changes in river bed levels leads to a time-dependent scalar partial differential equation describing the sediment continuity. This scalar equation may be solved by finite-difference methods. The paper evaluates a number of currently used finite-difference schemes in which their linear stability, wave propagation and truncation error properties are considered. The results of numerical experiments are given in which the performances of the various schemes are compared for a problem involving strong non-linearity and shock propagation.

1. NOTATION

1.1 General Symbols

A vector given by Equation 6
B matrix given by Equation 7
c velocity of propagation
c_r relative velocity of propagation
C Chezy coefficient
d numerical damping factor per wave period
$f()$ sediment transport function
F Froude number
g acceleration due to gravity
h water depth
i complex unity $\sqrt{-1}$; bed slope
k wave number
m coefficient in Equation 10
n exponent in Equation 10
p,q,r addresses for truncation errors
s sediment transport rate per unit width
t time
u water velocity
w vector given by $(u,h,z)^T$; general variable
W Fourier coefficient vector in Equation 5
x longitudinal distance
z river bed level
Δh change in water depth with iteration
Δt time step
Δx distance step
ε error in d or c_r
λ truncation error coefficient
θ weighting factor in 6-point scheme
σ Courant number, $c\,\Delta t/\Delta x$
ρ complex amplification factor
ξ ratio of distance step to wave length

1.2 Grid Notation

x_o abscissa where the time difference of w is evaluated
x_j $x_o + j\,\Delta x$
w_o value of w at (x_o,t)
w^o value of w at $(x_o,t + \Delta t)$
$\Delta^t w = w^o - w_o$
$\Delta_{j+\frac{1}{2}} w = w_{j+1} - w_j$
$\Delta_o w = w_{j+1} - w_{j-1}$
$\zeta = \Delta_{-\frac{1}{2}}x/\Delta_{\frac{1}{2}}x$
λ mesh ratio, $\Delta t/\Delta x$

1.3 Subscripts

x,t,u indicate partial differentiation with respect to x, t, and u respectively
o indicates an initial value

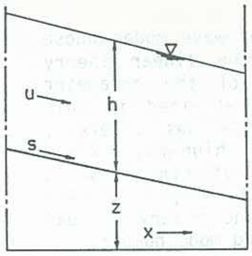

Fig. 1 Definition Diagram

2 FORMULATION

With reference to Figure 1, the one-dimensional formulation describing the bed-level changes in a unit width channel with moderate Froude number is given by:

$$(uh)_x = 0 \tag{1}$$

$$(\tfrac{1}{2}u^2 + gh + gz)_x + u|u|(C^2h)^{-1} = 0 \tag{2}$$

$$z_t + s_x = 0 \tag{3}$$

$$s = f(u, \ldots.) \tag{4}$$

in which a functional representation for the sediment transport relation (Eq.4) has been adopted. These equations were first presented by de Vries (1959, 1965). A discussion of these equations and the assumptions on which they are based can be found in Jansen et al (1979), or Vreugdenhil (1982).

Equations 1 to 4 can be composed as a pure convection problem by writing

$$A\,w_t + B\,w_x = 0 \tag{5}$$

in which $A = (0,\ 0,\ 1)^T$ (6)

and
$$B = \begin{bmatrix} u & g & g \\ h & u & o \\ f_u & o & o \end{bmatrix} \tag{7}$$

A wave-like solution to Equation 5 can be written as

$$w = W \exp\{ik\,(x - ct)\} \tag{8}$$

Substitution of Equation 8 into Equation 5 leads to three eigenvalue solutions c_1, c_2 and c_3 which are the wave propagation or characteristic velocities of small disturbances. Two of these eigenvalues, c_1, and c_2, are associated with the water wave disturbances. The third eigenvalue, c_3, which describes the sediment bed disturbances, is given by:

$$c = c_3 = u\, f_u\, h^{-1}\, (1 - F^2)^{-1} \qquad (9)$$

It can be shown from a more general analysis, in which the time dependency of u and h is considered, that the above quasi-steady formulation is appropriate for $F < 0.7$, more or less, for most practical problems.

