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A STUDY OF DEGENERATE & NON-DEGENERATE CRITICAL POINTS

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An examination of critical points which occur away from "no-slip" boundaries shows that the inclusion of time dependence or higher order terms in the series expansion for velocity can change the whole character of the flow pattern and this extends right to the origin of the critical point. Without these effects present or included, the pattern is degenerate. However, degenerate critical points on a "no-slip" boundary remain degenerate even with the inclusion of higher order terms. This study is relevant to three-dimensional separation and eddying motions.

Introduction

A critical point in a flow field is a point where the instantaneous streamline slopes are indeterminate and the velocities are zero. Oswatitsch (1958) and Lighthill (1963) examined viscous flow close to a rigid boundary and classified certain types of critical points. Smith (1972) applied the theory of critical points to the study of conical flows. Perry & Fairlie (1974) applied it to the study of inviscid flow with slip at the boundaries and inviscid rotational flow far from boundaries. Hunt et al. (1978) extended the work of Perry & Fairlie and introduced the Poincare-Bendixson theorem. They made studies of flow around obstacles attached to surfaces. Cantwell et al. (1978) applied it to the geometry of turbulent spots. Perry, Lim & Chong (1980) and Perry & Watmuff (1981) used it to explain various three-dimensional eddying motions. Recently Tobak & Peake (1982) and Hornung & Perry (1982) have applied the theory to complex three-dimensional separation patterns which occur on missile shaped bodies at an angle of attack.

A knowledge of the classification and properties of all possible critical points which are asymptotically exact local solutions of the Navier-Stokes and continuity equations gives one a "topological language" or vocabulary for describing complex flow fields in an intelligible manner.

Free-slip critical points

Let use consider critical points far from "no-slip" boundaries and refer to these as "free-slip" critical points. Close to critical points the velocity field is expressed as a Taylor series expansion, the leading terms of which are

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (1)$$

or $\quad \underset{\sim}{U} = \underset{\approx}{F} \cdot \underset{\sim}{x}$

where $\underset{\sim}{U} = \dot{\underset{\sim}{x}} = \underset{\sim}{i}u + \underset{\sim}{j}v + \underset{\sim}{k}w$ is the velocity of fluid particles at the point $\underset{\sim}{x} = \underset{\sim}{i}x + \underset{\sim}{j}y + \underset{\sim}{k}z$.

The series expansion is substituted into the incompressible Navier Navier-Stokes and continuity equations and the coefficients must be interrelated in such a way so that the expansion is a solution. If the expansion is a solution, it must yield an expression for the pressure distribution. An arbitrary series expansion will not necessarily do this. A solution exists only if the coefficients of the expansion are related in a certain way. The elements of the matrix F in equation (1) can be expressed in terms of the local vorticity and second derivatives of the static pressure at the critical point.

If the eigenvalues of the matrix F are real then there will exists three real eigenvectors which define the three planes (the eigenvector planes). These planes contain locally a set of streamlines close to the critical point. If we denote one of the planes as the x_1x_2 plane then for x_1 and $x_2 \to 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (2)$$

or $\quad \dot{\underset{\sim}{x}}_{1,2} = \underset{\approx}{F}_{1,2} \cdot \underset{\sim}{x}_{1,2}$

The patterns in the x_1x_2 plane can then be classified in the same way as phase-plane portraits in the study of non-linear dynamical systems (for details, see Kaplan, 1958 for example). The classification is done with the aid of the p-q chart as shown in figure 1 where

$$p = -(a+d) = -\mathrm{tr}\,\underset{\approx}{F}_{1,2}$$
$$q = (ad-bc) = \det \underset{\approx}{F}_{1,2} \qquad (3)$$

