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HIGH ORDER BOUNDARY INTEGRAL MODELLING OF CAVITATION BUBBLES

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SUMMARY When a collapsing cavitation bubble is sufficiently close to a boundary, a high speed liquid jet is observed to thread the bubble in the latter stages of collapse. A high order boundary integral method is developed to simulate the time-dependent shape of the cavitation bubble and to accurately predict the jet speed. The paper discusses the development of the technique, compares predictions against the well-known Rayleigh bubble solution in an infinite fluid and models the growth and collapse of a cavitation bubble near a rigid boundary. Our calculations indicate that the growth phase has an important bearing on the form of the cavitation bubble during the collapse phase yielding substantially different results for the speed and location of the jet than previously well-documented results. Experimental observation appears to substantiate our theory.

1 INTRODUCTION

It is well-known that a cavitation bubble collapses asymmetrically when near a rigid boundary yielding a high speed liquid jet that is directed towards the boundary (Benjamin and Ellis 1966, Plesset and Chapman 1971, Gibson and Blake 1980). If the bubble is sufficiently close to the boundary, the high speed jet may cause mechanical damage ('pitting') of the surface. Experimental observation (Benjamin and Ellis 1966, Gibson 1968, Gibson and Blake 1980) and theoretical calculation (Plesset and Chapman 1971, Prosperetti 1982) support this view.

However, theoretical developments so far have only considered the collapse of an initially spherical bubble, a state which is probably not realised in practice. It is our contention that the growth phase must also be important, especially if close to a boundary. Thus, in this paper, we consider both the growth and collapse phase of a cavitation bubble, initiating the growth of a cavitation bubble from a small sphere with the fluid being given a known finite kinetic energy. A high order boundary integral method is developed to allow us to obtain an accurate simulation of the time dependent bubble shape and, during collapse, the speed of the liquid jet.

In the next section we summarise the relevant aspects of the high order boundary integral technique which are developed more extensively in Taib, Doherty and Blake (1983) and Taib (1984). In later sections we illustrate the power of the technique by comparing our results against the well-known Rayleigh bubble solution. Results of our calculations for the growth and collapse of a cavitation bubble near a rigid boundary are compared against some previous experimental results (Gibson and Blake 1980) and theoretical predictions (Plesset and Chapman 1971, Prosperetti 1982).

2 BOUNDARY INTEGRAL METHOD

2.1 Boundary Integral Formulation

For any sufficiently smooth function ϕ which satisfies Laplace's equation within a domain Ω, having piecewise smooth surface S, Green's Integral Formula may be expressed as (Jaswon and Symm, 1977)

$$c(p)\phi(p) + \int_S \phi(q) \frac{\partial}{\partial n}\left(\frac{1}{|p-q|}\right) ds = \int_S \frac{\partial}{\partial n}(\phi(q)) \frac{1}{|p-q|} ds \qquad (1)$$

where $p \in \Omega + S$, $q \in S$, $|p-q|$ denotes the distance between p and q, $\partial/\partial n$ denotes the normal derivative in the direction of the outward normal to S and

$$c(p) = \begin{cases} 4\pi & \text{if } p \in \Omega \\ 2\pi & \text{if } p \in S \\ 0 & \text{if } p \notin \Omega . \end{cases} \qquad (2)$$

If either ϕ or $\frac{\partial\phi}{\partial n}$ is prescribed on S then solution of the integral equation,

$$c(p)\phi(p) + \int_S \phi(q) \frac{\partial}{\partial n}\left(\frac{1}{|p-q|}\right) ds = \int_S \frac{\partial}{\partial n}(\phi(q)) \frac{1}{|p-q|} ds \qquad (3)$$

$p, q \in S$ determines both ϕ and $\frac{\partial\phi}{\partial n}$ on S. The potential at any interior point can be computed using (1).

2.2 Geometric Modelling

As part of the numerical development it is necessary to parameterise the surface S. For axisymmetric problems, as we have here, all quantities are independent of circumferential location and thus we need only specify a half-contour which maybe rotated through 2π to form the surface. We approximate the surface S by N strips, say S_j $(j = 1,\ldots,N)$. In the case of a cavitation bubble, a typical strip is the surface of a frustrum of a cone. This development of the model allows us to perform one analytical integration in advance of the numerical solution.

