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A LAGRANGIAN STATISTICAL MODEL OF TURBULENT CONCENTRATION FLUCTUATIONS

B.L. SAWFORD

DIVISION OF ATMOSPHERIC RESEARCH

CSIRO, MORDIALLOC, VIC. 3195 AUSTRALIA

SUMMARY Lagrangian statistical models for the trajectories of pairs of marked fluid particles are used to calculate the variance of concentration fluctuations for a passive scalar in a range of turbulent flows.

1 INTRODUCTION

The prediction of fluctuations in the concentration of a passive tracer in turbulent flows is a problem of wide-ranging interest with application to air quality modelling, turbulent chemical reactors (both engineering and geophysical), flames and combustion processes - indeed wherever turbulent mixing processes occur. Such processes are most naturally treated in Lagrangian terms. Of course, an exact treatment is not possible, but Lagrangian models avoid some of the physically unsound approximations (e.g. the gradient-transfer assumption) made in the Eulerian approach (Corrsin, 1974) and expose the role played by basic properties of the flow, such as the energy spectrum, length scale, turbulence intensity and mean shear.

2 LAGRANGIAN STATISTICAL THEORY

The Lagrangian statistical theory is based on relations between moments of the concentration field and the displacement statistics of marked fluid particles. The ensemble-mean concentration field is given by (Tennekes and Lumley 1972, p.236)

$$\langle C(\underline{x},t)\rangle = \int_{-\infty}^{t}\int_{-\infty}^{\infty} P_1(\underline{x}',t';\underline{x},t)\, S(\underline{x}',t')d\underline{x}'dt' \quad (1)$$

where $P_1(\underline{x}',t';\underline{x},t)$ is the probability density that a particle which is at $\underline{x}$ at time t came from $\underline{x}'$ at time t' and $S(\underline{x}',t')$ is the source distribution. Similarly the two-point covariance is

$$\langle C(\underline{x}_1,t)C(\underline{x}_2,t)\rangle = \iint_{-\infty}^{t}\iint_{-\infty}^{\infty} P_2(\underline{x}_1',\underline{x}_2',t_1';t_2';\underline{x}_1,\underline{x}_2,t)$$

$$\times\ S(\underline{x}_1',t_1')S(\underline{x}_2',t_2')d\underline{x}_1'd\underline{x}_2'dt_1'dt_2', \quad (2)$$

where $P_2(\underline{x}_1',\underline{x}_2',t_1',t_2';\underline{x}_1,\underline{x}_2,t)$ is the probability density that particles which are at $\underline{x}_1$ and $\underline{x}_2$ at time t came from $\underline{x}_1'$ and $\underline{x}_2'$ at time t_1' and t_2' respectively. Thus as discussed by Corrsin (1952) P_1 and P_2 are probability density functions for reversed diffusion in that they describe the previous distribution of particles which later pass through chosen points. The distinction between reversed and forward diffusion is non-trivial whenever the underlying process is non-stationary, e.g. in nonstationary turbulence or stationary inhomogeneous turbulence and for P_2, even in stationary homogeneous turbulence since relative dispersion is nonstationary.

For computational covenience we consider the instantaneous area sources

$$S(\underline{x},t) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_o}\exp\left(-\frac{z^2}{2\sigma_o^2}\right)\delta(t) \quad (3)$$

or

$$S(\underline{x},t) = \frac{1}{2d_o}\delta(t) \qquad |z| \le d_o$$
$$= 0 \qquad |z| > d_o \quad (4)$$

so that the integrations over x',y' and t' are trivial and (1) and (2) reduce to one-component forms.

In order to apply (1) and (2) we require either a model (or approximation) for P_1 and P_2 or else a model for particle and particle-pair trajectories. We pursue the latter course here.

3 MODELLING PARTICLE TRAJECTORIES

In general particle motions and displacement probability densities depend on the whole history of the motion. However, the Uhlenbeck-Ornstein (UO) process, $\mathcal{U}(t)$, (Wax, 1954) serves as a reasonably realistic model for one component of the velocity of a single particle in high Reynolds number three-dimensional turbulence. This process is the solution of the Langevin stochastic differential equation

$$d\mathcal{U} = -\mathcal{U}dt/t_L + \sigma_w\sqrt{2/t_L}\,dW_t \quad (5)$$

where dW_t is a Gaussian white noise process, σ_w^2 is the velocity variance and t_L the Lagrangian integral time scale or (perhaps more appropriately) the decorrelation time scale. $\mathcal{U}(t)$ is a Gaussian-Markov process with an exponentially decaying correlation function

$$\langle\mathcal{U}(t)\ \mathcal{U}(t+\tau)\rangle/\sigma_w^2 = \exp(-\tau/t_L) \quad (6)$$

and thus models 'memory' in the turbulent motion of a particle. That it does so in an appropriate way may be seen by noting that the spectrum corresponding to (6) has the universal (ω^{-2}) inertial subrange behaviour.

