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MATHEMATICAL MODELLING OF LAG EFFECTS IN BED LOAD TRANSPORT

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SUMMARY A numerical model incorporating lags in bedload transport and able to predict bed levels and sediment transport rates under non-equilibrium sediment, steady and non-steady flow conditions is described. Local erosion rates are assumed proportional to the difference between actual and capacity sediment transport rates. A coefficient of proportionality is derived based on Einstein's bedload transport theories. Simulation of non-equilibrium bedload transport downstream of an abrupt transition from a fixed to an erodible bed is described. Account is taken of the separation zone immediately downstream of such a transition. Hence the maximum scour depth occurs someway downstream. Laboratory results are used to calibrate the spatial lag coefficient and the performance of the model is discussed.

1 INTRODUCTION

The sediment phase of an alluvial system cannot respond immediately to imposed changes in discharge. A certain time or distance is required before the sediment transport rate and bed form geometry can adjust to the new flow regime. This can be viewed as a lag effect.

Experimental evidence for the temporal lag associated with the imperfect adjustment of bed forms to changes in flow has been obtained by Simons, Richardson and Haushild (1962), Jensen (1973) and Griffiths (1976) for flood waves, while Gee (1975), Raichlen and Kennedy (1965) and Yalin (1975) used stepped changes in discharge.

Bell (1980) studied bed form changes induced by step changes in discharge and by the passage of flood waves as part of an investigation into non-equilibrium transport under these flow patterns. Comparisons between steady and non-steady flows showed the importance of bed form response to sediment transport rates, flow resistance and stage-discharge relations.

Mathematical models which do not recognise the lag effects inherent in situations where there is a significant lag between changes in flow conditions and changes in sediment transport cannot be reasonably applied to such conditions. A numerical model which incorporates lags and is able to predict sediment transport rates under non-equilibrium, steady and non-steady conditions has been assembled. The basis of this model is Wellington's (1978) model which accounts for lags in the suspended bed material transport but assumes that bed load transport reacts instantaneously to changes in local flow conditions. The present model incorporates lags in the bed load transport phase but does not treat suspended load transport.

2 NUMERICAL MODEL

A one-dimensional, unsteady, water and sediment routing model (UWASER) has been assembled. Such a model requires four basic equations. The first two describe the fluid flow; namely the continuity and momentum equations. One of the two remaining equations must be an equation representing sediment continuity and, in this model, the fourth equation characterises lag effects in bed load transport. These equations, their associated boundary conditions and the methods used to solve them are detailed below.

2.1 Flow Routing

The one-dimensional equations of continuity and momentum for gradually varied, unsteady flow in a non-prismatic channel are given by

$$B\frac{\partial z}{\partial t} + \frac{\partial Q}{\partial x} - q_\ell = 0 \qquad (1)$$

and

$$\frac{\partial Q}{\partial t} - \frac{QB}{A}\frac{\partial z}{\partial t} + \frac{\alpha Q}{A}\frac{\partial Q}{\partial x} - \frac{\alpha Q^2}{A^2}\left(B\frac{\partial z}{\partial x} + \left[\frac{\partial A}{\partial x}\right]_z\right) + gA\frac{\partial z}{\partial x} + \frac{Q^2}{2A}\frac{\partial \alpha}{\partial x} + gA\,S_f = \frac{-q_\ell Q}{A} \qquad (2)$$

where A = cross sectional area of flow;
B = channel width at surface;
g = gravitational acceleration;
Q = discharge;
q_ℓ = lateral inflow;
S_f = energy or friction slope;
x = longitudinal axis in flow direction;
z = stage above a horizontal datum; and
α = velocity distribution correction factor.

The term $[\partial A/\partial x]_z$ represents the departure from non-prismatic conditions, and is evaluated as the difference in flow area corresponding to the surface elevation midway between adjacent nodes.

