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Synthesis of Two- and Three-Dimensional Separation Bubbles

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SUMMARY

An algorithm has been developed which enables local Taylor series expansion solutions of the Navier-Stokes and continuity equations to be generated to arbitrary order. The algorithm can be used to synthesize nonlinear viscous flow patterns with certain required properties and can be applied to the construction of two- and three-dimensional flow separation patterns. These patterns are asymptotically exact solutions of the equations of motion close to the origin of the expansion.

1 INTRODUCTION

A critical point in a flow field is a point where the streamline slope is indeterminate, i.e. $(u_i/u_j) = 0/0$ where $i \neq j$ and u_i is the velocity. By Taylor series expanding the velocity field u_i about the critical point in terms of the space coordinate x_j and substituting the expansion into the Navier-Stokes and continuity equations, certain relationships between the coefficients of the expansion can be found. All possible patterns close to a critical point can be derived and classified. Sectional streamline patterns form saddles, nodes or foci. Oswatitsch (1957) was the first to carry out a systematic analysis of critical points located at a no-slip boundary and derived the various three-dimensional separation and reattaching flows close to such points. Lighthill (1963) discussed further the solutions of Oswatitsch and Perry & Fairlie (1974) applied phase-plane techniques to the description of critical points. Critical points which occur away from no-slip boundaries (the so-called free-slip critical points) require different formulations and have been studied by Perry & Fairlie (1974) and recently, in greater detail, by Perry (1984a).

Critical points are the salient features of a flow pattern. If their position and type is known, the rest of the pattern is known qualitatively since there are a limited number of ways the streamlines can be joined between the points. The basic topology and qualitative transport properties of the pattern can be understood by using the critical point concept.

A series expansion up to second order about a critical point (e.g. the Oswatitsch solution) is limited to describing the flow in the immediate vicinity of the critical point. Dallmann (1983) has recently shown that if the series expansion can be extended to higher orders, a flow field consisting of a cluster of critical points can be described in one formulation. Perry & Chong (1986a) has developed an algorithm which enables local solutions of the Navier-Stokes and continuity equations to be generated to arbitrary order. The algorithm is such that the necessary algebraic manipulations required to generate the relevant equations can be carried out on a computer. The algorithm has been applied to the study of three-dimensional separation patterns of the type recently observed and classified by Bippes & Turk (1983), Hornung & Perry (1984) and discussed by Dallmann (1983).

A brief summary of the theory is given below. Further details can be obtained from Perry & Chong (1986b) and full technical details can be obtained from Perry (1984b) and Perry & Chong (1986a).

2 THEORY

The Navier-Stokes equations for incompressible, constant density flow can be expressed as a single tensor equation thus:

$$\frac{\partial u_i}{\partial t} + u_q \frac{\partial u_i}{\partial x_q} = -\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_q \partial x_q} \tag{1}$$

where $P=p/\rho$ is the kinematic pressure, p is the pressure, ρ is the fluid density, ν is the kinematic viscosity, u_i is the velocity tensor and x_i is the space coordinate tensor (Cartesian coordinates). The continuity equation is

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

A Taylor series expansion of the velocity can be expressed as

$$u_i = A_i + A_{ij}x_j + A_{ijk}x_jx_k + A_{ijkl}x_jx_kx_l + A_{ijklm}x_jx_kx_lx_m + \dots \tag{3}$$

Equation (3) can be written as

$$u_i = \sum_{n=0}^{N} RA_{i(111\dots,222\dots,333\dots)}\, x_1^a\, x_2^b\, x_3^c \tag{4}$$

where $a+b+c = n$ in every possible combination and permutation. a, b and c are positive whole numbers or zero. R = number of possible permutations of the indices in (111...,222...,333...) where 1 is repeated 'a' times, 2 is repeated 'b' times and 3 is repeated 'c' times.

By substituting the series expansion (3) into the Navier-Stokes and continuity equations relationships between the various unknowns coefficients can be generated (see Perry 1984b and Perry & Chong 1986a,1986b). A corresponding set of equations for two-dimensional flow can also be derived.

