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STABILITY OF FULLY DEVELOPED TURBULENT CHANNEL FLOW

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SUMMARY Hydrodynamic stability of fully developed turbulent channel flow has been examined employing a non-homogeneous Orr - Sommerfeld equation incorporating a Reynolds number dependent velocity profile. Applied disturbance is characterized by high wave numbers. A sufficient condition obtained for the flow to be stable shows that the flow isstable when the non-dimensional wave number exceeds a value of 0.7.

1. INTRODUCTION

Study of stability of fully developed turbulent flow as a first step can be expected to provide some insight into the complex phenomenon of reverse transition. Malkus (1956) predicted the fully developed turbulent mean velocity profile in a channel through suitable postulates and constraints. Reynolds and Tiderman (1967) evaluated the ideas of Malkus (1956) by obtaining neutral stability curve following along the lines of classical hydrodynamic stability theory for laminar flow and concluded that the success of Malkus'(1956) theory to be fortuitus. Betchov and Criminale (1964) studied the stability of turbulent boundary layer over a flat plate employing a non-homogeneous Orr-Sommerfeld equation concluded that the instability does not occur when the non-dimensional wave number is in the range 0.6 to 2.5. Landahl (1967) while evaluating the wave propagation constants for the disturbance caused by turbulence break down, considered the terms due to variation in Reynolds stress (in non-homogeneous Orr-Sommerfeld equation) to be having in a sense a destabilizing effect. Kutateladge (1971) has shown that the terms due to variation in Reynolds stresses are not important in the sublayer as well as in the logarithmic region. Thus there exists considerable controversy as to the role of Reynolds stresses on stability of turbulent flow.

Present work attempts to clarify the role of Reynolds stresses on the stability of fully developed turbulent channel flow, in particular when considered in relation to higher order effects in the mean velocity profile as described by Afzal and Yajnik (1973). Analysis is carried out employing the method of matched asymptotic expansions. Features incorporated relevant to turbulent flow are, dependence of velocity profile on Reynolds number, largeness of wave numbers and variation of Reynolds stresses which leads to a non-homogeneous Orr-Sommerfeld equation.

2. MATHEMATICAL ANALYSIS

2.1 Governing Equations

Physical model and the coordinate system are shown in Fig.1. When a two dimensional disturbance is applied to a turbulent fully developed channel flow and recognizing that now the flow field will consist of a mean part, a fluctuating part and a disturbance, the governing equation for the stream function of the applied disturbance can be obtained in the form as given by Phillips (1967) as,

$$\begin{aligned}(U-C)(\phi'' - \alpha^2 \phi - U''\phi) = \\ -(i/\alpha R)(\phi'''' - 2\alpha^2\phi'' + \alpha^4 \phi) \\ -(T'_{xx} - T'_{yy}) + (i/\alpha)(T''_{xy} + \alpha^2 T_{xy}) \end{aligned} \quad (1)$$

In eqn(1) $U(y)$ is the turbulent mean velocity, $C(= C_r + iC_i)$ is the wave velocity, in general complex. α is the wave number and R is the Reynolds number based on the centre line velocity and half channel width. $\phi(y)$ is the function defining the disturbance stream function $\psi(x, y, t) = \phi(y) \exp[i\alpha(x - ct)]$. x,y are the coordinate along and

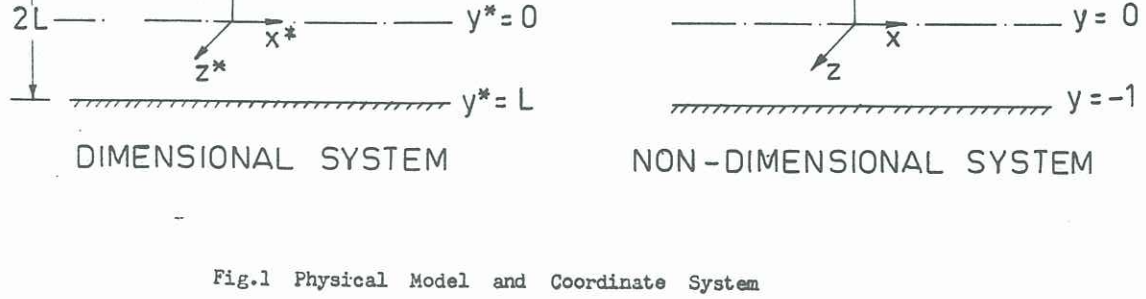

Fig.1 Physical Model and Coordinate System

perpendicular to the direction of the flow. Primes denote ordinary differentiation with respect to y. 'T' terms represent the variation in Reynolds stress defined, for example, as follows. If u_1, v_1 are the disturbance velocity components and u', v' are the turbulence velocity fluctuations, the shear stresses are,

$$\tau_{xy1} = \overline{u_1 v_1} \quad \text{and} \quad \tau_{xy} = \overline{u'v'} \, . \qquad (2)$$