Equations (1) and (2) are solved using an implicit finite difference scheme in conjunction with a double sweep algorithm. These equations were discretized using the modified Priessmann (SOGREAH) scheme as reported by Zoppou (1979). Two boundary conditions are also needed in order to solve these equations. The upstream boundary condition is assumed to be a discharge hydrograph, $Q = Q(t)$, and the downstream boundary condition to be a rating curve, $Q = f(z)$.

2.2 Sediment Routing

A sediment continuity equation for one-dimensional sediment routing in an alluvial channel is given by Wellington (1978) as

$$\frac{1}{u_s}\frac{\partial g_s}{\partial t} - \frac{g_s}{u_s^2}\frac{\partial u_s}{\partial t} + (1-m)\frac{\partial A_b}{\partial t} + \frac{\partial g_s}{\partial x} = 0 \qquad (3)$$

where A_b = area of deposition/scour of the bed, at a section;
g_s = sediment transport rate, (by volume);
u_s = mean sediment velocity; and
m = bed porosity.

If one assumes that the transport capacity of the flow is reached instantaneously at every point in time and space then the second sediment equation needed to form a determinate set of equations, would be

$$g_s = g_{se} = f(Q,\ S_f,\ \text{geometry}) \tag{4}$$

A typical relation for equilibrium sediment transport rate, g_{se}, obtained from Bell's (1980) data for transport of 2.11 mm grains in a rectangular flume on a slope of 0.002, is

$$\underset{(N/s/m)}{g_{se}} = 10.8\ \underset{(m/s)}{(U - U_c)^{2.57}} \tag{5}$$

where U = mean flow velocity, and
U_c = critical mean flow velocity.

However, as this equation does not recognise lag effects another form of equation needs to be adopted.

Einstein (1968) suggested that the local rate of erosion or deposition was proportional to the difference between the actual transport rate and the sediment transport capacity of the flow. The sediment transport capacity being evaluated from local flow parameters as though the flow were steady and uniform. Thus

$$(1-m)\ \frac{\partial A_b}{\partial t} = C\ (g_s - g_{se}) \tag{6}$$

Wellington (1978) analysed Einstein's (1950) formulation of a bedload transport equation and derived an equation for the spatial lag coefficient, C, under bedload conditions. This equation is

$$C = \frac{1 - P}{\lambda d} \tag{7}$$

where d = average grain diameter;
λ = Einstein's step length constant; and
p = Einstein's probability of erosion, given by

$$p = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{T} e^{-r^2/2}\ dr \tag{8}$$

and

$$T = 0.1925\ \psi - 2.75 \tag{9}$$

for sediment sizes greater than approximately 1.0 mm. Einstein assumed the step length constant, λ, to equal 100. Equations (3) and (6) were discretized using a modified Priessmann scheme and solved using Gaussian elimination. Two sediment boundary conditions are also required.

Downstream of a fixed bed, with no sediment input from upstream, a mobile bed is scoured giving a scour hole and flow field of the type, shown in Fig. 1, (Bell (1980), Dietz (1969)). Previous mathematical models have not accounted for the zone of separation encountered within the scour hole; consequently they predict that the local maximum depth occurs at the fixed to mobile bed interface.

In this model, account is taken of the zone of separation by using an upstream sediment boundary condition of the form

$$g_s = g_s(t) \tag{10}$$

where $g_s(t)$ is calculated from information on scour hole geometry. The boundary condition is applied at the downstream end of the zone of separation i.e. at the point of local maximum scour and it aims to maintain compatability between sediment transport rate and scour depth at this point. Since the point of local maximum scour moves downstream with time this boundary is mobile.

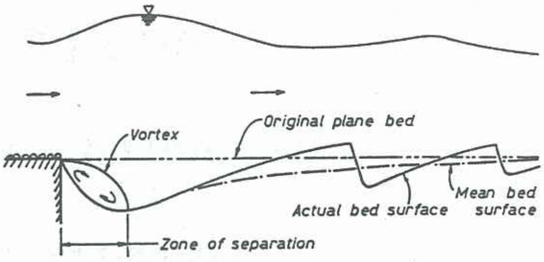

Figure 1 Flow Field in a Local Scour Hole Downstream of a Fixed Bed.