In the generation of the Navier-Stokes relationship we are effectively constructing vorticity transport equations to various orders (since pressure has been eliminated). These turn out to be first order ordinary differential equations for the series expansion coefficients. With the aid of the algorithm developed, it is possible to generate all the Navier-Stokes and continuity relationships between the coefficients to any order. The authors have developed a computer program which generates the equations because of the enormous amount of algebra required.

The number of unknowns always exceed the number number of equations generated using the algorithm. Hence, in order to obtain a solution, additional equations must be supplied from boundary conditions. In steady flow the problem is to solve a set of simultaneous algebraic equations. The continuity relationships are simple linear algebraic equations and the Navier-Stokes relationships consists of linear (viscous) terms and quadratic (convective) terms.

It has been found that in steady flow patterns, the specifications of boundary conditions on boundaries which passes through the origin of the expansion (canonical boundary conditions) lead to a very simple solution procedure. Because of the sequence in which certain coefficients are determined, the procedure leads to sets of Navier-Stokes relationships which are linear in the remaining unknown coefficients (all quadratic terms contain at least one known coefficient). All equations are then effectively linear in terms of the coefficients and can be solved by substitution. In other types of boundary condition specifications, the Navier-Stokes generated relationships remain nonlinear with terms involving products of unknown coefficients.

The specification of boundary conditions as a series introduces problems of redundancy. Certain coefficients determined from boundary conditions must also satisfy the equations of motions otherwise a contradiction occurs. In all computations carried out, coefficients determined from the equations of motion take priority if the same coefficients can also be determined from boundary conditions. These redundant boundary condition equations are ignored.

The above algorithm has been tested by Perry (1984b) using a number of simple three-dimensional test cases which have known solutions, e.g. the solutions given by Perry (1984a) and Hornung (1983). A further test of the algorithm using a simple potential flow problem is given in Perry, Chong & Hornung (1985) and in Perry and Chong (1986). Several methods were also developed for determining the region of accuracy which is defined as the region where the full Navier-Stokes and continuity equations agree with the generated truncated set of equations. This region will be a finite zone surrounding the origin of the expansion. The authors have found that the most convient way of determining the region of accuracy is to compare the truncated value of $|\mathrm{grad}\ P|$, i.e. $|\nabla P|_T$ with $|\mathrm{grad}\ P|$ obtained by substituting the truncated solution of the velocity field into the full Navier-Stokes equation, i.e. $|\nabla P|_F$. A suitable criterion for the region of accuracy can be formulated as follows

$$\{|\nabla P|_F - |\nabla P|_T\}/|\nabla P|_F \leq 10\ \% \tag{5}$$

3 SYNTHESIS OF SEPARATED FLOW PATTERNS

Istead of specifying boundary conditions to generate separated flow patterns, a general procedure has been developed for the synthesis of two- and three-dimensional flow separation. The surface vorticity must first be specified such that various critical points on the surface are defined. Points of separation and reattachment are critical points. By shifting the origin of the series expansion to various critical points located on the surface and by specifying the properties of these critical points, sufficient equations relating the various coefficients can be generated which allow the surface flow patterns to be synthesized. The flow pattern above the surface which would generate a particular type of surface flow pattern is not unique and a variety of separation flow patterns could be generated. Further conditions need to be specified. These conditions are usually the angles of separation and reattachment and various locations and properties of critical points above the surface. In general, the higher the order of the series expansion, the more conditions need to be specified for closure.

Figure 1 shows a typical example of a fifth order two-dimensional separation bubble which can be synthesized. A suitable region of accuracy is also shown in the figure. As an additional check on the algorithm, these solutions were used to generate boundary conditions for u_1 and u_3 along the x_1 and x_3 axes. These were then used with the algorithm and the resulting solution agreed with the synthesized solution to great accuracy over the entire flow field.

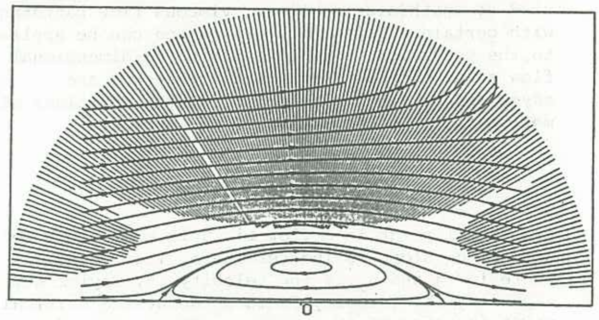

Figure 1. Fifth order two-dimensional separation bubble. The region of accuracy is the shaded zone about 0.