For the purpose here, the sediment transport relation is given in the generic form

$$s = m\, u^n \qquad (10)$$

in which m is a function of the physical properties of the sediment (grain size, density, etc). For the analysis here, it may be regarded to be constant. Equation 10 corresponds to the Engelund-Hansen formula for $n = 5$.

Equation 3 is highly non-linear due to the strong dependency of sediment transport on velocity. This non linearity is even stronger for problems involving longitudinal variations in sediment grain size, river width, roughness, etc. This non-linearity gives rise to the special problems in the application of numerical schemes used to solve Equations 1 to 4. Spatial variations in the wave propagation velocity, for example, may lead to the development of shock features in the bed profile represented by wave numbers at or near the grid resolution limit. Due to numerical errors, the shock feature will be smeared and secondary waves may develop. For some finite-difference schemes, these numerical effects may come to dominate the physical answer.

A number of currently used finite-difference schemes, used to solve the sediment continuity equation, are studied below in terms of their linear and non-linear properties. Four of these schemes stem from the American literature and have not, apparently, been studied in this way before in the context of river morphology problems. The presentation here is styled after Vreugdenhil (1982).

3. SOLUTION PROCEDURE

When the wave propagation velocity for the bed disturbances is much smaller than those for the water flow, it is appropriate to use an alternating step procedure for the solution of Equations 1 to 4 (Jansen, et al, 1979), namely

Step I: Compute u with Eqs 1 and 2 for known z
Step II: Compute z with Eqs 3 and 4 for known u

Step I is a conventional backwater calculation. Any of the well known procedures may be applied. For the results presented here, a standard step modified Euler solution procedure has been used with a convergence criterion of $|\Delta h/h| < 0.001$.

Step II represents the finite-difference solution of the sediment continuity equation (Eq.3). Solution algorithms are presented in the next section.

4. NUMERICAL SCHEMES

Our attention is now concentrated on the solution of Equation 3 using finite-difference methods. Six numerical schemes are investigated below, the algorithms for which are summarised in Table 1. The grid notation is given in Section 1.

The Godunov, Fromm, KUWASER, and HEC-6 schemes are explicit. The FLUVIAL-11 and 6-point schemes are implicit. For the implicit schemes an iterative procedure is used to solve for terms at the $t + \Delta t$ time step.

For the 6-point scheme, Holley et al (1984) use $\theta = 0$ for the first iteration and $\theta = \frac{1}{2}$ for subsequent iterations. An earlier reference to the 6-point scheme can be found in Karim and Kennedy (1982) in which $\theta = 0$ was adopted. Chang and Hill (1977) have used a Newton-Raphson iteration procedure to solve the FLUVIAL-11 scheme. For the analysis here, we have used the Godunov scheme for the first iteration step for the 6-point and FLUVIAL-11 schemes. This is convergent for $\sigma < 1$.

References to the KUWASER, HEC-6, and FLUVIAL-11 schemes are to be found in NRC (1982) in which the results of field study comparisons are reported.

For the latter four schemes summarised in Table 1, the original references indicate that non-uniform distance steps are typically used. We have assumed a constant distance step, however, except in some instances when discussing the HEC-6 scheme.

TABLE 1: NUMERICAL SCHEMES

NAME	ALGORITHM
Godunov	$\Delta^t z = -\lambda \Delta_{-\frac{1}{2}} s$
Fromm	$\Delta^t z = -\lambda \Delta_{-\frac{1}{2}} s$
	$-(\lambda/4)\ \{(1-\sigma)\ \Delta_{\frac{1}{2}} s - (1-\sigma) \Delta_{-\frac{3}{2}}\ s\}$
KUWASER	$\Delta^t z = -(\lambda/4)\ (\Delta_{\frac{1}{2}} s + 3\Delta_{-\frac{1}{2}} s)$
HEC-6	$\Delta^t z = -2\lambda\ (1+\zeta)^{-1}\ \Delta_o\ s$
FLUVIAL-11	$\Delta^t z = -(\lambda/2)\ (\Delta_{-\frac{1}{2}} s + \Delta_{-\frac{1}{2}} s^o)$
6-point	$\Delta^t z = -(\lambda/2)\ \{\theta\ \Delta_o\ s^o + (1-\theta)\ \Delta_o\ s\}$

