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The Hydraulics of Stratified Flow over Topography

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ABSTRACT

A procedure for determining the flow which results from introducing a large two-dimensional obstacle of finite height into an arbitrary stable stratified shear flow of finite depth is described. The method is based on a generalization of the known results for two-layer flows, described in Baines (1984), and the assumption that mixing in the fluid is negligible. For a given initial flow, obstacles with height greater than a critical height must generate disturbances which alter the incident flow far upstream. These disturbances may take the form of an upstream hydraulic jump or a time-dependent rarefaction, or both, depending on the non-linear dispersive properties of the system. This behaviour may be manifested by more than one mode in turn, as the obstacle height increases. A specific example of a three-layer flow with a relatively thin upper layer (approximating conditions in some fjords) is described in detail.

INTRODUCTION

For the most part, the subject of hydraulics is concerned with the flow of homogeneous fluid (water) through a system where the flow changes gradually with the horizontal co-ordinate so that the flow is hydrostatic, except for particular regions where it may change abruptly (eg. hydraulic jumps and energy dissipation structures). In many systems of geophysical and engineering interest the fluid is not homogeneous but is density stratified, implying that the density of the fluid increases with increasing depth. This density variation introduces a range of dynamical phenomena which are associated with internal gravity waves and have corresponding time scales, and which are important in a number of situations, such as flow in stratified fjords, straits, reservoirs and estuaries, as well as the atmosphere. Many of these phenomena have their analogues in open channel hydraulics, but the variety of phenomena in the stratified case is much richer.

In this paper we will discuss the hydraulic flow of stratified fluid over bottom topography, where the term "hydraulic" is taken to imply that the topographic variations are gradual so that the flow is mostly hydrostatic. We will approximate the stratified fluid by a number of homogeneous layers n, numbered from the bottom upwards. Above layer n we may have either a rigid horizontal surface or an infinitely deep layer of uniform density. In principle, a sufficient number of suitably chosen layers will give a satisfactory approximation to any continuously stratified stable shear flow. We proceed to discuss the properties of single and multi-layered models, and, based on experience with one - and two-layer systems which have been studied in detail (eg. Baines 1984) we describe a general procedure which may be used to determine the behaviour of multi-layered fluids over long obstacles of finite heights. We then give some typical results for a three-layer system which illustrates a broad range of flow types. More details of the method and its results are given in Baines (1986) and Baines & Guest (1986), and an up-to-date overall review of stratified flow over topography is given in Baines (1987).

SINGLE-LAYER HYDRAULICS

As a point of comparison, we first briefly summarize the hydrostatic flow of a single layer over topography. Flow states which result from the impulsive commencement of flow with velocity u of a layer of depth d_o over an obstacle of maximum height h_m depend on two dimensionless numbers: the initial Froude number $F_o = u_o/(gd_o)^{\frac{1}{2}}$ and $H=h_m/d_o$; corresponding steady-states are shown in Fig.1 of Baines (1984). Regions where the flow is sub or supercritical, partially blocked upstream or totally blocked may be seen. There is even a region where the flow may be partially blocked or supercritical, the state obtained depending on the past history of the flow (for example, the speed of start-up); this implies a hysteresis in the system. When the flow is partially or totally blocked the upstream disturbance has the character of a hydraulic jump and the flow is "controlled" by the condition that $F=u/(gd)^{\frac{1}{2}}=1$ at the obstacle crest. Using the Bernoulli and mass conservation equations, the properties of hydraulic jumps and this control condition, the detailed properties may be calculated. Our objective in this paper is to show how to obtain the corresponding information for multi-layered systems.

PROPERTIES OF LAYERED MODELS

The equations governing hydrostatic motion of n incompressible layers of fluid with horizontal velocity $u_i(x,t)$, thickness $d_i(x,t)$ and density ρ_i may be expressed as (eg. Lee & Su 1977)

$$\frac{\partial d_i}{\partial t} + \frac{\partial}{\partial x}(u_i d_i) = 0, \qquad (1)$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x}\left[\tfrac{1}{2} u_i^2 + g\left(h + \sum_{j=1}^{i} d_j + \sum_{j=i+1}^{n} \frac{\rho_j}{\rho_i} d_j\right) + \frac{p_s}{\rho_i}\right] = 0 , \quad i=1,\ldots,n , \qquad (2)$$

where p_s is the pressure at the top of the nth layer and h is the height of the bottom topography. For simplicity we will restrict consideration to systems with a rigid upper boundary; the treatment for a free upper boundary follows similar lines. For steady-state flows equations (1) and (2) may be integrated to give

$$d_i u_i = D_i U_i \;, \qquad i = ,\ldots,n, \tag{3}$$

$$\tfrac{1}{2}\,(u_i^2 - U_i^2) + g \sum_{j=1}^{n} \rho_{ij}(d_j - D_j) + \frac{1}{\rho_i}\,(p_s - P_s) + gh = 0 \tag{4}$$