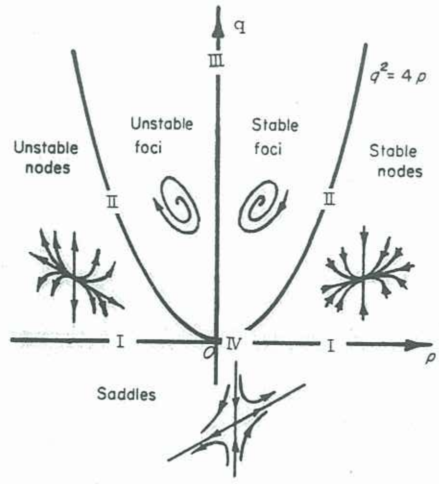
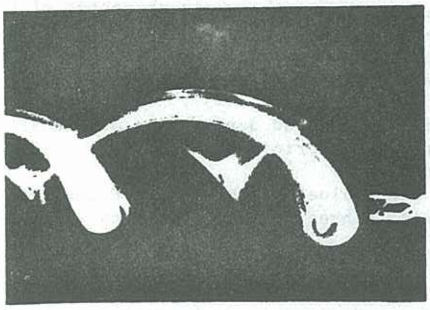


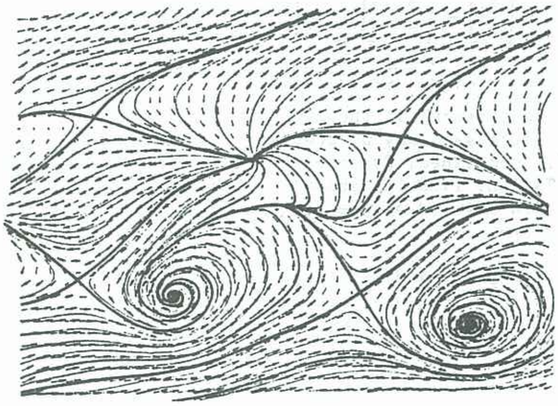
Figure 1. Classification of critical points.

One can see that this figure is divided up into certain zones, the boundaries of which are defined by the p-q axes and the parabola $p^2 = 4q$. There are regions of saddles, nodes and foci and each of the three eigenvector planes will have patterns corresponding to one of the zones given on the p-q chart. These then constitute three "views" of a pattern and one can get an insight into the three-dimensional properties of the pattern. If the eigenvalues of $\underset{\approx}{F}$ are complex, then we will obtain only one real eigenvector. There will then be only one plane containing streamlines and this pattern will be a focus.

Such patterns classified on the p-q chart abound in three-dimensional fluid flow in both steady and unsteady situations and an example is shown in figure 2. This is an experimentally determined instantaneous vector field of periodic eddying motions obtained in a coflowing wake as seen by an observer moving with the eddies. The streamlines were actually determined by integrating the velocity field which was firstly curve fitted in zones by Taylor series expansions of up to the 5th order.



(a)



(b)

Figure 2. Coflowing negatively buoyant wake pattern.
(a) Smoke pattern.
(b) Experimentally determined vector field. (Done with D.K.M. Tan, University of Melbourne).

There exists a certain class of critical points which lie on the axes of the p-q chart or on the parabola $p^2 = 4q$. These are "degenerate" critical points and a classification of these patterns are shown in figure 3.

If the vorticity at a critical points is finite, then using the truncated form given by equation (1), it is found that streamlines will be contained in a plane normal to the vorticity vector. The matrix $\underset{\approx}{F}_{x,y}$ say, for this plane will be such that we are confined completely to the q axis of the p-q chart as shown in figure 4.

The flows are always (locally) two-dimensional and if the eigenvalues of F are real there will exists other planes whose patterns are described by points on the p axis as given in figure 3, case I. Thus all critical points are degenerate and the patterns in figure 4 range from saddles through pure shear to solid body rotation.

If the flow is irrotational then the formulation (1) yields real eigenvalues for $\underset{\approx}{F}$ but all eigenvectors are orthogonal. Thus the patterns for all three eigenvector planes can lie anywhere shown shaded in figure 5(a).

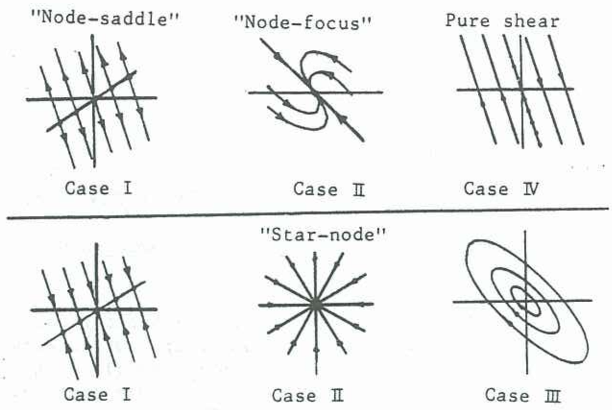


Figure 3. Degenerate or "border-line" cases on the "boundaries" of figure 1.