Breusers (1965) measured typical scour hole profiles in the zone of separation. These profiles, Fig. 2, can be approximated by

$$\frac{y}{H} = 1.5\ \left(\frac{x}{L}\right)^3 - 4.0\ \left(\frac{x}{L}\right)^2 + 3.5\ \left(\frac{x}{L}\right) \tag{11}$$

where L = distance from the fixed bed to the point of local maximum scour;
H = local maximum scour depth;
x = longitudinal axis in flow direction;
y = vertical axis (+) downwards.

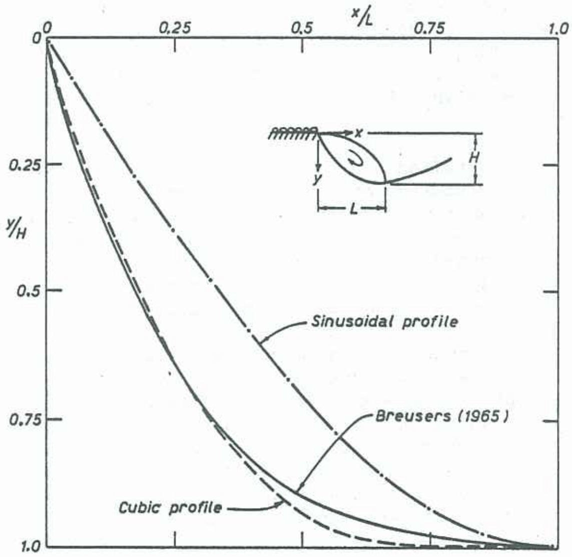

Figure 2 Scour Hole Profile in Zone of Separation.

Integration of Eq. 11 from $x = 0$ to $x = L$ gives a total scour volume of 0.791 HL/unit width.

Dietz (1969) presented experimental data which can be represented by

$$\frac{L}{H} = 65.6\ \delta^{-0.509};\ \delta < 150$$
$$= 5.1\ ;\ \delta \geq 150 \tag{12}$$

where

$$\delta = \frac{U_1 - U_{c1}}{w} \left[\frac{\nu^2}{(S_s - 1)gd^3}\right]^{1/3} \tag{13}$$

in which ν is the kinematic viscosity of the fluid, S_s is the specific gravity of the sediment, w is the sediment fall velocity and subscript 1 refers to upstream values at the fixed/mobile bed interface.

With reference to Fig. 3; during an increment of time Δt the point of maximum scour depth moves from A to B and from section 2 to section 2'. The aim of the boundary condition is to predict the location of section 2' under given flow conditions whilst maintaining compatability between the transport rate due to the scoured volume upstream of section 2' and the rate of scour, at section 2', given by Eq. 6. Thus, at the end of time increment Δt, the sediment transport rate at section 2', g_s, is assumed to be

$$g_s = \Delta V_b \ (1-m) \ b/\Delta t \tag{14}$$

$$\text{where} \quad \Delta V_b = 0.791 \ L'H' - 0.791 \ LH - H(L' - L); \tag{15}$$

b = bed width, and where L, H are known; L',H' are related by Eq. 12 and $H' = \Delta z_b + H$ (see Fig. 3). The calculated transport rate, g_s, from Eq. 14 is then substituted in the spatial lag equation (Eq. 6) and section 2' is located by an iterative procedure such that Eqs. 14, 12 and 6 are compatible at section 2'. Once section 2' is found, the boundary sediment transport rate, g_s, can be found from Eq. 14. This then is the boundary condition $g_s(t)$ given in Eq. 10.