Using (5) to model the vertical component of the velocity of a single particle gives (with an obvious adjustment of notation)

$$P_1(z,t;z_o,0) = \left[(2\pi)^{\frac{1}{2}}\sigma_z\right]^{-1}\exp\left[-(z-z_o)^2/2\sigma_z^2\right] \quad (7)$$

and for the Gaussian source distribution (3),

$$\langle C(z,t)\rangle = \left[(2\pi)(\sigma_o^2+\sigma_z^2)\right]^{-\frac{1}{2}}\exp\left[-z^2/2(\sigma_o^2+\sigma_z^2)\right]. \quad (8)$$

The single-particle position variance, σ_z^2, is as a consequence of (6),

$$\sigma_z^2 = 2\sigma_w^2 t_L^2\left[\exp(-t/t_L) + t/t_L - 1\right] \quad (9)$$

The velocities of a pair of particles are not independent but are correlated because of the spatial structure of the turbulence, characterized by the Eulerian structure function. Durbin (1980) proposed a model in which this correlation is included by writing the particle velocities as appropriate combinations of two independent UO processes, U' and U''. In "centre-of-mass" and "separation" variables, $Z = (z_1+z_2)/\sqrt{2}$ and $\Delta = (z_1-z_2)/\sqrt{2}$ respectively, his model is

$$\frac{dZ}{dt} = \left\{2 - R(\Delta)\right\}^{\frac{1}{2}} U' \qquad (10a)$$

and
$$\frac{d\Delta}{dt} = R^{\frac{1}{2}}(\Delta)\ U'' \qquad (10b)$$

where $2\sigma_w{}^2 R(\Delta)$ is an Eulerian structure function (Townsend, 1976, p.11). A simple but appropriate form for $R(\Delta)$ is

$$R(\Delta) = \left\{\Delta^2/(\Delta^2 + L^2)\right\}^{1/3} \qquad (11)$$

where L is the Eulerian integral length scale of the turbulence. We use $t_L = L/\sigma_w$. Equation (11) interpolates between the inertial range form $R(\Delta)\sim\Delta^{2/3}$ for $\Delta \ll L$ (corresponding to the Eulerian velocity spectrum (-5/3) inertial subrange) and the large scale limit $R(\Delta) \to 1$. Equation (10b) is a non-linear stochastic differential equation in which the rate of separation is a function of the separation and is thus akin to Richardson's (1926) idea that the eddy diffusivity (for relative dispersion) is a function of separation. It yields a non-Gaussian distribution for Δ (Durbin, 1980)

$$P(\Delta,t;\Delta_o,0) = \frac{R^{-\frac{1}{2}}(\Delta)}{(2\pi)^{\frac{1}{2}}\sigma_z} \exp\left\{-\frac{\left(G(\Delta)-G(\Delta_o)\right)^2}{2\sigma_z{}^2}\right\} \qquad (12)$$

where $G(\Delta) - G(\Delta_o) = \int_{\Delta_o}^{\Delta} R^{-\frac{1}{2}}(\Delta')d\Delta'$.

Alternative models have been proposed (Lamb, 1981; Sawford 1982a,b) in which $P(\Delta)$ is Gaussian. The essence of these alternative models can be demonstrated by replacing $R(\Delta)$ by $\langle R(\Delta)\rangle$ in (10) to give

$$\frac{d\Delta}{dt} = \langle R(\Delta)\rangle^{\frac{1}{2}} U''\ ;\ \frac{dZ}{dt} = \left(2-\langle R(\Delta)\rangle\right)^{\frac{1}{2}} U' \qquad (13)$$

Here the rate of separation depends on the separation in a statistical sense only (on $\langle\Delta^2\rangle$ in fact) as proposed by Batchelor (1952).