An example of a three-dimensional separation bubble is shown in figure 2. The surface flow patterns (limiting streamlines in the x_1x_2 plane) has been classified as owl-face of the second kind by Hornung & Perry (1984) and Perry & Hornung (1984) from surface dye trace observations of flow behind missile-shaped bodies at various angles of attack (see Fairlie 1980 and Bippes & Turk 1983). The Reynolds numbers of the observed patterns are high (of order 10^5) but the flow patterns synthesized are at low Reynolds numbers (of order 10^2 or less. The Reynolds number here is based on the vorticity at the origin divided by viscosity. Nevertheless, in this preliminary investigation we have managed to synthesize patterns which are topologically similar to those observed (at least at the surface).

Unsymmetrical solutions can be obtained by generating the canonical boundary conditions from the synthesized symmetrical solutions and perturbing the resulting boundary conditions on the x_1x_3 plane so that the symmetry condition is violated. We then use the algorithm in combination with the new unsymmetrical canonical boundary conditions to solve for the three-dimensional flow pattern. This solution includes the limiting streamlines. This has been applied to the owl-face pattern of the first kind (see Hornung & Perry 1984 and Perry & Hornung 1984). The resulting pattern is shown in figure 3. Note in figure 3(a) how fluid shown shaded on one side of the centre plane finds its way to the focus on the other side of the centre plane. This pattern has undergone a major change in topology since the original symmetrical pattern was structurally unstable, i.e. it possessed a saddle-to-saddle connection by a separatrix streamline (see Tobak & Peake 1982 and Perry & Hornung 1984 regarding structural stability).

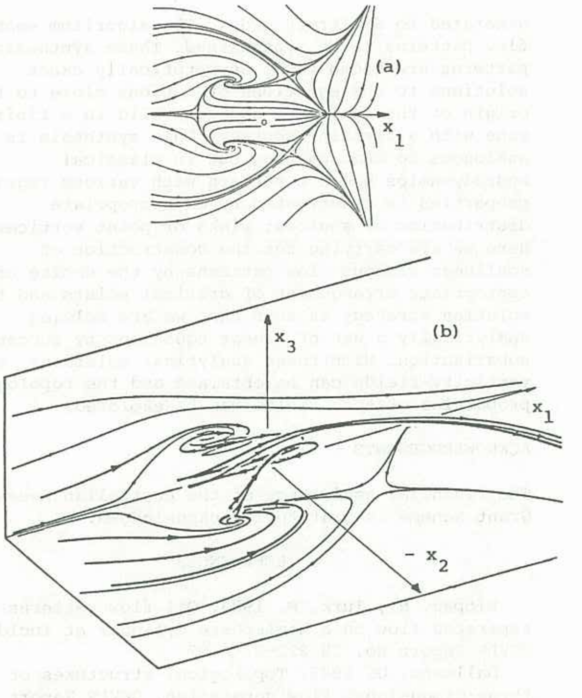

Figure 2. Fourth order owl-face of the second kind. (a) Surface flow pattern, i.e. limiting streamlines on the x_1x_2 plane. (b) Oblique view with some out-of plane trajectories added (shown heavy).

A preliminary attempt at solving a time-dependent two-dimensional separation bubble has also been carried out. If the flow is unsteady, these boundary conditions must be known functions of time which become forcing functions for the ordinary differential equations. Also, all coefficients must be known at some initial time. Solving the problem as a time-dependent one also overcomes the difficulty of nonlinearity which arises from a specification of noncanonical boundary conditions. Here we march in time and the various coefficients are updated by simple substitution without any iterative procedure at the end of each time step. An example of a flow pattern obtained is shown in figure 4(a). In this preliminary attempt the separation bubble is of a non-standard type as sketched in figure 4(b). The flow pattern is topologically correct. Work is still being carried out to produce a standard unsteady separation bubble of the type shown in figure 1.