NAME	REFERENCE
Godunov	Perdreau and Cunge (1973)
Fromm	van Leer (1977), Vreugdenhil (1982)
KUWASER	Simons, et al (1979)
HEC-6	Thomas and Prasuhm (1977)
FLUVIAL-11	Chang and Hill (1977)
6-point	Holley et al (1984)

5. LINEAR PROPERTIES

5.1 Modified Equations

Some insight into the properties of the finite-difference schemes can be obtained by examining their linear properties obtained by assuming $c = ds/dz =$ constant. This can be conveniently examined by taking Taylor series expansions for the dependent variable z about the centre point for the scheme. A partial differential equation involving terms in z_t, z_x, z_{tt}, z_{xx}, etc is then obtained. For implicit schemes, additional terms involving z_{tx}, z_{ttxx}, z_{txx}, etc, may also be present.

Using the auto-elimination procedure described by Warming and Hyett (1974), all second order partial differential terms involving time can be eliminated. There then results an equation

$$z_t + c z_x = \sum \frac{\lambda_p}{p!} \frac{\Delta x^p}{\Delta t} \frac{\partial^p z}{\partial x^p} \qquad p = 1, 2, \ldots \qquad (11)$$

The expression on the right hand side of Equation 11 is the accumulated truncation error. The λ_p are the truncation error coefficients. Algebraic expressions for λ , $p = 2, 3$, and 4 are given by Croad (1986) for any general explicit scheme. For the schemes summarised in Table 1, the expressions

TABLE 2: TRUNCATION ERROR COEFFICIENTS

SCHEME	λ_1	λ_2	λ_3	λ_4
Godunov	0	$\sigma(1-\sigma)$	$-\sigma(\sigma-1)(2\sigma-1)$	$-\sigma(\sigma-1)(6\sigma^2 - 6\sigma + 1)$
Fromm	0	0	$\frac{1}{2}\sigma(1-\sigma)(1-2\sigma)$	$-3\sigma(1-\sigma)(\sigma^2 - \sigma + 1)$
KUWASER	0	$\sigma(\frac{1}{2}-\sigma)$	$-\sigma(1-\frac{3}{2}\sigma + 2\sigma^2)$	$\sigma(\frac{1}{2} - \frac{19}{4}\sigma + 6\sigma^2 - 6\sigma^3)$
HEC-6	$-\sigma$	$-2\sigma(1-\zeta+2\sigma)$	$-2\sigma\{8\sigma^2 + 6(1-\zeta)\sigma + (1+\zeta^3)(1+\zeta)^{-1}\}$	$-2\sigma\{48\sigma^3 + 48(1-\zeta)\sigma^2 + 8(1+\zeta^3)(1+\zeta)^{-1}\sigma + 6(1-\zeta)^2\sigma + (1-\zeta)(1+\zeta)^2\}$
FLUVIAL-11	0	σ	$-\sigma(1+\frac{1}{2}\sigma^2)$	$\sigma(1+3\sigma^2)$
6-point	0	$(2\theta-1)\sigma^2$	$-\sigma-(2-6\theta+6\theta^2)\sigma^3$	$-2\sigma^2(1-2\theta)\{2+3(1-2\theta + 2\theta^2)\sigma^2\}$

for λ up to fourth order differential terms are summarised in Table 2. Truncation error coefficients for a different set of numerical schemes are given in Vreugdenhil (1982).

5.2 Consistency

The finite-difference scheme must converge to the original differential equation as the distance and time steps vanish ($\Delta x, \Delta t \to 0$). This is called the consistency requirement. Comparing Equations 3 and 11, the consistency requirements is achieved if $\lambda_1 = 0$. Only the HEC-6 scheme does not meet this requirement. Consequently simulations involving the HEC-6 scheme will contain errors which have the same order of magnitude as the solution.