$$\sum_{i=1}^{n} d_i + h = \sum_{i=1}^{n} D_i = D \;, \tag{5}$$

where

$$\rho_{ij} = 1 \;, \quad j \leq i, \qquad = \rho_j/\rho_i \;, \quad j > i, \tag{6}$$

and D_i, U_i, P_s denote, respectively, the values of d_i, u_i and p_s far upstream. The U_i will all be assumed to be positive, so that the flow is uni-directional. Equation (1) denotes simple mass conservation, whereas equ. (2) is in effect the Bernoulli equation for each layer. If equations (3) to (5) are differentiated with respect to x and derivatives of u_i eliminated, the resulting system may be expressed in the matrix form (see equation 7 below) where

$$f_i^2 = \frac{u_i^2}{g d_i} \;, \quad g_n = \rho_n(1+f_n^2) \;. \tag{8}$$

Hence, at a point in the flow where $\frac{dh}{dx} = 0$ we must have either

$$\frac{dd_i}{dx} = 0 \;, \quad \text{for all } i, \tag{9}$$

or

$$\det [\;\;] = 0 \tag{10}$$

Equation (9) normally implies that the flow is symmetric at the crest of an obstacle so that the downstream flow state is the same as the upstream flow. The other possibility, equ. (10), may be shown to be equivalent to the condition that some internal wave mode (n=m, say) has zero propagation speed (relative to the topography) at this point (Benton 1955, Baines 1986). This means that the flow is "critical" in the hydraulic sense with respect to this mode, and that this point acts as a "control" on the flow when this condition is satisfied. For a stratified (layered) flow, such a "control" is a much weaker condition than it is for a single layer. For an n-layered system we will have n-1 internal wave modes with a total of 2(n-1) wave velocities (positive and negative). We assume that all of these wave speeds are real, implying that the flow is stable to long-wave disturbances. The speed and structure of these various modes may be obtained from studying linear disturbances to equations (1) and (2). In particular, linear disturbances may have infinite wavelength; these are termed columnar disturbance modes.

For our general model of stratified flow over topography, treated as an initial-value problem, we expect that in some cases upstream hydraulic jumps will occur. Hydraulic jumps may be modelled as regions where the flow changes abruptly from one uniform stream to another. They propagate at constant speed, and their overall properties do not change with time. In order to establish relationships between conditions upstream and downstream of hydraulic jumps in systems with two or more layers, some assumptions are required. The question of the most appropriate assumptions is controversial (Wood & Simpson 1984), but the following are the most commonly used (since Yih & Guha 1955): i) each layer maintains its identity, density and mass flux through the jump, ii) the flow in the jump is hydrostatic, or sufficiently so for our purposes, and iii) the mean value of the ith layer thickness in the jump is equal to the mean of the upstream and downstream values. With these assumptions, equations relating the conditions upstream and downstream of the jump may be derived (Su 1976, Baines 1986). As jump amplitude tends to zero, jump speed approaches a linear wave speed. In general jumps dissipate energy, although this dissipation may vanish in some cases.

THE GENERAL METHOD

We now give an outline of the general procedure for calculating stratified flow over finite topography, and its physical basis. Space obviously precludes a detailed discussion, and the procedure is embodied in a set of computer programmes which permit its application to a wide range of situations. We begin with a known flow without topography and define a mean velocity $\bar{U} = Q/D$, where Q is the total volume flux; $\bar{U}$ will be a constant, independent of x and t. We may then define an initial Froude number F_o by

$$F_o = \frac{\bar{U}}{\bar{U} - c_1}$$

where c_1 is the velocity of the fastest linear internal wave mode propagating against the stream in the (rest) frame of the topography. c_1 may be positive (directed downstream) or negative (directed upstream). We describe the procedure in three stages, each corresponding to progressively higher obstacles, and assume that $o < F_o < 1$, so that at lease one mode may propagate upstream. An implicit assumption is that at no point does the mass flux in any layer become reversed in direction (relative to the topography) as a result of topographic effects.

$$\begin{bmatrix} \rho_1(1-f_1^2)-g_n & \rho_2-g_n & \rho_{n-1}-g_n \\ \rho_2-g_n & \rho_2(1-f_2^2)-g_n & \rho_{n-1}-g_n \\ \rho_{n-1}-g_n & \rho_{n-1}-g_n & \rho_{n-1}(1-f_{n-1}^2)-g_n \end{bmatrix} \begin{bmatrix} \frac{dd_1}{dx} \\ \frac{dd_2}{dx} \\ \frac{dd_{n-1}}{dx} \end{bmatrix} = -\frac{dh}{dx} \begin{bmatrix} \rho_1-g_n \\ \rho_2-g_n \\ \rho_{n-1}-g_n \end{bmatrix} \tag{7}$$

Stage 1
As the obstacle height is increased from zero, the steady-state solutions for the flow are only altered over the obstacle. At $dh/dx=0$, equ.(9) is satisfied, so that the upstream and downstream flows are the same and are equal to the initial undisturbed flow; changes which take place on the upstream slope are reversed on the downstream slope. This situation is obtained until h_m reaches a critical height h_c (which depends on F_o), at which point equ. (10) is satisfied, and a wave speed vanishes for some mode (say the ith). With this given upstream profile, obstacle heights $h_m > h_c$ are not possible.