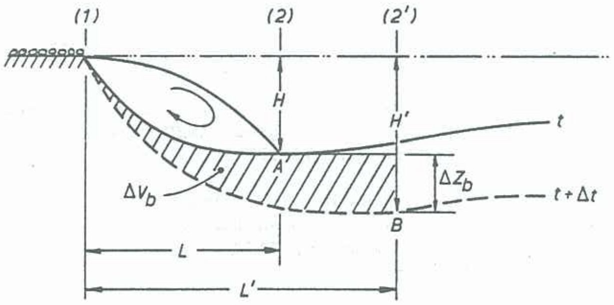

Figure 3 Definition Diagram for Upstream Sediment Boundary Condition.

The downstream boundary condition is assumed to be of the form of Eq. 6 applied at the downstream node.

3 MODEL PERFORMANCE

Data obtained by Bell (1980) was obtained for testing the numerical model and for investigating the spatial lag coefficient, C, defined in Eq. 7. Bell studied conditions downstream of a sudden transition from a rigid bed to an alluvial bed (Fig. 1). Sediment transport only occurred as bedload. In his ST series of runs he first established equilibrium transport conditions by injecting sediment at the appropriate rate at the upstream end of the alluvial reach. Following the instantaneous removal of this supply, details of the scour hole development and variations in sediment transport rate, both temporal and spatial, were recorded. A typical flow field and resultant bed shape are given in Fig. 1.

As some doubt surrounds the value of Einstein's step length constant, λ, the model was calibrated so that the local maximum (centre-line) scour depth matched experimental measurements, see Fig. 4. The best fit was obtained for $\lambda = 50$. However, before the model and experimental results were compared a correction, to account for transverse bed profile shape of the latter, was applied to the model results, H_m, to give the variation of the local maximum scour depth, with time, on the centre-line, $H_{℄}$. This correction is necessary because Bell's experiments were conducted in an 0.3048 m wide flume. Consequently, the side-walls influence the transverse bed profile giving a concave transverse bed profile rather than a uniform profile, as predicted by the model. This behaviour is particularly prominent within the scour hole. The following correction method was adopted

$$H_{℄} = H_m + \Delta H; \qquad t > 360 \text{ s}$$

$$H_{℄} = fH_m \quad ; \qquad t \leq 360 \text{ s} \tag{16}$$

where $\quad f = H_{℄}/H_m \quad$ at t = 360 s

and ΔH = a measured bed correction constant.

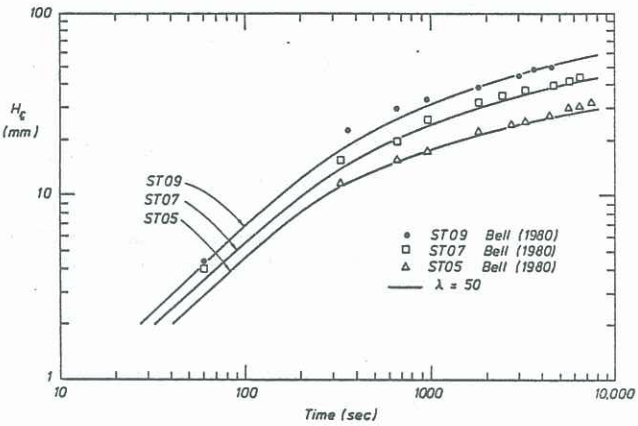

Figure 4 Local Maximum Centreline Scour Depths as a Function of Time.