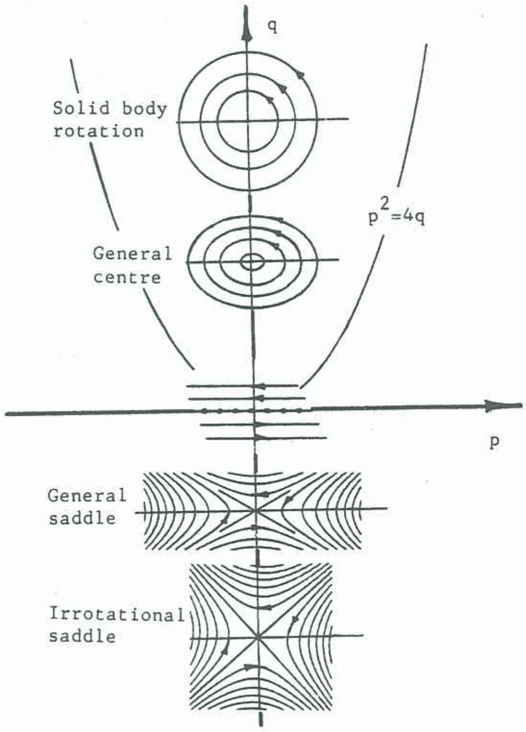


Figure 4. Degenerate critical points resulting from linear steady finite vorticity analysis.

These patterns range from axisymmetric stagnation point flow to plane stagnation point flow and the patterns in the xy plane range from the star node (1) through the general node (2) to the degenerate "node-saddle" (3), the locations of which are shown on the p-q chart.

In all of the cases treated so far, the viscous terms in the Navier-Stokes equation drop out and so these cases have been referred to as inviscid critical points (Perry & Fairlie). Centres corresponds to pressure minima and all saddle patterns correspond to pressure maxima.

The stretched vortex problem

An inspection of patterns obtained in practice (like those in figure 2) show that degenerate critical

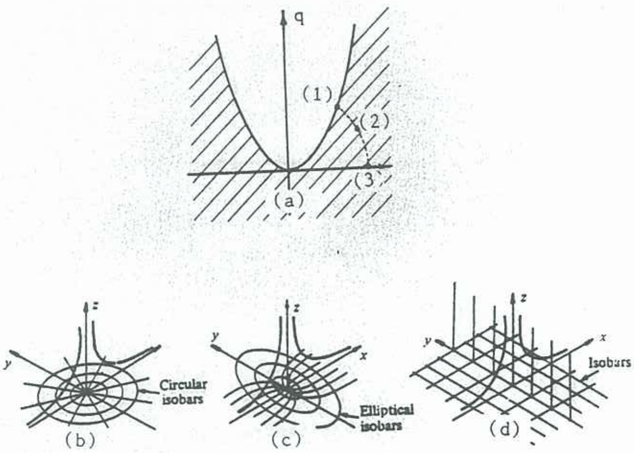


Figure 5. Critical points resulting from linear irrotational flow analysis.

points are very rare in practice. In particular, instead of centres we have foci. One is tempted to model such critical points as a combination of axisymmetrical stretching as shown in figure 5(b) with solid body rotation about the z axis as shown in figure 4. Each pattern is described by the form (1) and they separately constitute an exact solution of the Navier-Stokes and continuity equations. However, in combination, they are not solutions unless we introduce time dependence. In fact, an exact solution is

$$\underset{\sim}{U} = \begin{bmatrix} -a & -\omega & 0 \\ \omega & -a & 0 \\ 0 & 0 & 2a \end{bmatrix} \cdot \underset{\sim}{x} \qquad (4)$$

where $$\omega = (\zeta_o/2)\, e^{2at} \qquad (5)$$

where ζ_o is the vorticity aligned with the z axis at time t = 0, ω is the angular velocity of the solid body rotation component and a is a stretching rate constant. Since the flow is unsteady, we must think in terms of instantaneous streamlines and these are obtained by determining the integral curves for equation (3) for a fixed "frozen" or instantaneous value of ω. In the xy plane we obtain a non-degenerate focus.