Stage 2
If the obstacle height is increased very slightly (infinitesimally) above h_c by an amount Δh, the flow will adjust locally so that it is again critical at the obstacle crest for the same ith mode. This will require a small linear disturbance, in the form of a columnar disturbance mode, to be sent upstream and to alter slightly the oncoming velocity and density profiles in the new steady state. This disturbance will have the structure of the same ith mode, will have amplitude $\Delta\alpha$ (say), and will travel upstream at the long-wave speed of the ith mode, c_i.

If the obstacle height is increased by a further infinitesimal amount, this process will be repeated: the flow will adjust to a slightly different state at the obstacle crest, which will again satisfy $c_i = 0$, and a linear columnar disturbance mode will propagate upstream at the linear wave speed, altering the oncoming flow which approaches the obstacle. Here, however, we must make an important distinction between two different cases. The propagation speed of the new upstream disturbance may be written $c_i + \Delta c_i$, where Δc_i denotes the difference in speed from the previous value. This speed will be slightly different because the second disturbance will propagate on the slightly modified flow behind the first disturbance. Δc_i may be positive or negative. If Δc_i is positive or zero, the second disturbance will never catch up to the previous one, and the flow over the obstacle is determined by these physical processes alone. If, however $\Delta c_i < 0$, the second disturbance will catch up with the first one and increase its amplitude. This will form, in effect, an infinitesimal hydraulic jump. As discussed above, a hydraulic jump travels at a speed which is dependent on its amplitude, and jump conditions may be found which determine the nature of the flow on the downstream side. Once the jump has formed, this flow will in general be different from that which was present behind the second upstream disturbance. This difference will then be communicated back to the flow in the vicinity of the obstacle and cause further adjustments there. These changes will in turn affect the jump, and the flow will finally reach a steady state when the jump amplitude is adjusted so as to be consistent with a critical flow state at the obstacle crest.

If the obstacle height is increased still further and successive values of Δc_i all have the same sign, these processes will be repeated. The result in the first case ($\Delta c_i > 0$) will be a succession of upstream disturbances which become increasingly spread out, forming a rarefaction, and the result in the second case ($\Delta c_i > 0$) will be a progressively larger hydraulic jump. These two different types of upstream motion and the conditions determining them may be summarized by saying that

$$\frac{dc_i}{d\alpha} < 0 \text{ implies a hydraulic jump} \quad ,$$

and

$$\frac{dc_i}{d\alpha} > 0 \text{ implies a rarefaction} \quad ,$$

where c_i denotes the propagation speed of the ith mode, which is the mode that is critical at $dh/dx = 0$, and α denotes the upstream amplitude of this mode. Expressions for $dc_i/d\alpha$ may be obtained in terms of the mean flow properties and the structure of the relevant eigenfunction. It is important to note that c_i and α are cumulative variables, in the sense that the following disturbances propagate on and add to previous ones. The structure of the corresponding eigenfunction also changes continuously. If the upstream amplitudes of these disturbances are small the resulting flows calculated assuming one or the other flow type will be similar, but as the amplitude increases the flow properties will diverge. Both of these physical processes may be calculated numerically with the use of algorithms given by Su (1976) and Lee & Su (1977).

Stage 3
The above procedure may be followed to give the flow over progressively higher obstacles until one of two things happens. These are (i) the flow immediately upstream of the obstacle may become critical (with respect to the ith mode, so that $c_i = 0$ just upstream) or (ii) the velocity of some layer U_i may become zero just upstream. We now discuss each of these situations in turn.

(i) Critical flow upstream; c_i decreases to zero. Here the current upstream disturbance must be a rarefaction. When c_i becomes zero upstream the flow over the topography must be supercritical with respect to this mode. If the ith mode is in fact the fastest upstream mode, this will be applicable for obstacle heights up to the maximum $h = D$. If the ith mode is not the fastest, then the next-fastest mode (say the i-1th) will become critical at the obstacle crest (i.e., $c_{i-1} = 0$ there) at some value for the obstacle height.

