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Time Series Synthesis for Wave Simulation

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INTRODUCTION

The simulation of a given sea state in a laboratory flume is generally achieved by using a wave generator which can be controlled to produce any desired motion. The sea state is produced by generating the signal which controls the wave generator such that the resulting waves produced correspond to the desired sea state.

Several methods are available for the production of the wave generator control signal:

1. Superposition of sine waves
2. Digital filtering of random numbers
3. Suitable scaling of real wave records
4. Hardware realisation of a combination of the first two methods

Variations of each of these broad categories exist within each method but generally most schemes can be included in one of the above methods. The basic scheme for each type of simulation is detailed below.

In all methods it is assumed that it is desired to simulate a sea state defined by the one sided spectral density function $S(f)$. The variance of the final time series (waves) is related to the spectral density function by

$$\sigma^2 = \sum_{n=0}^{N-1} S(n)\,\Delta f \qquad (1)$$

This is a useful expression as a check can be made of the specified variance and the variance of the time series.

Furthermore the spectral density function has been derived from a discrete fourier transform $X(x)$ by the relationship

$$S(n) = 2\,X^2(n)\,\Delta f \qquad (2)$$

where Δf is the spacing in the frequency domain of the fourier transform.

SUPERPOSITION METHOD

This method uses an equation of the form

$$x(k) = \sum_{n=0}^{N-1} C_n \cos\,(2\pi kn/N - \phi_n) \quad k = 0\ ..\ N = 1 \qquad (3)$$

in which C_n are determined explicitly from the required spectrum and ϕ_n are random phase angles uniformly distributed between 0 and 2π.

The coefficients C_n are related to the spectral density function

$$C_n = \sqrt{(2\,S(f_n)\,\Delta f)} \qquad (4)$$

This is readily illustrated by taking a single sine wave component and recalling that the variance of a sine wave is the amplitude squared and divided by two ($a^2/2$), and setting this equal to the right hand side of equation (1) yielding equation (4) for a single value of n.

In practice this synthesis process is done by means of a fast fourier transform.

DETERMINISTIC SPECTRAL AMPLITUDE SIMULATION (DSA)

Expanding the equation (3)

$$x(k) = \sum_{n=0}^{N-1} [C_n \cos\,(2\pi kn/N) \cos\phi_n + C_n \sin\,(2\pi kn/N) \sin\phi_n] \qquad (5)$$

and recalling that the discrete fourier transform is

$$x(k) = \Delta f \sum_{n=0}^{N-1} [X_1(n) \cos\,(2\pi kn/N) + X_2(n) \sin\,(2\pi kn/N)\ + j\,\Delta f \sum_{n=0}^{N-1} X_1(n) \sin\,(2\pi kn/N) + X_2(n) \cos\,(2\pi kn/N)] \qquad (6)$$

where $X_1(n)$ and $X_2(n)$ are the real and imaginary parts of $X(n)$.

Now if $X_1(n)\,\Delta f = C_n \cos\phi_n$ and $X_2(n)\,\Delta f = C_n \sin\phi_n$,

remembering that the Δf term in equation (6) is accounted for as part of the discrete fourier transform. It is evident that if $x(k)$ is to be real valued then there must exist some special relationship between $X_1(n)$ and $X_2(n)$.

The relationship is that

$$X(N-n+1) = X^*(n) \text{ for } n = 1\ ..\ N/2-1 \qquad (7)$$

where * denotes the complex conjugate. $X(0)$ = mean value of time series. The value at $X(N/2)$ is the Nyquist value. Text vary on the way n is specified either from zero to N-1 or from 1 to N. Noting that the discrete transform implicitly uses negative values of frequency and time but folds these portions into the high coefficients between N/2+1 and N-1. Because of this folding the single sided spectrum must be divided by two before determining the coefficients to be used in the synthesis.

A time series simulating a given wave spectra is synthesised in the following manner: a complex data series is generated in the frequency domain such that upon inverse fourier transformation, the time series resulting is real valued. To achieve this the real part of the function must be an even function and the complex part an odd function. For the discrete fourier transform with N values then $X(n) = X^*(N-n+1)$ for n = 1 to N/2-1. When this series is transformed to the time domain a real time series results. The time series will have a single sided spectral density function determined by the amplitude of the values used for the frequency function. The relationship between the desired spectral function S(n) and the amplitude |X(n)| is as follows:

$$|X(n)| = \sqrt{(0.5\ N\Delta t\ S(n))} = \sqrt{(S(n)/(2\Delta f)} \qquad (8)$$

This follows directly from the definition of equation (2), and it should be noted that there is a discrepancy between the coefficients from equation (8) and the relationship found by equating equations (5) and (6). This is because the fourier transform method includes energy above the Nyquist frequency in the time series, whereas equation (3) does not. In fact for the normal case the summation in equation (3) need only be performed between 0 and N/2, assuming the usual relationship between Δf and Δt.

NONDETERMINISTIC SPECTRAL AMPLITUDE (NSA) SIMULATION

In this case the amplitude and phase of the components are determined randomly and the time series obtained from a fast fourier transform in the same manner as for the DSA method.

Pairs of normally distributed random numbers are created with zero mean and unit variance. A series of N/2 pairs is required, let these be α_n and β_n. These are then multiplied by the coefficients defined by equation (8) to yield a_n and b_n thus:

$$a_n = \alpha_n \sqrt{(0.5\ N\ \Delta t\ S(n))}$$
$$b_n = \beta_n \sqrt{(0.5\ N\ \Delta t\ S(n))} \qquad (9)$$

X(n) is then assembled using an amplitude of $\sqrt{(a_n^2 + b_n^2}$ and a phase angle of $\tan^{-1}(b_n/a_n$. As in the case of the DSA simulation the requirement that $X(N-n+1) = X^*(n)$ must be satisfied in order to produce a real time series. The complex series X(n) is then transformed using the inverse discrete fourier transform to produce the required time series.