Then variation in Reynolds stress is,

$$T_{xy} = \tau_{xy} - \tau_{xy1} \, , \qquad (3)$$

and

$$T_{xy} = T_{xy}(y) \, e^{i\alpha(x - ct)} \, . \qquad (4)$$

Boundary conditions on the disturbance velocity component walls ($y = \pm 1$) imply,

$$\phi = 0 = \phi' \quad \text{at} \quad y = \pm 1 \qquad (5)$$

$$T_{xx} = 0 = T_{yy} = T_{xy} = T'_{xy} = T'_{yy} \quad \text{at} \quad y = \pm 1. \qquad (6)$$

Eqn.(1) together with the boundary conditions eqns.(5) and (6) govern the applied disturbance.

2.2 Turbulent Velocity Profile

From Afzal and Yajnik (1973) a uniformly valid expansion for the mean velocity U can be written as,

$$U = U_*\Big[1 + U_1(y) + u_1(\eta) + \delta\{U_2(y) + u_2(\eta)\} + o(\delta) - \{\tfrac{1}{K}\ln\eta + C_{1\infty} + \delta(C_{2\infty} - \tfrac{1}{K\eta}) + o(\delta)\}\Big] \, . \qquad (7)$$

In eqn. (7) U_* is the friction velocity, K the von Karman constant and η is the inner stretched coordinate defined by $\eta = (1 \mp y)/\delta$ where $\delta = (1/U_* R)$. Functions U_1, U_2, u_1 and u_2 are defined in Afzal and Yajnik (1973).

2.3 Analysis at High Reynolds Numbers

Wave numbers of interest in the case of turbulent flow can be infered to be large from the results of Betchov and Criminale (1964) and Reynolds and Tiderman (1967). In the present study, the wave number α, is assumed to be of order ($U_* R$). To study the stability at high Reynolds numbers by the method of matched asymptotic expansions, an outer limit and an inner limit are considered.

2.3.1 Outer limit

Outer limit is defined as $R \to \infty$ for fixed y ; same as Rayleigh limit in the classical stability analysis. Outer limit is valid in the region away from the wall. Following outer expansions along with eqn. (7) for U are called for.

$$\phi(y;R) = \phi_o^{(o)}(y) + E_1(R)\,\phi_1^{(o)}(y) + \cdots \qquad (8)$$

$$T(y;R) = U_*^2[\Gamma_1(y) + e_2(R)\,\Gamma_2(y) + \cdots] \qquad (9)$$

E, e are gauge functions. With $\alpha = AU_* R$, when A is a constant of O(1) and $E_1 = (U_*/R)$, it follows,

$$\phi_o^{(o)} = 0 \quad \text{and} \quad \phi_1^{(o)} = \frac{i}{A}\Gamma_{xy1} \qquad (10)$$

E_1 is chosen to be ($U_* R$) in order that ϕ in the outer layer is non-trivial and is bounded. Since $\phi_o^{(o)} = 0$, exponentially small behaviour can be expected for $\phi(y;R)$. Letting $\phi_o(y;R) = \exp(\alpha g)$, where $g = g_o + o(1)$ and substituting in eqn.(1) gives,

$$g_o = \pm y + B_o \qquad (11)$$

and hence

$$\phi(y;R) = e^{\alpha[(\pm y + B_o) + o(1)]} \qquad (12)$$

Since ϕ has to vanish to first order as $R \to \infty$, the solution to be considered is with the negative sign.