5.3 Linear Stability

In order to be stable, a finite-difference scheme must be dissipative, i.e. the wave amplitudes must not grow with time. Therefore for first order accurate schemes, it is required that $\lambda_2 > 0$. This is the Hirt (1968) heuristic stability condition. This has been generalised for higher order consistent schemes by Warming and Hyett (1974) to

$$(-1)^{r-1} \lambda_{2r} > 0 \qquad (12)$$

in which r is a natural number such that r corresponds to the order of consistency. Based on Equation 12, the stability limits for the various schemes can be summarised as:

- Godunov, Fromm, $\sigma < 1$
- KUWASER, $\sigma < \frac{1}{2}$
- HEC-6, $\sigma < \frac{1}{2}$ $(\zeta - 1)$
- FLUVIAL-11, any value of σ
- 6-point, $\theta > \frac{1}{2}$, any value of σ

Formally the 6-point scheme, as implemented by Holley et al (1984), is unconditionally unstable due to the explicit first iteration solution step. In practice, however, convergence is often still obtained for $\sigma < 1$.

For the explicit schemes, $\sigma < 1$ is the well known Courant-Friedrichs-Lewy (CFL) condition. For the HEC-6 and KUWASER schemes, more stringent stability requirements apply. For the HEC-6 scheme, it would be impractical to achieve formal stability in a real problem since there will be locations where $\zeta < 1$.

Formally, the stability condition given by Equation 12 applies to solutions represented by small wave numbers (i.e. smooth solutions). For well conditioned schemes it gives an excellent guideline for stability even under highly non-linear conditions.

Other stability criteria compared to Equation 12 (e.g. the von Neumann condition may be applied. Sometimes these will lead to different stability limits. The treatment of the boundaries may also govern the stability of the scheme.

5.4 Shock Wave Propagation

Vreugdenhil (1969) has demonstrated that the amount of smoothing over a shock region, and the association of secondary waves with the shock, depends on the truncation error of the numerical scheme. Odd order truncation errors ($\lambda_1, \lambda_3, \ldots$) lead to phase shift and even order truncation errors ($\lambda_2, \lambda_4, \ldots$) lead to amplitude changes in the solution.

Schemes which are first order accurate, and in which the λ_2 truncation errors predominate, exhibit a large amount of smearing in the presence of a shock front and secondary wave development tends to be suppressed. As $\lambda_2 \to 0$, the secondary waves associated with the λ_3 truncation errors begin to predominate.

The relative magnitudes of λ_2, λ_3, and λ_4 for the various schemes can be seen in Figure 2. For the Fromm scheme, $\lambda_2 = 0$ and dissipation is achieved through the λ_4 term. The 6-point scheme shows large first order errors for $\sigma \to 1$.

The Godunov and Fromm schemes have very small λ_3 terms which pass through zero as $\sigma = \frac{1}{2}$. These schemes are therefore referred to as zero-average-phase-error schemes. The remaining schemes exhibit

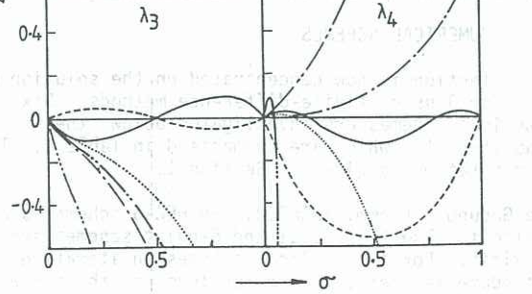

Fig. 2 Truncation Error Coefficients

large odd-order truncation error coefficients which could allow the secondary waves to dominate the solution as $\lambda_2 \to 0$. These waves will grow unboundedly and cause the scheme to "blow-up" if $\lambda_2 < 0$ or $\lambda_2 = 0$ and $\lambda_4 > 0$.

