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THE STABILITY OF TWO-DIMENSIONAL LINEAR FLOW

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SUMMARY The paper reports a theoretical investigation of the stability of a viscous incompressible fluid undergoing two-dimensional flow in which the velocity field is a linear function of position. Such flows are approximately generated by the four-roll mill device of G.I. Taylor and are described by a parameter λ ranging from $\lambda = 0$ for simple shear flow to $\lambda = 1$ for pure extensional flow. An instability criterion relating the initial disturbance wave vector $\underset{\sim}{\alpha}$ to the steady flow strain rate E, kinematic viscosity ν and the parameter λ is obtained showing that for all admissible values of E and ν, a wave vector $\underset{\sim}{\alpha}$ may be found which corresponds to disturbances which grow exponentially in time.

INTRODUCTION

A linear stability analysis is the usual first step taken to investigate the stability of a laminar flow. In this approach only infinitesimal disturbances to the basic flow are studied which considerably simplifies the mathematical analysis. Further simplification of the mathematics is often provided by assuming that the flow is unbounded in one or more directions in space. The immediate objective of the analysis is to obtain a criterion involving the relevant physical parameters which determines whether the solutions of the linearized disturbance equations grow with time or ultimately decay to zero. There will be in general a discrepancy between the linear stability theory and the experimental observations of instability in real physical systems due to nonlinear effects and the presence of boundaries. The linear stability analysis can however contribute to an understanding of the physical processes which are associated with the instability of the base flow.

We consider here a linear stability analysis for an unbounded incompressible Newtonian fluid which is undergoing a two-dimensional lineal flow. The analysis applies to a certain class of two-dimensional lineal flows ranging from simple shear to pure extensional flow. These flows can be produced approximately in the regions between the rollers of two- and four-roll mills of G.I. Taylor (1934). These devices consist of two or four rollers which are positioned at the corners of a square and immersed in a tank of fluid. One primary reason for the interest in the flows generated by the two- and four-roll mills is that they can be regarded as models for the complicated calendering and extrusion processes of polymeric liquids. They are also being increasingly used for the study of drop formation of macromolecules in solution and floc stability in flow.

In spite of the importance of these flows we find no systematic stability investigation for them, except in the limiting cases of simple shear and elongational flows.

The stability of simple shear flows between two parallel plates (plane Couette flow) has undergone thorough investigations. Hopf (1914) carried out an asymptotic analysis of the Orr-Sommerfield equation. Gallagher and Mercer (1962) obtained numerical solutions and Reid (1979) supplied an exact solution for the same equation. All of these studies point out that the plane Couette flow is stable to all infinitesimal disturbances at all Reynolds numbers (of the base flow).

The other limiting case of pure extensional flow has been considered by Pearson (1959) in a paper concerned with the behaviour of weak homogeneous turbulence subjected to a uniform distortion. He found that the total energy associated with the turbulence increases without limit when the mean flow is an unbounded pure extensional flow. It may then be inferred that unbounded pure extensional flow is unconditionally unstable to infinitesimal disturbances.

In this paper we present a straightforward linear stability analysis for the general class of unbounded two-dimensional lineal flows ranging from simple shear to pure extensional flows. Our purpose is to investigate the effects of flow types and flow strength on stability in order to connect the results obtained for the limiting cases. Criteria for determining the stability of the base flow with respect to spatially periodic disturbance are obtained. A qualitative interpretation of the physical mechanism for instability is also given.

THE GOVERNING EQUATIONS AND FORMAL SOLUTION

The velocity field $\underset{\sim}{U}$ for a steady linear flow is of the form $\underset{\sim}{U} = \underset{\sim}{\Gamma}\underset{\sim}{x}$, where $\underset{\sim}{\Gamma}$ is a second-order tensor which is independent of the position vector $\underset{\sim}{x}$. We shall consider these two-dimensional linear flows for which the components of $\underset{\sim}{\Gamma}$ referred to a Cartesian system are given by

$$[\underset{\sim}{\Gamma}] = \frac{E}{2}\begin{pmatrix} 1+\lambda & 1-\lambda & 0 \\ -1+\lambda & -1-\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (1)$$

where E is the magnitude of the local velocity gradient (the flow strength), and λ specifies the type of flow as it varies between ± 1. The cases $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$ correspond to, respectively, pure rotational, simple shearing and pure extensional flow.

The principal strain rates and axes for the base flow are determined by the eigenvalues and eigenvectors of the strain rate tensor $\underset{\sim}{\Gamma} + \underset{\sim}{\Gamma}^T$. Here we find that the principal axes of strain coincide with x_1-, x_2- and x_3-axis and the associated principal strain rates are $E(1+\lambda)$, $-E(1+\lambda)$ and 0, respectively. The base vorticity field $\underset{\sim}{\Omega} = \nabla x \underset{\sim}{U}$ has components $\{0, 0, -E(1-\lambda)\}$ so that the ratio of the magnitude of the vorticity to the magnitude of the strain rate is given by $(1-\lambda)/(1+\lambda)$. This ratio varies monotonically from ∞ to 0 as λ ranges from -1 to +1.