The question now arises as to whether such a focus is possible in steady flow. If we model this as a straight vortex undergoing axisymmetrical stretching with a constant strain rate, the only way to obtain a steady focus is to include viscosity and this means extending the form (1) to higher order terms. The solution is

$$\underset{\sim}{U} = \begin{bmatrix} -a & -\zeta_o/2 & 0 \\ \zeta_o/2 & -a & 0 \\ 0 & 0 & 2a \end{bmatrix} \cdot \underset{\sim}{x} + \frac{\zeta_o a}{8\nu} \begin{bmatrix} x^2 y + y^3 \\ -(x^3 + y^2 x) \\ 0 \end{bmatrix} \qquad (6)$$

and $$\zeta = \zeta_o - (\zeta/2)(a/\nu)(x^2 + y^2)$$

where ζ_o is the steady vorticity at the critical point and the local vorticity varies quadratically with radius from the z axis. This solution is asymptotically exact for small radius and represents the case where viscous diffusion and vortex stretching effects are in balance. A complete exact similarity solution for this case is given in Batchelor (1967) where the vorticity is a Gaussian function of radius.

Thus a steady focus is possible but elements of the matrix for the first order terms could not be determined without accounting for the higher order terms. This is because gradients in the viscous stress terms are as important as the inertia and pressure gradient terms.

Thus it is seen that whenever we obtain a degenerate solution or "border line" case, this signals the need to look into the possible effects of either unsteadiness or higher order terms.

No-slip critical points

Consider flows at a flat rigid boundary where the no-slip condition applies. Let z be the distance normal to the boundary. For such flows, the leading terms in a Taylor series expansion are

$$\begin{bmatrix} u/z \\ v/z \\ w/z \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (8)$$

or $\underset{\sim}{U}/z = \underset{\approx}{F} \cdot \underset{\sim}{x}$ or $\dot{\underset{\sim}{x}}/z = \underset{\approx}{F} \cdot \underset{\sim}{x}$

Note that we must use the variable $\underset{\sim}{U}/z$ so as to satisfy the no-slip condition, i.e. $\underset{\sim}{U} = 0$ for $z = 0$ for all x and y.

Substituting equation (8) into the Navier-Stokes and continuity equations, the elements of $\underset{\approx}{F}$ can be determined.

These elements can be expressed in terms of the first derivative of vorticity and pressure (see Perry & Fairlie). The author has recently extended the Taylor series expansion to the third order and found that the degenerate case of two-dimensional separation is unaffected by the inclusion of higher order terms. The pattern (although modified away from the critical point) remains degenerate and the elements in the matrix $\underset{\approx}{F}$ in equation (8) are unaltered.

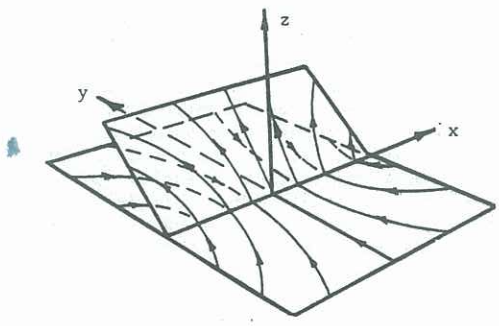


Figure 6. Plot of equation (2).
A degenerate no-slip critical point with higher order terms included. (After Hornung, 1983).

A simple example of this third order solution for $\underset{\sim}{U}/z$ is

$$\begin{aligned} u &= -Exyz^2 \\ v &= B_2 yz + B_3 z^2 + Ex^2 zy \\ w &= -(B_2/2)\, z^2 - (E/2)x^2 z^2 + (E/2)y^2 z^2 \end{aligned} \qquad (9)$$

This solution was used by Hornung (1983) in his study of surface bifurcation lines. Figure 6 shows the solution he plotted for the case of E = 1.0, B = -2.0 and $B_3 = 0$. Space does not permit a more general form of equation (9) to be presented here.