2.3 Integrals to be Evaluated Analytically

Let p and $q \in S$ with coordinate $(r_0,0,z_0)$ and (r,θ,z) respectively, we have

$$\frac{1}{|p-q|} = \frac{1}{[(r+r_0)^2+(z-z_0)^2-4rr_0\cos^2\frac{\theta}{2}]^{\frac{1}{2}}} . \qquad (4)$$

By using (4), we obtain

$$\int_{S_j} \frac{ds}{|p-q|} = \int_0^1 \frac{4r(\xi)\left[\left(\frac{dz}{d\xi}\right)^2+\left(\frac{dr}{d\xi}\right)^2\right]^{\frac{1}{2}} K(k)}{[(r(\xi)+r_0)^2+(z(\xi)-z_0)^2]^{\frac{1}{2}}} d\xi \qquad (5)$$

where

$$k^2(\xi) = \frac{4r(\xi)r_0}{(r(\xi)+r_0)^2+(z(\xi)-z_0)^2} \qquad (6)$$

and with $K(k)$ being the complete elliptic integral of the first kind.

This is the integral over one strip defined by $r = r(\xi)$, $z = z(\xi)$ for ξ in the interval [0,1] where ξ is a parameter convenient for defining the surface shape.

One can obtain

$$\int_{S_j} \frac{\partial}{\partial n}\left(\frac{1}{|p-q|}\right) ds = 4\int_0^1 \frac{r(\xi)d\xi}{[(r(\xi)+r_0)^2+(z(\xi)-z_0)^2]^{\frac{3}{2}}} \tag{7}$$

$$\times\left\{\left[\frac{dz}{d\xi}(r+r_0)-\frac{dr}{d\xi}(z-z_0)-\frac{2}{k^2}\frac{dz}{d\xi}r_0\right]\frac{E(k)}{1-k^2}+\frac{2}{k^2}\frac{dz}{d\xi}r_0K(k)\right\}$$

where $E(k)$ is the complete elliptic integral of one second kind, approximations for which available in (Hastings, 1955),

$$\begin{aligned} K(k) &\simeq P(x) - Q(x)\ln(x) \\ E(k) &\simeq R(x) - S(x)\ln(x) \end{aligned} \tag{8}$$

where $x = 1-k^2$, P, Q, R and S are polynomials.

2.4 Constant Elements

The potential and its normal derivatives are assumed constant on each segment and the boundary integral equation is written at the mid point of each segment. Equation (3) for a given point i, becomes

$$2\pi\phi_i + \sum_{n=1}^{N}\phi_j\int_{S_j}\frac{\partial}{\partial n}\left(\frac{1}{|p_i-q_j|}\right)ds = \sum_{j=1}^{N}\frac{\partial}{\partial n}(\phi_j)\int_{S_j}\frac{1}{|p_i-q_j|}ds \tag{9}$$

$$i = 1,2,\ldots,N$$

Equation (9) may now be expressed as

$$2\pi\phi_i + \sum_{j=1}^{N}\hat{H}_{ij}\phi_j = \sum_{j=1}^{N}G_{ij}\frac{\partial}{\partial n}(\phi_j) \quad i = 1,2,..,N \tag{10}$$

where

$$\hat{H}_{ij} = \int_{S_j}\frac{\partial}{\partial n}\left(\frac{1}{|p_i-q_j|}\right)ds \quad \text{and}$$

$$G_{ij} = \int_{S_j}\frac{1}{|p_i-q_j|}ds .$$

By defining $H_{ij} = \hat{H}_{ij} + 2\pi\delta_{ij}$ we can write equation (10) as

$$\sum_{j=1}^{N}H_{ij}\phi_j = \sum_{j=1}^{N}G_{ij}\frac{\partial}{\partial n}(\phi_j) \quad i = 1,2,..,N \tag{11}$$

or in the matrix form, simply

$$H\Phi = GQ \tag{12}$$

2.5 Linear Elements

The surface is replaced by N segments of a cone such that the nodes are now at the intersection of these segments. For $0 \le \xi \le 1$ we let

$$\begin{aligned} M_1(\xi) &= 1 - \xi \\ M_2(\xi) &= \xi \end{aligned}$$

and consider the surface approximation defined by

$$\begin{aligned} r(\xi) &= \sum_{i=1}^{2} M_i(\xi)r_i \\ z(\xi) &= \sum_{i=1}^{2} M_i(\xi)z_i . \end{aligned} \tag{13}$$