5.5 GRID SIZE REQUIREMENTS

Following the conventions of Vreugdenhil (1982), we compare the approximate solution to Equation 5, namely:

$$w = W\rho^n \exp(ijk\,\Delta x) \tag{13}$$

with the exact solution given by Equation 8 using the following two quantities:

damping factor per wave period, $d = |\rho|^{2\pi/\sigma\xi}$ (14)
relative propagation velocity,

$$c_r = -(\sigma\xi)^{-1} \arg(\rho) \tag{15}$$

in which k is the wave number and $\xi = k\,\Delta x$. The number of grid points per wave length is $n_x = 2\pi/\xi$. If we define some error ε, then we can compute the required n_x to achieve $|1-d| < \varepsilon$ and $|1-\varepsilon_r| < \varepsilon$.

The complex propagation factor ρ (actually the eigenvalues of the amplification matrix, see Abbott, 1979) can be conveniently expressed in terms of the truncation error coefficients. The procedure is described in Croad (1986) giving general estimates for n_x to achieve an amplitude error ε as:

$$n_x^{2q-1} = (\sigma\varepsilon)^{-1} \{(2q)!\}^{-1} (-1)^q (2\pi)^{2q} \lambda_{2q} \tag{16}$$

in which 2q is the address of the lowest non-zero even-order truncation error. Similarly, for phase errors:

$$n_x^{2r} = (\sigma\varepsilon)^{-1} \{(2r+1)!\}^{-1} (-1)^r (2\pi)^{2r} \lambda_{2r+1} \tag{17}$$

in which 2r+1 is the address of the lowest non-zero odd-order truncation error. It is assumed that $\lambda_1 = 0$ for consistency, otherwise the error cannot be controlled by adjusting the grid size.

Only positive values of ε are allowed in Equation 16 otherwise an instability in the finite-difference scheme is implied. The sign for ε in Equation 17 must be chosen to give a positive value for n from which it can be determined if the phase error is leading or lagging.

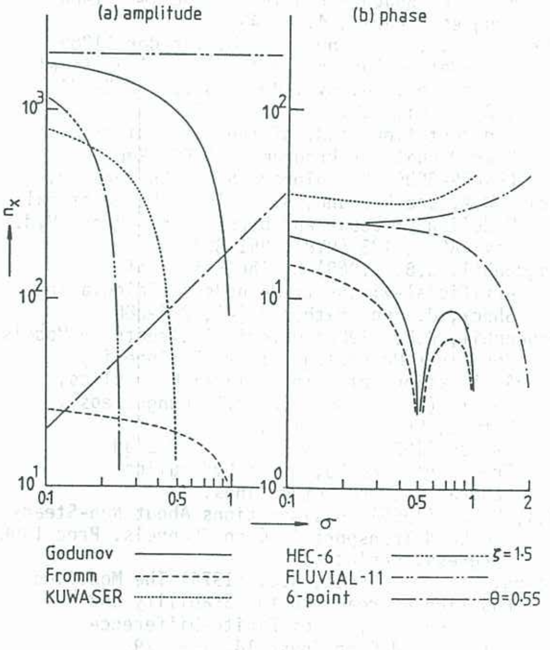

Fig. 3 Grid Size Requirements to Achieve an Accuracy of 1 Percent in (a) Amplitude and (b) Phase Errors

Figure 3 gives the number of grid points per wave lengths n for $|\varepsilon| = 0.01$ in d and c as a function of the Courant number for the schemes summarised in Table 1. It can be concluded that:

(a) Generally the amplitude errors of the first order schemes are quite high relative to the second order Fromm scheme. The FLUVIAL-11 amplitude errors are generally the largest of those schemes studies which are stable.

(b) Similarly, the second order Fromm scheme generally has the smallest phase errors of the schemes studied.

(c) The implicit scheme's give large amplitude errors for $\sigma > 1$. Consequently, the often quoted benefit of an implicit scheme, of being stable with large time steps, is not apparent in the context of the river morphology problem due to the associated large truncation errors. The errors will be most severe in shock regions due to the presence of high σ values.