As a consequence of the form of (11) all of these models produce the inertial range relative dispersion law, $\langle\Delta^2\rangle\sim t^3$.

4 CONCENTRATION FLUCTUATIONS IN HOMOGENEOUS TURBULENCE

4.1 Without Mean Shear

Sawford (1983) has calculated concentration fluctuations in stationary homogeneous turbulence using Durbin's model (10) and a Gaussian model (Sawford 1982a,b). The main results of this investigation can be seen from Figure 1 which shows the centreline (z = 0) squared intensity of fluctuations, $s^2 = \langle C^2\rangle/\langle C\rangle^2-1$, as a function of time for the Gaussian source distribution (3) with a range of source sizes. Because in (10) Δ and Z are only weakly dependent, Z is very nearly Gaussian and (1) and (2) can be partly integrated to give

$$s^2(o,t) \approx \frac{(2\pi)^{\frac{1}{2}}(\sigma_o{}^2+\sigma_z{}^2)}{(\sigma_o{}^2+\sigma_z{}^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} P(\Delta',t;o,o)\ S(\Delta')d\Delta' \qquad (14)$$

where $\sigma_z{}^2 = \langle Z^2\rangle$.
This relation is exact for Gaussian models. It shows clearly the role played by source size in determining the intensity of fluctuations. For small time such that $\sigma_o \gg \sigma_\Delta$ (the standard deviation of the separation) both models show a similar dependence on σ_o and t as can be seen from Figure 1. The importance difference

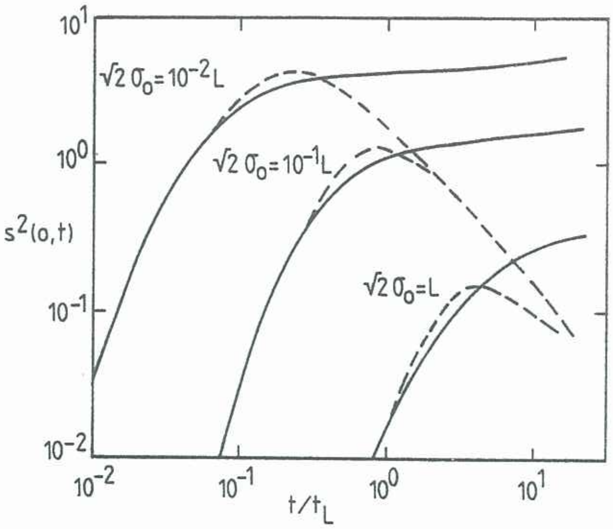

Figure 1 Intensity of fluctuations in unsheared homogeneous turbulence. (——) non-Gaussian; (----) Gaussian.

arises in the large-time limit where for Durbin's model $s^2(o,t)$ approaches a constant limit which is a function of σ_o,

$$s^2(o,t) \to 1{\cdot}35\ \sigma_o{}^{-1/3}-1 \qquad t\to\infty;\ \sigma_o \ll L \qquad (15)$$

while for the Gaussian model it is independent of source size and vanishes according to

$$s^2(o,t) \to \tfrac{1}{2}\rho^2. \quad \sim t^{-1} \qquad t\to\infty \qquad (16)$$

where $\rho = 1 - \sigma_\Delta{}^2/\sigma_z{}^2$ is the correlation between the position of the two particles. Wind tunnel measurements in grid turbulence support a constant asymptotic level as in (15) (Stapountzis et al., 1983) but the source-size dependence has yet to be shown directly.

By calculating separately the fluctuations due to meandering (i.e. bulk motion of the cloud of contaminant) and those due to the internal structure of the cloud (i.e. in coordinates relative to the cloud centre-of-mass) Sawford (1983) explained this difference in asymptotic behaviour by showing that Gaussian models smooth out the internal structure of the cloud i.e. at large time the ensemble averaging in (13) is equivalent to spatial averaging over the cloud. Then concentration fluctuations are due entirely to meandering and consequently vanish. The non-linearity of Durbin's model plays an essential role in properly accounting for the internal structure of the cloud and fluctuations in relative coordinates.

It is apparent from Figure 1 that the timescale for development of fluctuations is a function of source size. Sawford and Hunt (1983) have shown that for small sources, $\sigma_o \ll L$, this timescale is $(\sigma_o/L)^{2/3}\ t_L$ - a consequence of inertial range scaling.