We approximate ϕ and $\frac{\partial\phi}{\partial n}$ by

$$\begin{aligned} \phi &= \sum_{i=1}^{2} M_i(\xi)\phi_i \\ \frac{\partial\phi}{\partial n} &= \sum_{i=1}^{2} M_i(\xi)\frac{\partial}{\partial n}(\phi_i) . \end{aligned} \tag{14}$$

Equation (3) can now be written as

$$c_i\phi_i = \sum_{j=1}^{N+1}\int_{S_j}\phi\frac{\partial}{\partial n}\left(\frac{1}{|p_i-q_j|}\right)ds = \sum_{j=1}^{N+1}\int_{S_j}\frac{\partial\phi}{\partial n}\frac{1}{|p_i-q_j|}ds \tag{15}$$

which leads to a matrix equation of the form defined in (12).

2.6 Quadratic Elements

More accurate approximations can be obtained by using quadratic elements. In this case ϕ, $\frac{\partial\phi}{\partial n}$ and the surface are approximated by quadratic functions. For $0 \le \xi \le 1$, we let

$$\begin{aligned} M_1(\xi) &= (\xi-1)(2\xi-1) \\ M_2(\xi) &= 4\xi(1-\xi) \\ M_3(\xi) &= \xi(2\xi-1) \end{aligned} \tag{16}$$

and consider the surface approximation to be defined by

$$\begin{aligned} r(\xi) &= \sum_{i=1}^{3} M_i(\xi)r_i \\ z(\xi) &= \sum_{i=1}^{3} M_i(\xi)z_i \end{aligned} \tag{17}$$

where $r_i = r(p_i)$, $z_i = z(p_i)$ $(i = 1,2,3)$ and where p_1 and p_3 are at the end points and p_2 at the mid point of the segment. We approximate ϕ and $\frac{\partial\phi}{\partial n}$ by

$$\begin{aligned} \phi &= \sum_{i=1}^{3} M_i(\xi)\phi_i \\ \frac{\partial\phi}{\partial n} &= \sum_{i=1}^{3} M_i(\xi)\frac{\partial\phi_i}{\partial n} \end{aligned} \tag{18}$$

where $\phi_i = \phi(p_i)$ $\quad \frac{\partial\phi_i}{\partial n} = \frac{\partial}{\partial n}(\phi(p_i))$ $\quad i = 1,2,3.$

Equation (3) for a given point i becomes

$$c_i\phi_i + \sum_{j=1}^{2N+1}\int_{S_j}\phi\frac{\partial}{\partial n}\left(\frac{1}{|p_i-q_j|}\right)ds = \sum_{j=1}^{2N+1}\int_{S_j}\frac{\partial\phi}{\partial n}\frac{1}{|p_i-q_j|}ds \tag{19}$$

$$i = 1,2,3,..,2N+1$$

again leading to a matrix equation of the form given in (12).

2.7 Numerical Evaluation of Integrals

The integrals can be evaluated using Gauss quadrature formulae, except when the interval over which we are integrating contains the point under consideration. In this case the logarithmic singularity in $K(k)$ and $E(k)$ must be accommodated. A special case exists when this point lies on the axis. The detailed calculations required in this section are far too extensive to be included in this paper but may be found in either Taib, Doherty and Blake (1983) or Taib (1984).

We now apply the techniques outlined in this section to the Rayleigh bubble problem and to the growth and collapse of a cavitation bubble near a rigid boundary.

3 RAYLEIGH BUBBLE SOLUTION

To test the performance of the method, we used quadratic elements to simulate the collapse of a single spherical bubble in an infinite fluid, the well-known Rayleigh problem. The calculation started to develop instability for a bubble radius 1.2520×10^{-3} at time $t = 0.9167$ which is in reasonable agreement with the exact theoretical collapse time of $t_c = 0.91479$. Greater accuracy could be obtained if additional elements or more accurate time stepping procedures were used.