After approximately six minutes, measurements of transverse bed profiles indicated that the bed correction was constant with time but that the correction, ΔH, varies with discharge. At times less than six minutes a multiplicative correction factor was considered more appropriate for modelling non-uniform transverse bed profiles. Measurements gave the following bed corrections for the ST runs considered in the calibration studies (Fig. 4).

$$\begin{aligned} &\text{Run ST09}, \quad q = 0.160 \ m^2/s, \quad \Delta H = 2.3 \text{ mm} \\ &\text{Run ST07}, \quad q = 0.127 \ m^2/s, \quad \Delta H = 2.9 \text{ mm} \\ &\text{Run ST05}, \quad q = 0.097 \ m^2/s, \quad \Delta H = 4.1 \text{ mm} \end{aligned} \tag{17}$$

where q is the water discharge/unit width. The corrected model results were then compared with Bell's results and a best fit λ value was found to be $\lambda = 50$. The model was then run and the temporal and spatial variations of sediment transport and bed elevation compared with Bell's ST09 results. These are given in Fig. 5 and Fig. 6.

The predicted sediment transport rates are in reasonable agreement with those measured by Bell. The lack of agreement at station 0.74 m may be due to the difficulty in measuring sediment transport rates deep within the scour hole. The model predicts well the onset of reduced sediment transport rate at stations 5.3 m and 9.3 m although the model results at station 9.3 m are noticeably higher than the experimental values.

The agreement between the predicted and measured centreline bed profiles, displayed in Fig. 6, is good. Once again, a correction to allow for transverse bed profiles within the scour hole was applied to the model results.

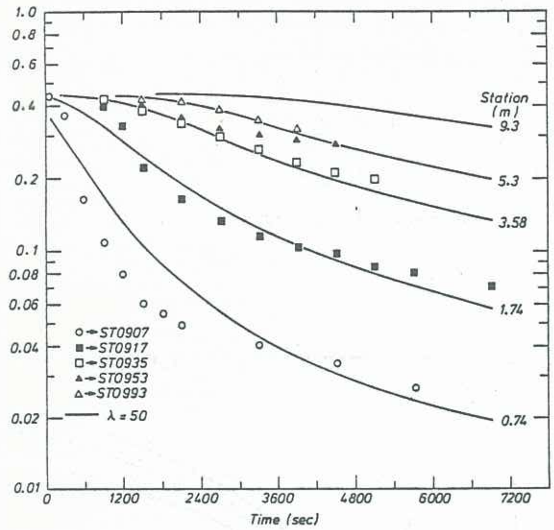

Figure 5 Sediment Transport Rate as a Function of Time at Different Sections. Data from Bell (1980) - Run ST09.

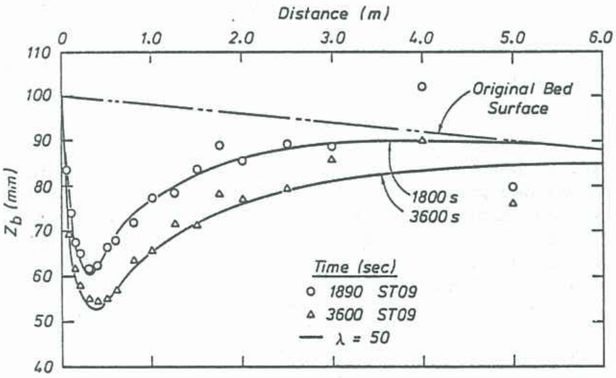

Figure 6 Variation of Longitudinal Centre-line Bed Profiles with Time. Data from Bell (1980).

4 CONCLUSIONS

A numerical model and its application to bedload transport downstream of a sudden transition from a rigid bed to an alluvial bed has been described. Two innovative features of this model are:

1. The upstream sediment boundary condition takes account of the zone of separation immediately downstream of the rigid bed. Consequently, the local maximum scour depth occurs someway downstream from the rigid/mobile bed interface; and

2. The model incorporates a spatial lag equation in the formulation of bedload transport. This equation assumes that the local erosion/deposition rates are proportional to the difference between actual and capacity transport rates. The assumed form of the coefficient of proportionality incorporates Einstein's step length parameter which was found to have a value of 50 to give the best fit between measured and calculated local maximum scour depths.

Further experimental studies are being conducted to test the validity of the spatial lag equation and to establish the validity of the model calibration.

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