Conclusions and discussion

In the case of free-slip critical points, if the vorticity is finite, then to the linearized approximation, it must be uniform in space. In the absence of vorticity gradients there is no diffusion and so the viscous terms in the Navier-Stokes equation drop out and there are no viscous stress gradients. This is the situation if form (1) is used.

Furthermore, if vorticity is finite and uniform, the solution is always degenerate and is confined to the q axis of the p-q chart. This means that the flow is two-dimensional. This is a consequence of the fact that the solution has been assumed to be steady. If the flow were three-dimensional, we would have vortex stretching and in the absence of diffusion this would lead to unsteady flow. However, the instantaneous streamlines for this unsteady flow are not confined to degenerate cases and further examples show that solutions can occur anywhere on the p-q chart.

In the case of irrotational flow, steady solutions are permitted and are also non-degenerate. However, the eigenvectors are always real and orthogonal. The presence of vorticity is manifested by non-orthogonal vectors or else by the presence of foci.

This now leads us to the very important question of steady three-dimensional vortical flow. It would seem that it cannot exists near a critical point but practical experience shows this to be unlikely. When a solution is degenerate, it is often a signal that high order terms should be included in the analysis and that these higer order terms are somehow going to affect the lower order terms in such a way as to remove the degeneracy. However, this is not always true. For instance, if equation (1) were to be substituted into the Euler equations and the continuity equation, any higher order terms would have no effect on the coefficients of equation (1). This would imply that such an inviscid analysis would lead to the pattern shown in figure 7. Far from the region of vorticity, trajectories are allowed to spiral in (as would be given by three-dimensional potential vortex analysis) but these must, according to the Euler equations asymptote to centres (closed trajectories) since the critical points must be locally two-dimensional. This means that the axial velocity w must have zero gradient with respect to z as shown in figure 7. This is the only steady critical point with complex eigenvalues permitted by the Euler equations. In order to depart from such a pattern (i.e. to move off the q axis of the p-q chart) and still have steady flow, we have to include viscous effects and these enter only if we introduce third order terms. This changes the whole character of the patterns including the first order terms and this leads to the more likely pattern shown in figure 8.

If the pattern of figure 7 were to be a steady solution to the Navier-Stokes and continuity equations, this would imply that viscous effects enter the problem only beyond a certain radius. This would require the lowest order nonlinear terms to be higher than third order. This could well be possible but appears to be too special to serve as a generally applicable solution.

To summarize, steady three-dimensional flow in regions of finite vorticity cannot exist according to the Euler equations, but can exist according to the Navier-Stokes equation and so viscosity must always play an important part in steady free-slip vortical flow theory.

The question now naturally arises with the no-slip critical points at a surface. Is it necessary to include higher order terms close to a degenerate critical point? It can be seen from equations derived that the case of simple two-dimensional separation (which is degenerate) is not affected by the higher order terms and close to the critical point, the pattern remains degenerate.

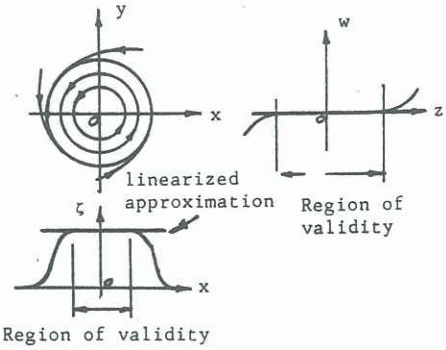


Figure 7. Degenerate centre.

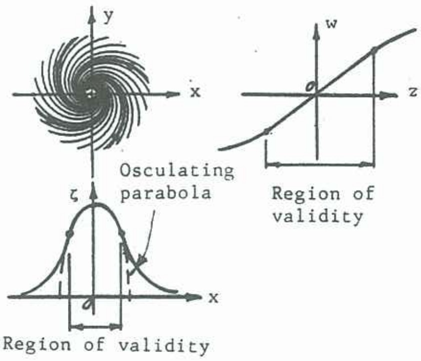


Figure 8. Non-degenerate focus.

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