4 RIGID BOUNDARY

The growth and collapse phases of a cavitation bubble near a rigid boundary are modelled using linear elements. We will assume the fluid is incompressible, inviscid and that surface tension and gravitational effects are negligible (Plesset and Chapman 1971, Blake and Gibson 1981). With these assumptions we may present the velocity as the gradient of a potential which satisfies Laplace's equation. That is

$$\underset{\sim}{u} = \nabla\phi, \quad \nabla^2\phi = 0 \tag{20}$$

where $\underset{\sim}{u}$ is the cartesian velocity vector and ϕ is the potential. The conditionsat infinity are

$$\underset{\sim}{u} \to 0 \quad \text{and} \quad p \to p_\infty \tag{21}$$

where p is the pressure and p_∞ is the constant pressure at infinity. On the bubble surface we have

$$\underset{\sim}{u}_s = \underset{\sim}{u}, \quad p = p_c \tag{22}$$

where $\underset{\sim}{u}_s$ is the velocity of a particle on the surface and p_c is the saturated vapour pressure in the bubble which is assumed constant. The Bernoulli condition gives

$$p_c = p_\infty - \rho\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho|\underset{\sim}{u}|^2 \tag{23}$$

where ρ is the density of the fluid.
For the rigid boundary, there is no flow across it, thus,

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \,. \tag{24}$$

Initially, the growth starts from a small spherical bubble of radius R_0 with the potential given by (Blake and Gibson 1981)

$$\phi_0 = -R_0\left\{\frac{2}{3}\left(\frac{p_\infty - p_c}{\rho}\right)\left[\left(\frac{R_m}{R_0}\right)^3 - 1\right]\right\}^{\frac{1}{2}} \tag{25}$$

where R_m is the maximum bubble radius which is determined by p_∞ and p_c.

In our computation, we introduce an image bubble so that the condition (24) will be automatically satisfied so it is no longer necessary to discretise the rigid boundary.

The strategy of solution is straightforward. Since, initially we know the position of the bubble surface and the value of ϕ on the surface, we can now solve one of the discretised versions of the integral equation to give us the value of $\partial\phi/\partial n$ on the bubble surface. With knowledge of ϕ, we can calculate the tangential derivative $\partial\phi/\partial s$. Knowing $\partial\phi/\partial n$ we can proceed to calculate the particle velocities, u_R and u_Z where u_R and U_Z are the particle velocities in the R and Z direction respectively. Immediately, we can use a simple Euler scneme to calculate the next position of the particle, a time Δt later,

$$z_j(t+\Delta t) = z_j(t) + u_{Z_j}(t)\Delta t + O(\Delta t^2) \tag{26}$$

$$r_j(t+\Delta t) = r_j(t) + u_{R_j}(t)\Delta t + O(\Delta t^2) \,. \tag{27}$$

The potential ϕ can be updated as follows,

$$\phi_j(t+\Delta t) = \phi_j(t) + \left[\frac{p_\infty - p_c}{\rho} + \frac{1}{2}\underset{\sim}{u}^2\right]\Delta t + O(\Delta t^2) \tag{28}$$

where $j = 1,2,\ldots,M$, M is the number of nodes. The time increment Δt is carefully chosen so that the potential ϕ can only change by at most a specified fixed amount $\Delta\phi$ (Gibson and Blake, 1982).

In our computations, linear dimensions are made dimensionless with respect to R_m. Thus the axial co-ordinate z, the radial co-ordinate r and the initial distance between the bubble centroid and the rigid boundary h, now become (Blake and Gibson, 1981)

$$Z = z/R_m, \quad R = r/R_m, \quad \gamma = h/R_m \,. \tag{29}$$

By using the characteristic collapse velocity $\left((p_\infty - p_c)/\rho\right)^{\frac{1}{2}}$, we can scale the time unit to get a dimensionless time T as follows,

$$T = t/t^*, \quad t^* = R_m\left(\frac{p_\infty - p_c}{\rho}\right)^{-\frac{1}{2}} . \tag{30}$$

5 RESULT AND DISCUSSION

The growth and collapse of bubbles near a rigid boundary was simulated for a number of cases and we include two illustrative examples here. In case 1, the parameter γ is 1.0 and in case 2, γ is 1.5. Calculations were stopped when the jet touched the opposite side of the bubble. The bubble shape for selected dimensionless time T for each case are shown in figure 1. Table 1 lists the dimensionless time T for each shape

Shape	Figure 1a	Figure 1b
	Time T	Time T
A	0.001553	0.001553
B	1.018353	1.016221
C	1.947809	1.880133
D	2.026057	2.000004
E	2.056111	2.028881
F	2.097495	2.050086
G	2.125974	2.068861
H	2.148931	2.087584
I	2.167770	2.098546

Table 1 Table of dimensionless time T for each shape illustrated in figure 1.