Denoting the velocity field and the pressure field by $\underset{\sim}{U}(\underset{\sim}{x}) + u(\underset{\sim}{x},t)$ and $P(\underset{\sim}{x}) + p(\underset{\sim}{x},t)$, respectively, we have, from the Navier-Stokes equations and neglecting body forces and non-linear terms,

$$\nabla\cdot\underset{\sim}{U} = 0$$
$$\underset{\sim}{U}\cdot\nabla\underset{\sim}{U} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\underset{\sim}{U} \tag{2}$$

and

$$\nabla\cdot\underset{\sim}{u} = 0$$
$$\partial_t\underset{\sim}{u} + \underset{\sim}{U}\cdot\nabla\underset{\sim}{u} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underset{\sim}{u} \tag{3}$$

where ρ is the density and ν is the kinematic viscosity of the fluid. The linearized disturbance equations (3) are subjected to the initial condition

$$\underset{\sim}{u}(\underset{\sim}{x},0) = \underset{\sim}{u}^o(\underset{\sim}{x}) \ . \tag{4}$$

The vector field $\underset{\sim}{u}^o(\underset{\sim}{x})$ specifies an arbitrary initial disturbance and is assumed solenoidal and spatially bounded for physical reasons.

Most linear stability analyses begin with the assumption that the disturbance velocity $\underset{\sim}{u}(\underset{\sim}{x},t)$ is of the form $e^{\sigma t}$ function of $\underset{\sim}{x}$. Direct substitution of this into (3) yields an eigenvalue problem for σ. The flow is judged stable or unstable depending upon whether the real part of σ is negative or positive, respectively. This usual separation of variables approach will fail to work for the case of the general two-dimensional flow given by (1), however. To discover what functional form $\underset{\sim}{u}(\underset{\sim}{x},t)$ must take we Fourier transform (3,4) to obtain

$$\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}} = 0 \ ,$$
$$\partial_t\hat{\underset{\sim}{u}} - \left(\underset{\sim}{\Gamma}^T\underset{\sim}{k}\right)\cdot\partial_{\underset{\sim}{k}}\hat{\underset{\sim}{u}} = -\underset{\sim}{\Gamma}\hat{\underset{\sim}{u}} + \frac{i}{\rho}\hat{p}\underset{\sim}{k} - \nu k^2\hat{\underset{\sim}{u}} \ , \tag{5}$$

and

$$\hat{\underset{\sim}{u}}(\underset{\sim}{k},0) = \hat{\underset{\sim}{u}}^o(\underset{\sim}{k}) \ . \tag{6}$$

Here, the circumflex denotes a Fourier transform, e.g.

$$\hat{\underset{\sim}{u}}(\underset{\sim}{k},t) = \int e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\,\underset{\sim}{u}(\underset{\sim}{x},t)\,d\underset{\sim}{x} \ ,$$

$\partial_{\underset{\sim}{k}}$ is the gradient with respect to $\underset{\sim}{k}$, $k^2 = \underset{\sim}{k}\cdot\underset{\sim}{k}$ and the superscript T denotes a transpose.

The pressure $\hat{p}$ can be eliminated from (5) and one has

$$\partial_t\hat{\underset{\sim}{u}} - \left(\underset{\sim}{\Gamma}^T\underset{\sim}{k}\right)\cdot\partial_{\underset{\sim}{k}}\hat{\underset{\sim}{u}} = -\underset{\sim}{\Gamma}\hat{\underset{\sim}{u}} + \frac{2}{k^2}\left(\underset{\sim}{k}\cdot\underset{\sim}{\Gamma}\underset{\sim}{u}\right)\underset{\sim}{k} - \nu k^2\hat{\underset{\sim}{u}} \ . \tag{7}$$

Equations (7) and (6) can be solved by the method of characteristics. Along a characteristics one has

$$\underset{\sim}{k}(t) = e^{-t\underset{\sim}{\Gamma}^T}\underset{\sim}{\alpha} \ ,$$

where $\underset{\sim}{\alpha}$ is a constant vector and $\hat{\underset{\sim}{u}}\left(\underset{\sim}{k}(t),t\right)$ takes the value $\tilde{\hat{\underset{\sim}{u}}}'(\underset{\sim}{\alpha},t)$ which is governed by

$$\partial_t\hat{\underset{\sim}{u}}' = -\underset{\sim}{\Gamma}\hat{\underset{\sim}{u}}' + \underset{\sim}{B}(t)\,\hat{\underset{\sim}{u}}' - \nu k^2(t)\,\hat{\underset{\sim}{u}}' \tag{8}$$

Here $\hat{\underset{\sim}{u}}'$ takes on the initial value (6) and

$$\underset{\sim}{B} = \frac{2}{k^2}\underset{\sim}{k}\underset{\sim}{k}\cdot\underset{\sim}{\Gamma} \ .$$

Thus, if $\underset{\sim}{\Phi}(\underset{\sim}{\alpha},t)$ is the fundamental matrix solution of (8) and (6), then the disturbance velocity is given by