2.3.2 Inner limit

Following inner variables have been introduced to consider the inner limit defined as $R \to \infty$, for fixed ξ. $\xi = (1-y)/\Delta$, $\phi_i = \phi/\pi$, $u_i = U/U_*$, $C = U(y_c,R)/U_*$ and $T_i = T/U_*$. y_c denotes the location of the critical layer. Employing the inner variables in eqn.(1) and after straight forward algebra leads to the following considerations. If (Δ/δ) is of order unity, all the terms of homogeneous Orr-Sommerfeld equation are of unity and the non-homogeneous Orr-Sommerfeld equation can be written as,

$$(u_i - C_i)(\phi_i'' - A^2\phi_i) - u''\phi_i + (i/A)(\phi''''-2A^2\phi_i''+A^4\phi_i)$$
$$= -(1/\pi R)\,[(T'_{xxi} - T'_{yy}) - (i/A)(T''_{xyi} + A^2 T_{xyi})] \, . \qquad (13)$$

Eqn.(13) clearly brings out the role of Reynolds stresses. If the disturbance is of order unity, then π is of order unity and all the terms due to Reynolds stresses are of order (1/R). However, there can be a case when π is of order (1/R). In such a situation the Reynolds stress terms are of equal importance to lowest order. In the analysis to follow, the former case will be studied. Considering the following inner equations,

$$\phi_i(\eta;R) = \phi_o(\eta) + \delta\phi_1(\eta) + U_*\delta\phi_2(\eta) + \cdots \qquad (14)$$

$$u_i(\eta;R) = u_1(\eta) + \delta u_2(\eta) + o(\delta) \qquad (15)$$

$$C(\eta_c;R) = u_1(\eta_c) + \delta u_2(\eta_c) + o(\delta) \qquad (16)$$

$$T_i(\eta;R) = T_{i1}(\eta) + o(1) \, , \qquad (17)$$

and substituting in eqn. (13), equations for various orders are as follwing.

$$\phi_o'''' - [2A^2 + iA(u_1 - u_1^c)]\,\phi_o'' + [A^4 + iA^3(u_1 - u_1^c) + iAu_1'']\,\phi_o = 0. \qquad (18)$$

$$\phi_1'''' - [2A^2 + iA(u_1 - u^c)]\,\phi_1'' + [A^4 + iA^3(u_1 - u_1^c) + iAu_1'']\phi_1$$
$$= iA[-\{A^2(u_2 - u_2^c) + u_2''\}\phi_o + (u_2 - u_2^c)\,\phi_o''] \qquad (19)$$

$$\phi_2'''' - [2A^2 + iA(u_1 - u_1^c)]\,\phi_2'' + [A^4 + iA^3(u_1 - u_1^c) + iAu_1'']\,\phi_2$$
$$= iA[T'_{xxi1} - T'_{yyi1} - \{T''_{xyi1} + A^2 T_{xyi1}\}] \qquad (20)$$

u_1^c and u_2^c are the values of u_1 and u_2 at $\eta = \eta_c$ the critical layer location. The boundary conditions at the upper wall transform to,

$$\phi_o = \phi_o' = 0; \quad \phi_1 = \phi_1' = 0 \quad \text{at} \ \eta = 0 \qquad (21)$$

2.3.3 Asymptotic solutions of the inner equations

Asymptotic solutions for eqn.(18) for large values of η can be readily obtained by considering ϕ_o to be of the forms $\phi_o \sim \exp(m\eta)$ and $\phi_o \sim \exp(\int g d\eta)$. Each of these forms gives two solutions and the four fundamental solutions to eqn.(18) for large η are,

$$\phi_{o1} \sim \exp(a\eta), \quad \phi_{o2} \quad \exp(-A\eta) \tag{22}$$

$$\phi_{o3} \sim \exp(-\lambda\eta\sqrt{\ln\eta}), \; \phi_{o4} = \exp(\lambda\eta\sqrt{\ln\eta}) \tag{23}$$

where

$$\lambda = \sqrt{iA/K} \tag{24}$$

Though eqn.(19) is non-homogeneous, the solution to ϕ_1 will be exactly of the same form as given by eqns.(22) and (23) for large values of η, since r.h.s of eqn.(19) vanishes faster.

2.3.4 Philosophy of eigen value determination

Matching of inner and outer solutions leads to the consideration of ϕ_{o2} and ϕ_{o3} only for the general solution to ϕ_o. Homogeneous boundary conditions require the vanishing of coefficient determinant which can be formally written as,

$$f(\eta_c, A) = 0. \tag{25}$$

Thus the zeroth order solution relates A(i.e., the wave number for a given $U_* R$) and η_c (the location of the critical layer, i.e., essentially C_r for a marginally stable disturbance). Fig.2 shows the variation of η_c with $A(= \alpha\delta)$, obtained by numerical intrigation of eqn.(18).

To introduce Reynolds number dependence, higher order effects as incorporated in eqn.(19) are to be considered. Numerical results for this part of the investigation will be reported later.