Another aspect of this work which is still undergoing development is the generation of the vorticity field from the series expansion of the velocity field. Integrating the vorticity field produces critical points of vorticity and the topological features of the vorticity field can be explored. Unlike the velocity field, the vorticity field are often complex and there is no systematic method of mapping out the vorticity field except for low order of series expansions. For example, the vorticity field which corresponds to the owl-face of the first kind (as classified by Perry & Hornung 1984) and shown in figure 5(a) is given in figure 5(b). This is for a 3rd order expansion of the velocity field. At this low order of series expansion a line of zero vorticity forms the "backbone" on the x_1x_3 plane. Trajectories originating from points which lie on a horizontal line passing through the backbone and normal to the x_1x_3 plane form cylindrical sheets as shown in figure 5(b). This is more obvious in the plan view and side view shown in figure 5(c) and (d) respectively. It should also be noted that the vorticity vectors on the surface are orthogonal to the limiting streamlines as pointed out by Lighthill (1963). At higher orders of series expansions, the vortex lines no longer lie in well

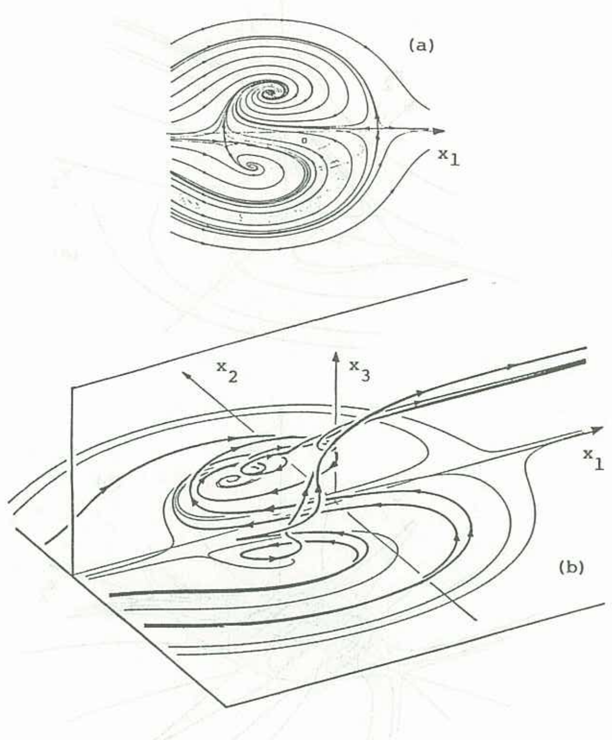

Figure 3. Fifth order unsymmetrical owl-face of the first kind. (a) Surface flow pattern, i.e. limiting streamlines on the x_1x_2 plane. (b) Oblique view with some out-of-plane trajectores added (shown heavy).

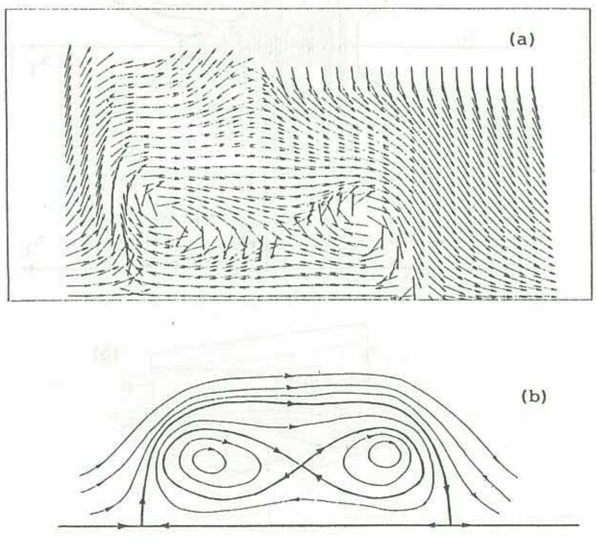

Figure 4. Ninth order two-dimensional separation bubble solved as a time dependent problem. (a) Computed vector field. (b) Conjectured flow pattern of (a).

defined sheets. The structure of the vorticity field is unlike the conjectured model of the vortex skeleton suggested by Perry & Hornung (see figure 5(e)). Perhaps at the low order of series expansions (which suggests low Reynolds numbers), the vorticity is diffused and it is only at much higher Reynolds numbers that the vorticity concentrates into vortex rods which forms the vortex skeleton model. This aspect of the work is still being developed.