4.2 Sheared Homogeneous Turbulence

For simplicity we consider a constant mean shear, $\gamma = d\bar{u}/dz$ and ignore dispersion due to the streamwise component of the turbulence. Then the streamwise velocities of the pair of particles are

$$\frac{dx_1}{dt} = \gamma z_1 \quad ; \quad \frac{dx_2}{dt} = \gamma z_2 \qquad (17)$$

As soon as the particles separate in the vertical due to that component of the turbulence, mean shear also causes them to separate in the streamwise direction. Extending our previous notation we have

$$\frac{d\Delta_x}{dt} = \gamma\Delta_z \qquad (18)$$

with Δ_z given by (10b). However, now the correlation between the particle velocities depends on both the streamwise and vertical separations and we model this by replacing Δ^2 by $\Delta_x^2+\Delta_z^2$ in the structure function, $R(\Delta)$.

As $t \to o$, $\langle\Delta_x^2\rangle \sim O(t^5) \ll \langle\Delta_z^2\rangle$ so dispersion takes place initially as it would in the absence of shear. As $t \to \infty$, pairs of particles tend to move independently and (18) can be integrated to give $\langle\Delta_x^2\rangle \sim t^3 \gg \langle\Delta_z^2\rangle$ (Durbin 1980). Thus in this limit the structure function is dominated by Δ_x and is essentially independent of Δ_z so that (10b) reduces to a linear stochastic equation and $P(\Delta)$ is Gaussian. Concentration fluctuations are smoothed out and the intensity of fluctuations vanishes, in agreement with the results shown in Figure 2 of Travoularis and Corrsin (1981).

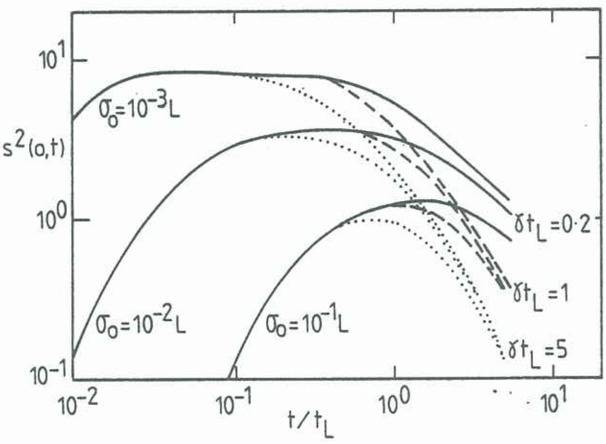

Figure 2 Effect of mean shear on intensity of fluctuations in homogeneous turbulence

Figure 2 shows $s^2(o,t)$ for three source sizes and three values of the mean shear. Two timescales are relevant; in the absence of shear fluctuations develop (i.e. s^2 approaches the limit (15)) on the time scale $(\sigma_o/L)^{2/3}t_L$, while the time scale for shear to become important is γ^{-1}. Thus the evolution of concentration fluctuations depends on the source size and the strength of the shear. When $(\sigma_o/L)^{2/3}\gamma t_L \ll 1$ (i.e. for sufficiently small sources or weak shear) the asymptotic level (15) is observed before shear begins to reduce fluctuations. Conversely for large sources or strong shear, $(\sigma_o/L)^{2/3}\gamma t_L \gg 1$, this constant intensity of fluctuations does not occur. We see also that for $\gamma t \gg 1$ (i.e. when shear is effective in smoothing fluctuations) the source-size dependence disappears.

5 INHOMOGENEOUS TURBULENCE

As a simple example we consider the constant stress region of a boundary layer flow in which the length scale is proportional to distance from the boundary. The effect of this increasing scale of turbulence is modelled simply by using local values of t_L and L (evaluated at the centre-of-mass of the pair of particles, $Z_m = Z/\sqrt{2}$ in (5), (10) and (11). For a logarithmic mean-flow profile (18) becomes

$$\frac{d\Delta_x}{dt} = \frac{u_*}{k\sqrt{2}} \ln(z_1/z_2) \ . \qquad (19)$$

where u_* is the friction velocity and k is von Karman's constant.