(ii) Blocking. The addition of upstream disturbances always (i.e., in all known experimental cases) has the effect of reducing the oncoming velocity in the lowest layer. If condition (i) does not occur, this velocity may decrease until the lowest layer (or possibly some higher layer) comes to rest. If the lowest layer is blocked, experimental observations show that additional upstream disturbances (caused, say, by raising the height of the obstacle) will result in this layer remaining at rest and will reduce the oncoming velocity of the second-lowest layer. For this to occur, the upstream disturbances must be more complicated than previously. In general they will consist of two disturbances - a faster mode and a slower mode, of which either may be a jump or a rarefaction, and which together result in the lowest layer remaining at rest, but with an altered thickness. The treatment of these situations in practice is more complex and will not be gone into here.

A SPECIFIC EXAMPLE

We describe here the results for a three-layer system when the density increments across each interface are the same, but the upper-most layer has half the thickness of each of the other two. This situation is a realistic approximation to the stratification in some fjords and gulfs. The results of applying the above procedures are

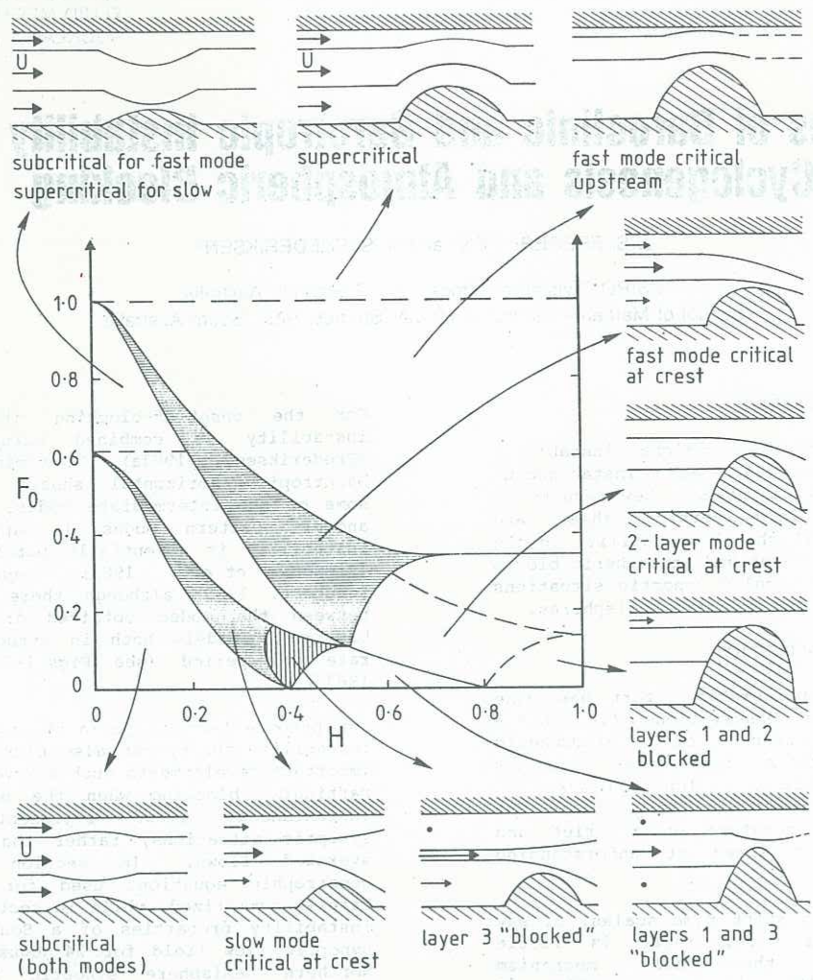

Fig. 1: Regime diagram for steady-state flow patterns for a three-layer hydrostatic flow, impulsively started from rest at speed U. The initial layer depths are 0.4 D, 0.4 D, 0.2 D, and the two density increments $\Delta_{1\rho}$, $\Delta_{2\rho}$ are equal, with $(\Delta_{1\rho} + \Delta_{2\rho})/\rho << 1$. In the shaded regions the flow is critical with respect to one of the modes at the obstacle crest, and the resulting upstream disturbances are of the rarefaction type.

shown in Fig.1; this diagram shows the various flow regions in terms of F_o and $H = h_m/D$.

The upstream disturbances in the shaded regions are of the rarefaction type. The region where the topmost layer is stagnant or "blocked" is of particular interest. Layer 3 becomes blocked before layer one because the upstream columnar "slow mode" reduces the layer 3 velocity more than layer 1 in this non-uniform density structure. There is even a regime where both layers 1 and 3 are at rest upstream.

In conclusion, I have given a brief outline of a general procedure for calculating the flow properties of stratified fluids over long topography of arbitrary amplitude, along the lines of "conventional" hydraulics. Attention has been focussed on flow properties on the upstream side and over the obstacle, where effects due to strong mixing events downstream (should they occur) would be minimal. The procedure is based on sound mechanistic principles extrapolated from verified properties of one and two layer systems.

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