This approach does not produce an exact representation of the required sea state, as is the case with the DSA simulation, but it fluctuates about the required spectrum. This is due to the randomness introduced in selecting the amplitude.

DIGITAL FILTERING OF RANDOM NUMBERS

This method is perhaps aesthetically more attractive than the superposition of sinusoidal components, since it does include some random selection into the process. Originally this approach was the attempt to copy analogue techniques where band limited white noise was passed through a series of filters to produce a time series with the desired distribution of energy. The method consists of taking a time series consisting entirely of random numbers uniformly distributed between 0 and 1 and filtering these to produce a desired time series. The filtering may be done in the time or frequency domain.

In the frequency domain this may be represented by the relationship between the Spectral density function of the white noise $S_x(n)$ and the filtered spectrum $S_y(n)$ by

$$S_y(n) = |H(n)|^2\ S_x(n) \qquad (10)$$

where H(n) is a frequency response if a suitable filter. The use of fast fourier transforms makes this process more efficient in the frequency domain. Suppose that the fourier transform of the random numbers is X(n) then this will have a value which will fluctuate about unity. For an infinitely long series of random numbers the value should be exactly unity, but because of the finite length of time series used and sampling errors there will be some difference. It is this difference which gives the method its attraction, since every time series generated will reflect some characteristics of the random numbers. If X(n) is multiplied by a suitable filter or frequency response a further frequency function Y(n) is produced. This function is then just the fourier transform of the desired time series provided the transfer function has been chosen correctly. Thus the target time series is given by

$$y(k) = \Delta f \sum_{n=0}^{N-1} X(n)\ H(n) \exp(j2\pi nk/N) \qquad (11)$$

the required frequency response H(n) of the filter is given by

$$H(f) = \sqrt{(S(n)/2)} \qquad (12)$$

where S(n) is the target spectrum.

Once again some care is required in arranging that the frequency function $Y(n) = H(n)\ X(n)$ has the correct form to produce a real valued time series. As before $Y(N-n+1) = Y^*(n)$.

This method will yield similar results to the NSA method.

SCALING WAVE RECORDS

One of the problems with any of the above methods is that it produces normally distributed amplitudes with no flattening of troughs or sharpening of peaks that occurs in real wave records. It may be argued that if waves are to be generated in the laboratory from these synthesised time series, then the physical process will produce these distortions naturally. This is certainly so as is obvious to any observer of waves in a laboratory flume. However the degree of modification is difficult to predict and measure.

To overcome this shortcoming, real wave records may be scaled in amplitude and frequency to produce a time series for use in the laboratory. However if this is used to drive the wave generator then the wave will again be modified by the generation process. The degree of modification is again likely to be unpredictable. So although this is still a viable alternative the end result may not be any better than that obtained by using synthesised time series.

HARDWARE

Several types of analogue or digital simulation machines exist which essentially sythesise time series by one of the above techniques but using dedicated hardware. Many of these are very successful but lack the flexibility of a software implementation. They will not be considered in detail here.

PITFALLS OF CURRENT METHODS

For a given target spectrum the maximum amplitude likely for a fixed time interal is determined by a Rayleigh distribution with a mean value of $\sqrt{(\pi\sigma/2)}$ where σ^2 is the variance of the complete time series. This follows from Longuet-Higgins (1952). Thus for a longer record, more extreme cases are likely and will mean a larger paddle motion. This must be considered when determining scale factors.

The average maximum amplitude for a sample of N waves is shown to be

$$a_{max}/\sigma = \sqrt{(2\ \ln N)} + 0.5772/(2\ \ln N) \qquad (13)$$

Thus the longer the time series generated the greater will be the maximum wave height. So whilst it is desirable to generate the longest possible wave sequence and also to model at the largest scale possible, these two requirements are to a certain extent contradictory. Additionally the usual criteria of relating the maximum wave height to the variance can lead to a problem with the maximum stroke available on the wave generator.

PACKET SUPERPOSITION METHOD

This method has been developed as a means of overcoming the shortfalls of the DSA method with regard to the maximum paddle amplitude required. Using any method which involves some statistical uncertainty means that the maximum stroke required is never exactly defined. The Packet Superposition Method (PS) has been evolved to overcome this problem and to provide a means of producing waves with larger amplitude for a given spectral frequency distribution than is possible with the DSA approach. This means that a larger scale may also be employed which is always an advantage.

Normal superposition methods use equation (3) as a basis for producing the control time series for the generator. The PS method uses the water in the flume as the superposition domain and generates packets of sinusoidal waves, the amplitude and frequency of which varies from packet to packet. Each packet moves with a different speed, and the packets will interact in a natural manner as they progress down the flume. The result will be a sea state of the required spectral composition provided the packets are chosen correctly. For a given spectrum one may choose certain frequency bands for the packets to be produced either at random or by some other scheme. The other variables involved are the number of waves in a packet and the order in which the packets are produced. Having decided upon these factors, the amplitude for each sinusoidal packet may be calculated using relationships outlined earlier in the paper.

The Packet Superposition method has proved a useful method if simulating a given sea condition and has improved the scale ratios for a number of model tests already. Further work is being done on the choice of parameters for each packet but results to date have been encouraging.

REFERENCES

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