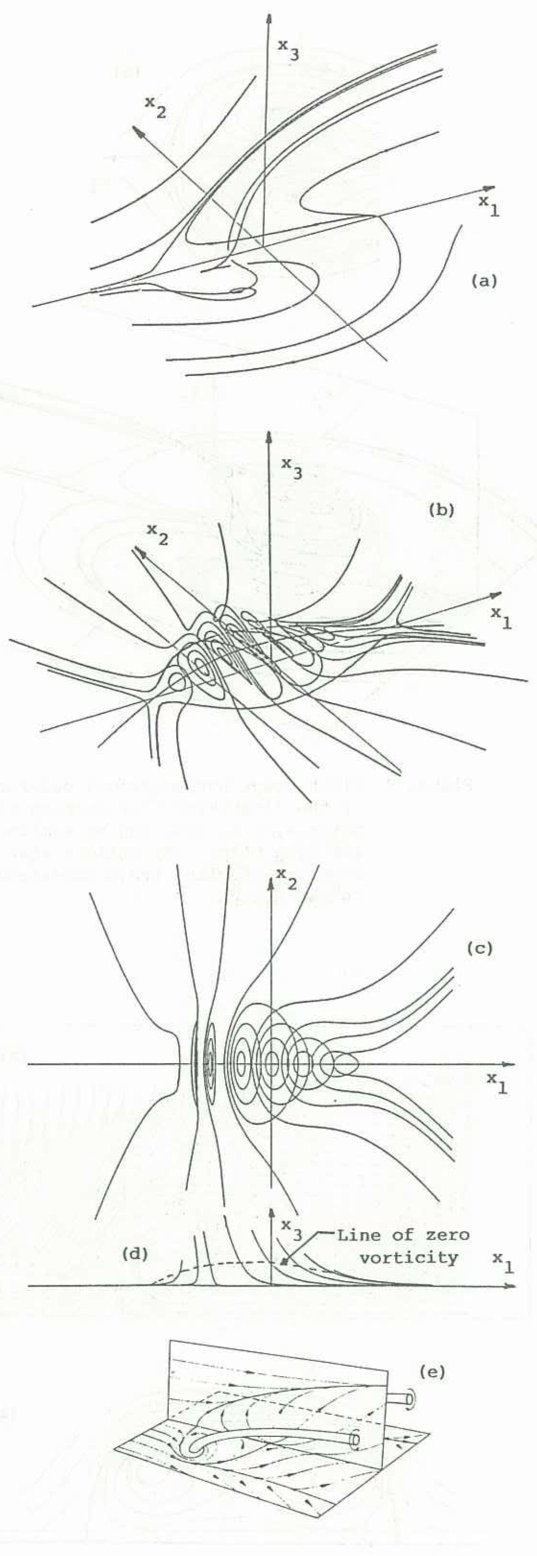

FIGURE 5. Third order owl face of the first kind.
(a) Oblique view of instantaneous streamlines with some out-of-plane trajectories.
(b) Oblique view of vorticity lines.
(c) & (d) Plan and side view of (b).
(e) Conjectured vortex skeleton model Perry & Hornung (1984b).

4 CONCLUSIONS AND DISCUSSIONS

An algorithm has been developed which enables the local Taylor series expansion solutions of the Navier-Stokes and continuity equations to be generated to arbitrary order. The algorithm enables flow patterns to be synthesized. These synthesized patterns are known to be asymptotically exact solutions to the equations of motions close to the origin of the expansion and are valid in a finite zone with a certain accuracy. This synthesis is analogous to that carried out in classical hydrodynamics where a pattern with various required properties is constructed by an appropriate distribution of sources, sinks or point vortices. Here we are carrying out the construction of nonlinear viscous flow patterns by the choice of an appropriate arrangement of critical points and the solution strategy is such that we are solving analytically a set of linear equations by successive substitution. With these analytical solutions, the vorticity fields can be obtained and the topological properties of such fields can be explored.

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