For the case $\gamma = 1.0$, Plesset and Chapman (1971) simulate the collapse of an initially spherical cavity. They obtained bubble shapes which are more elongated than those we obtained here indicating the importance of also considering the growth phase as well. We have separately considered the case $\gamma = 0.96$ where we obtain very good agreement with the experimental results reported in Gibson and Blake (1980), hence confirming our views.

For $\gamma = 1.5$, the bubble shapes we obtained are in general agreement with those obtained by Plesset and Chapman (1971) and those obtained by Guerri, Lucca and Prosperetti (1981). However in our model the collapse occurred much nearer to the rigid boundary again indicating the importance of the growth phase in determining the characteristic of the collapsing cavitation bubble.

In summary, we conclude that the boundary integral method is a powerful technique for modelling unsteady problems in fluid mechanics involving moving boundaries, as has been illustrated in this paper concerning the growth and collapse of cavitation bubbles.

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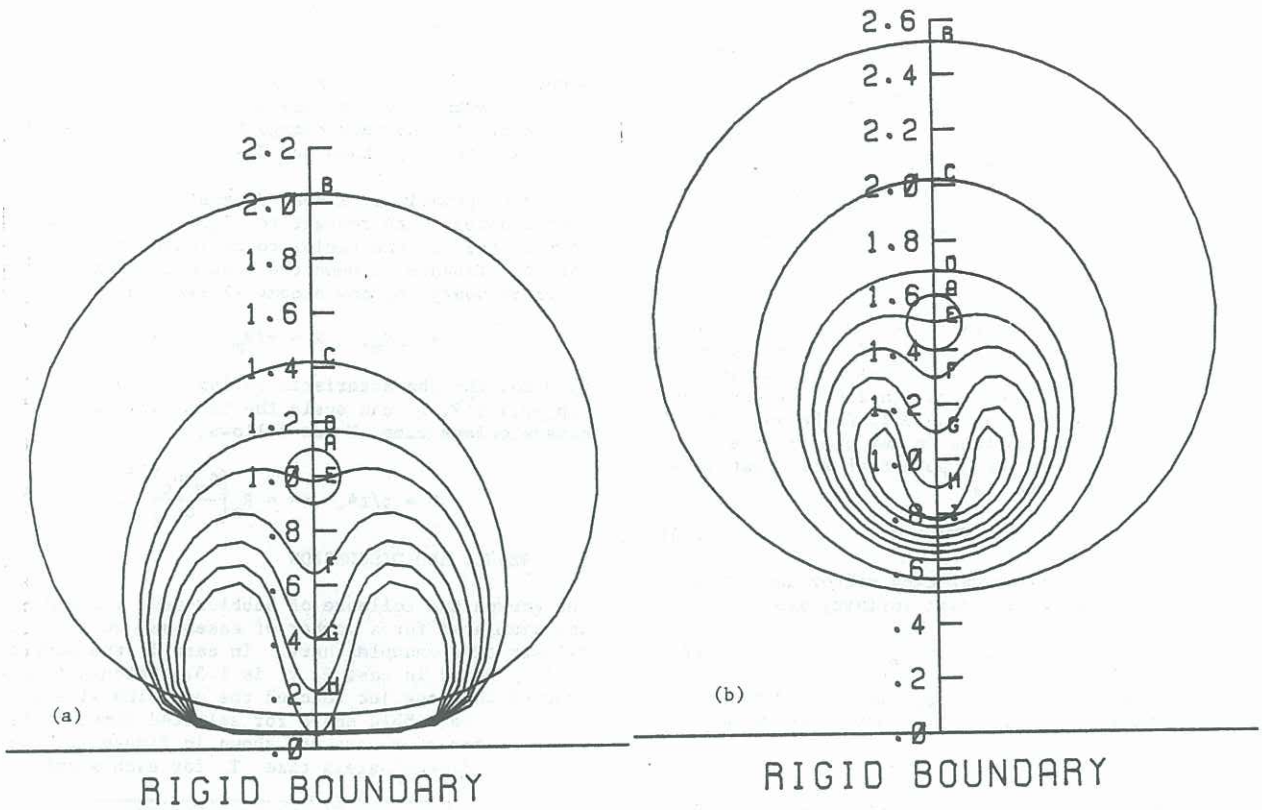

Figure 1 Bubble shapes for (a) $\gamma = 1.0$ and (b) $\gamma = 1.5$

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