3. SUFFICIENT CONDITION FOR A STABLE FLOW

By incorporating u_1^c, the velocity at the critical layer to be in general complex ($= u_{1r}^c + iu_{1i}^c$) in eqn. (18), and examining the sign of u_{1i}^c will determine the stability of the flow. The flow is stable

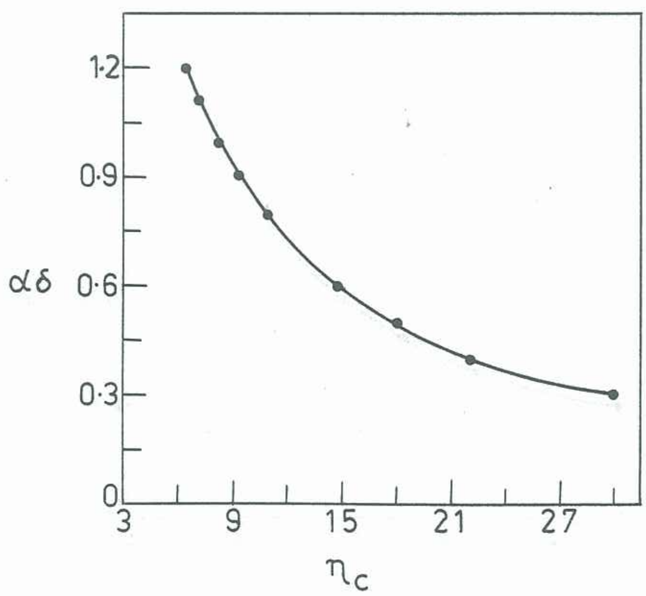

Fig. 2 $\alpha\delta$ Vs η_c

if $u_{1i}^c < 0$. Consider the first order inner equation, eqn.(18) with $u_1^c = u_{1r}^c + iu_{1i}^c$ and its adjoint equation as given below.

$$\phi_o'''' - [2A^2 + iA(u_1 - u_1^c)]\,\phi_o'' + [A^4 + iA^3(u_1 - u_1^c) + i\,A\,u_1'']\,\phi_o = 0 \tag{26}$$

$$\phi_o^{*''''} - [2A^2 - iA(u_1 - u_1^{c*})]\,\phi_o^{*''} + [A^4 - iA^3(u_1 - u_1^{c*}) - iAu_1'']\cdot\phi_o^* = 0 \tag{27}$$

Quantities with a * denote the complex conjugates. Multiplying eqn.(26) by ϕ_o^* and eqn. (27) by ϕ_o and integrating between 0 to ∞, on adding the resulting equations gives,

$$2[I_2 + 2A^2 I^2 + A^4 I_o^2] = iA(Q - Q^*) - 2Au_{1i}^c(I_1^2 + A^2 I_o^2). \tag{28}$$

In eqn. (28),

$$Q - Q^* = \int_0^\infty u_1'(\phi_o' \phi_o^* - \phi_o^* \phi_1)d\eta, \tag{29}$$

$$I_2^2 = \int_0^\infty |\phi_o''|^2 d\eta, \quad I_1^2 = \int_0^\infty |\phi_o'|^2 d\eta \tag{30}$$

$$I_o^2 = \int_0^\infty |\phi_o|^2 d\eta \tag{31}$$

Making use of Schwavtz's inequality,

$$|Q - Q^*| \leq 2\int_0^\infty |u_1'|\,|\phi_o'|\,|\phi_o|\, d\eta \leq q\, I_o\, I_1 \tag{32}$$

where

$$q = \text{Max}\, |u_1'|\,. \tag{33}$$

Using eqn. (32), eqn. (28) can be written as,

$$Au_{1i}^c(I_1^2 + A^2 I_o^2) \leq A\,q\,I_o\,I_1 - (I_2^2 + 2A^2 I_1^2 + A^4 I_o^2) \tag{34}$$

To find out when $u_{1i}^c < 0$ (i.e., the flow is always stable), if L and M are any two arbitrary real constants, it can be readily shown that,

$$\int(\phi_o + L\,\phi_o' + M\phi_o'')(\phi_o^* + L\phi_o^{*1} + M\,\phi_o^{*''})d\eta > 0. \tag{35}$$

which implies,

$$M^2 I_2^2 + (L^2 - 2M)\, I_1^2 + I_o^2 > 0\,. \tag{36}$$

Hence from eqn. (34),

$$M^2 A\, u_{1i}^c (T_1^2 + A^2 I_o^2) < M^2 q\, A\, I_o\, I_1 - I_1^2(2A^2 M^2 - L^2 + 2M) - I_o^2(A^4 M^2 - 1). \tag{37}$$