6. NUMERICAL EXPERIMENTS

In order to test the numerical schemes under non-linear conditions, numerical experiments were carried out on a test problem.

All computations started with the same initial conditions, namely $i_o = 1 \times 10^{-4}$, $h_o = 3.0$ m, $u_o = 1.0$ m/s, and $s_o = 2.2 \times 10^{-3}$ m^2/s. Equation 10 was assumed for the sediment transport function with $m = 2.2 \times 10^{-3}$ s^4/m^3 and $n = 5$. The Chezy roughness coefficient was $C = 57.7$ m /s. The flow conditions were uniform.

A step increase in the sediment transport rate Δs was introduced at the upstream boundary such that $\Delta s/s_o = 0.1$. A relatively accurate analytical hyperbolic model solution for the above problem, which includes a correction for the step change in sediment transport rate, is given by Ribberink and van der Sande (1985). The following grid parameters were adopted: $\Delta x = 50.0$ m, and $\Delta t = 0.01$, 0.1 days corresponding to initial Courant numbers $\sigma_o = 0.066$, 0.657 respectively. For the 6-point scheme, $\theta = 0.55$ was adopted.

The results of the numerical experiments are shown in Figure 4 giving "snapshot" views at t = 8 days (the results for the HEC-6 scheme are given at t = 6 days due to the "blow-up" of the scheme). The analytical solution from Ribberink and van der Sande (1985) is also plotted (broken line). The following general observations are drawn from the results:

(a) The Fromm scheme gives the most accurate results.

(b) The smearing effects due to the relatively high truncation errors in the first order schemes can be seen. The relative amounts of smearing are consistent with that indicated from the linear analysis (see Fig. 3).

(c) The presence of secondary waves can be seen for the Fromm, KUWASER, and 6-point schemes. The secondary waves for the 6-point scheme are rather severe but could be reduced by increasing θ although the amount of smearing would also be increased.

(e) The HEC-6 scheme was unstablle for all cases consistent with the results of the linear stability analyis. Also convection rates for the sediment transport process are over predicted by a factor of two for this scheme due to the zeroeth order truncation errors.

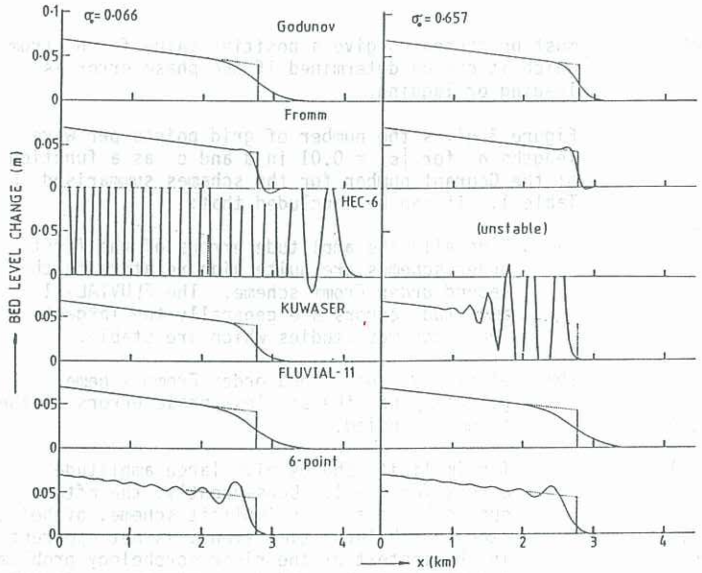

Fig. 4: Results of Numerical Experiments

Morphological models of the type described by Equations 1 to 4 require calibration in order to be applied to problems in nature. Particularly for solutions represented by large wave numbers, it is important that the amplitude and phase errors of the numerical scheme are minimal in order than an adequate calibration can be carried out. The first order schemes are less satisfactory from this point of view.

7. SUMMARY AND CONCLUSIONS

A number of currently used finite-difference schemes, used to solve the sediment continuity equation for river morphology problems, have been presented. An analysis of their linear properties, considering truncation errors, amplitude and phase shift properties and stability, has been given based on the modified equations which represent the finite-difference schemes.