As with the single particle case (Durbin and Hunt, 1980) the ensemble dispersion statistics depend on the release height, H, and an appropriate time scale is H/u_*. For $t \ll H/u_*$ dispersion is unaffected by the inhomogeneity of the turbulence. On the other hand, for $t \gg H/u_*$ all memory of the release height is lost and dispersion proceeds as for a ground-level release. In this asymptotic stage the cloud grows linearly with time and, as shown by Hunt and Weber (1979a), $\langle\Delta_z^2\rangle$ (effectively the mean square <u>instantaneous</u> width of the cloud) remains a constant fraction of the single-particle dispersion, $\langle z_1^2\rangle - \langle z_1\rangle^2$ (effectively the mean-square width of the <u>envelope</u> of the cloud). In physical terms, as the cloud grows the meandering motions of the centre-of-mass persist because increasingly large eddies are encountered as it rises through the boundary layer. Using the values $\sigma_w = 1{\cdot}3u_*$ and $L = 0{\cdot}3z$ suggested by Hunt and Weber, we find $\langle\Delta_z^2\rangle/(\langle z_1^2\rangle - \langle z_1\rangle^2) = 0{\cdot}72$, which is in good agreement with their rough theoretical estimate of 0·64.

Figure 3 shows the intensity of fluctuations at the source-height for a range of source-sizes (the top hat source (4) has been used here). To begin with we concentrate on the effect of the inhomogeneity by ignoring the effect of mean-shear. These results are shown as the solid lines. We see again, that at small time $t \ll H/u_*$, the inhomogeneity is unimportant and for small enough sources, $\{\sigma_o/L(H)\}^{2/3}\, t_L(H) \ll H/u_*$, the homogeneous asymptotic limit (15) (in fact the equivalent result for the top-hat source, (4)) is reached before the influence of the inhomogeneity is felt. However, for $t \gg H/u_*$ the ever-increasing scale of the turbulence and the persisting centre-of-mass motion cause the intensity of fluctuations to increase with time. Presumably, asymptotically the effect of source size disappears although this is not clear from the numerical results.

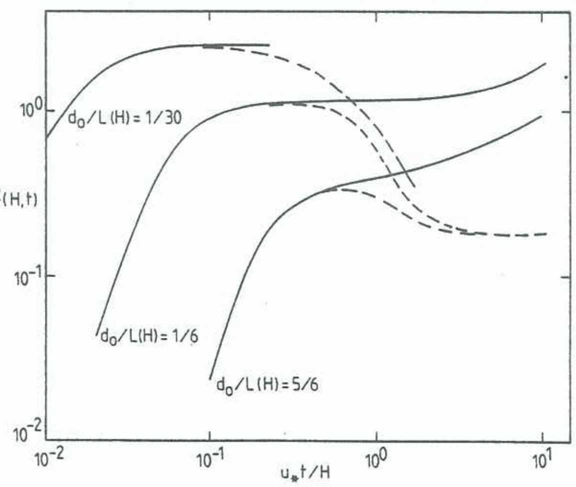

Figure 3 Intensity of fluctuations in inhomogeneous turbulence. (——) without mean shear; (----) logarithmic mean flow profile.

The intensity of concentration fluctuations when mean shear is modelled using (19) is shown as the dashed lines in Figure 3. Clearly mean shear too acts on the timescale H/u_* (as can also be seen by expanding (19) for $z_1 \approx z_2 \approx H$). As in the homogeneous case, mean shear smooths and reduces concentration fluctuations, but now it is competing against the tendency of the inhomogeneity to increase the intensity of fluctuations. The net result of these opposing influences is that for $t \gg H/u_*$, $s^2(H,t)$ is constant and (contrary to the homogeneous case) independent of source-size.

These results agree well with the wind-tunnel measurements of Fackrell and Robins (1982). Figure 3 reproduces all the qualitative features of their Figure 5 and indeed there is a remarkable similarity between the two. Quantitatively the asymptotic limit from Figure 3, $s^2(H,t) \simeq 0{\cdot}2$, agrees quite well with their measured value, 0·3 - 0·4, and for the quasi-homogeneous stage of dispersion their results confirm the inertial range timescale, $(\sigma_o/L(H))^{2/3}\, t_L$, and the $\sigma_o^{-1/3}$ dependence in (15).

6 CONCLUSIONS

The Lagrangian statistical model of concentration fluctuations presented here is simple in concept and provides a very direct connection between statistics of the concentration field and appropriate properties of the velocity field. Predictions from the model agree well with wind tunnel measurements in grid turbulence (with and without mean shear) and in a boundary layer.

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