If r.h.o of eqn. (37) is negative, $u_{1i}^c < 0$ and r.h.s of eqn. (37) will be negative, if

$$(M^2\, q\, A)^2 < 4(2A^2 M^2 - L^2 + 2M)(A^4 M^2 - 1). \tag{38}$$

when

$$(2A^2M^2 - L^2 + 2M) > 0 \quad \text{and} \quad (A^4 M^2 - 1) > 0 \tag{39}$$

or when both $(2A^2 M^2 - L^2 + 2M)$ and $(A^4 M^2 - 1)$ are negative. Eqn. (38) can be written in a convenient form as

$$1 < 4(2A^2 M^2 - L^2 + 2M)(A^4 M^2 - 1)/\,M^4 A^2 \tag{40}$$

It may be noted that in writing eqn. (40) $q = \max |u_1'|$

is replaced by unity, since for the present problem

Max $|u_1'|$ is unity at $\eta = 0$.

To proceed further, the quantity on r.h.s. of eqn.(40) is evaluated for different values of A, fixing L and M. For a given value of L, M, the minimum value of A at which the r.h.s. of eqn.(40) is greater than unity, gives that, if the value of A is greater than this minimum value, the flow is stable. The same procedure is repeated for different values of L and M. A plot of (L, M) versus A, satisfying eqn. (40) is shown in Fig.3. Fig.3 shows an extremum behaviour and the minimum values of A is 0.7. Thus the stability is ensured if A ($= \alpha\delta$, the non-dimensional wave number) is greater than 0.7.

4. DISCUSSION

Examination of stability of fully developed turbulent flow based on non-homogeneous Orr-Sommerfeld equation brings out that, the non-homogeneity is due to variation in Reynolds stresses arising due to fluctuations in the velocity field and the applied disturbance. In order to resolve the controversy as to the role played by Reynolds stresses, it is essential to consider the higher order effects in the turbulent mean velocity profile. From this consideration, when the applied disturbance is of the order of unity, the Reynolds stresses do not contribute to first two orders. In the second order, non-homogeneity arises, but due to higher order effects in the velocity profile. However, if the applied disturbance is of order R, Reynolds stresses are important to the lowest order. For large wave numbers, the inner limit leads to an equation, which contains all the terms of Orr-Sommerfeld equation, unlike the laminar stability problem, where, in the viscous region, the governing equation (Tollmien equation) though is of fourth order, does not contain all the terms of Orr-Sommerfeld equation.

Sufficient condition derived shows that the turbulent flow is stable if $\alpha\delta > 0.7$. In Fig.3 ($\alpha\delta$) vs.(L,M) is shown for descrete values of L,M. Several combinations of L,M have been tried and in no case, a value of ($\alpha\delta$) < 0.7 satisfying eqn. (40) could be found.

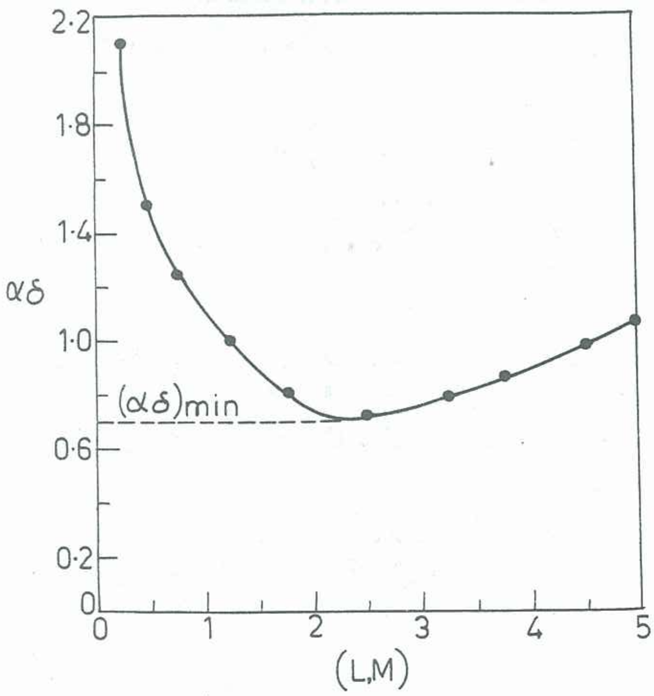

Fig.3 Sufficiency condition

5. ACKNOWLEDGEMENT

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