From numerical experiments on a test problem involving strong non-linearity and shock propagation, it has been shown that the conclusions to be drawn from the linear analysis are consistent with the findings in the non-linear experiments.

For problems involving large wave numbers, it is shown that the amplitude and phase errors of the first order schemes give rise to significant smearing and secondary wave development in the solution. This could cause difficulties with calibrating such models when applied to problems in nature.

Somewhat severe stability limits for the HEC-6 and KUWASER schemes, predicted from the linear analysis, are confirmed in the numerical experiments. For most practical problems, the HEC-6 will always exhibit instabilities and the results, therefore, will be unreliable.

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REFERENCES

Abbott, M.B. (1979) Computational Hydraulics, Pitman, London.
Chang, H.H. and Hill, J.C. (1977) Minimum Stream Power for Rivers and Deltas, J. Hyd. Div, ASCE, 103 (HY12), 1375-1389.
Croad, R.N. (1986) Grid Size Requirements for Accurate Finite-Difference Linear Convection Equation Solutions, Rep 3-86/6, Central Laboratories, MWD, Lower Hutt, NZ.
Hirt, C.W. (1968) Heuristic Stability Theory for Finite-Difference Equations, J. Comp. Phys., 2(4), 339-355.
Holley, F.M., Yang, J.C. and Karim, M.F. (1984) Computer-Based Prognosis of Missouri River Bed Degradation Refinement of Computational Procedures, Rep 281, IIHR, University of Iowa, Iowa.
Jansen, P.Ph. et al (1979) Principles of River Engineering. The Non-Tidal River. Pitman, London.
Karim, M.F. and Kennedy, J.F. (1982) Ialluvial: A Computer-Based-Flow-and-Sediment-Routing Model for Alluvial Streams and its Application to the Missouri River, Rep 250, IIHR, University of Iowa, Iowa.
Leer, B.van (1977) Towards the Ultimate Conservative Difference Scheme. III. Upstream-Centred Finite-Difference Schemes for Ideal Compressible Flow. J. Comp. Phys. 23, 253-275.
NRC (1983) An Evaluation of Flood-Level Prediction Using Alluvial-River Models, Report National Research Council, National Academy Press, Washington.
Perdreau, N. and Cunge, J.A. (1971) Sedimentation Dans Les Estuaries et Les Embouchures Bouchon Marin et Bouchon Fluvial. Proc 14th IAHR Congress, Paris, 4, 41-49.
Ribberink, J.S. and Sande, J.T.M. van der (1985) Aggradation Due to Overloading - Analytical Approaches. J. Hyd. Res. 23 (3), 373-284.
Simons, D.B., Li, R.M. and Brown, G.O. (1979), Sedimentation Study of the Yazoo River Basin - User Manual for Program KUWASER, Rep CERC79-80DBS-R, Colorado State University.
Thomas, W.A. and Prasuhn, A.L. (1977) Mathematical Modeling of Scour and Deposition. Proc. Hyd. Div, ASCE, 103 (HY8), 851-863.
Vreugdenhil, C.B. (1969) On The Effect of Artificial-Viscosity Methods in Calculating Shock, J. Eng. Math., 3 (4), 285-288.
Vreugdenhil, C.B. (1982) Numerical Effects in Models for River Morphology, from: Engineering Applications of Computational Hydraulics, Vol I, (M.B. Abbott and J.A. Cunge, eds), Pitman, London, 91-110.
Vries, M. de (1959) Transients in Bed-Load Transport. Rep R3, Delft Hydraulics Laboratory, The Netherlands.
Vries, M. de (1965) Considerations About Non-Steady Bed Load Transport in Open Channels. Proc IAHR Congress, Leningrad.
Warming, R.F. and Hyett, B.J. (1974) The Modified Equation Approach to the Stability and Accuracy Analysis of Finite-Difference Methods. J Comp Phys, 14, 159-179.