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \frac{1}{(2\pi)^3}\int \exp\left(-ie^{-t\underset{\sim}{\Gamma}^T}\underset{\sim}{\alpha}\cdot x\right)\underset{\sim}{\Phi}(\underset{\sim}{\alpha},t)\,\hat{\underset{\sim}{u}}^o(\underset{\sim}{\alpha})\,d\underset{\sim}{\alpha} \tag{9}$$

FUNDAMENTAL MODES

In the fundamental mode analysis the initial disturbance is taken to be

$$\underset{\sim}{u}^o(\underset{\sim}{x}) = e^{i\underset{\sim}{\alpha}\cdot x}\,\underset{\sim}{v}^o \ ,$$

where $\underset{\sim}{v}^o$ is a constant vector and $\underset{\sim}{\alpha} = \{\alpha_1, \alpha_2, \alpha_3\}$ is the wave vector.

It is clear from (9) that the stability of the basis flow is determined by the asymptotic nature of $\underset{\sim}{\Phi}(\underset{\sim}{\alpha},t)\,\underset{\sim}{v}^o$, which for a fixed value of $\underset{\sim}{\alpha}$ is the solution of the linear system

$$\frac{d}{dt}\underset{\sim}{y} = -\underset{\sim}{\Gamma}\underset{\sim}{y} + \underset{\sim}{B}(t)\underset{\sim}{y} - \nu k^2(t)\underset{\sim}{y} \ ,$$
$$\underset{\sim}{y}(0) = \underset{\sim}{v}^o \ . \tag{10}$$

By taking the inner product of (10) and $\underset{\sim}{y}$ one can show with the aid of Cauchy-Schwarz inequality that

$$\frac{d}{dt}\|\underset{\sim}{y}\| \leq \left(3\|\underset{\sim}{\Gamma}\| - \nu k^2(t)\right)\|\underset{\sim}{y}\| \tag{11}$$

where $\|\underset{\sim}{y}\| = \sqrt{\underset{\sim}{y}\cdot\underset{\sim}{y}}$ and $\|\underset{\sim}{\Gamma}\|$ is the spectral norm of $\underset{\sim}{\Gamma}$

$$\|\underset{\sim}{\Gamma}\| = \sup\left\{\frac{\|\underset{\sim}{\Gamma}\underset{\sim}{x}\|}{\|\underset{\sim}{x}\|} \ , \ \underset{\sim}{x}\in\mathbb{R}^3 \ , \ \|\underset{\sim}{x}\| \neq 0\right\} \ .$$

Since $\|\underset{\sim}{\Gamma}\| = E$ in this class of two-dimensional flow, one has immediately

$$\|\underset{\sim}{y}(t)\| \leq \|\underset{\sim}{v}^o\|\exp\left\{3Et - \nu\int_o^t k^2(\tau)\,d\tau\right\} \tag{12}$$

$\lambda = 0$

For the case of a simple shear flow where $\lambda = 0$, k^2 can be calculated and we conclude that $\|\underset{\sim}{y}\| \to 0$ as $t \to \infty$ if either of the following conditions are satisfied

$$\alpha_1 \neq 0$$

or

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2^2 + \alpha_3^2 > \frac{3E}{\nu} \ . \tag{13}$$

If $\alpha_1 = 0$ and $\alpha_2^2 + \alpha_3^2 < \frac{3E}{\nu}$ then the right hand side of (12) is unbounded; but whether or not $\|\underset{\sim}{y}\|$ is unbounded cannot be determined.

$0 < \lambda \leq 1$

In this case and after some algebra it can be shown that $\|\underset{\sim}{y}\| \to 0$ as $t \to \infty$ under either of the following conditions

$$\left(-1 + \sqrt{\lambda}\right)\alpha_1 + \left(1 + \sqrt{\lambda}\right)\alpha_2 \neq 0$$

or

$$\left(-1 + \sqrt{\lambda}\right)\alpha_1 + \left(1 + \sqrt{\lambda}\right)\alpha_2 = 0 \quad \text{and} \quad \alpha_3 > \sqrt{\frac{3E}{\nu}} \ . \tag{14}$$

The asymptotic behaviour of $\underset{\sim}{y}(t)$ cannot be determined if $(-1 + \sqrt{\lambda})\alpha_1 + (1 + \sqrt{\lambda})\alpha_2 = 0$ and $\alpha_3 < \sqrt{\frac{3E}{\nu}}$.

To complete the study the two "indeterminate" cases must be considered. To this end we define z(t) by

$$\underset{\sim}{y}(t) = \exp\left\{-\nu\int_0^t k^2(\tau)\,d\tau\right\} z(t) .$$

Then z(t) satisfies

$$\frac{d\underset{\sim}{z}}{dt} = -\underset{\sim}{\Gamma}\underset{\sim}{z} + \underset{\sim}{B}(t)\,\underset{\sim}{z} \tag{15}$$

$\lambda = 0, \quad \alpha_1 = 0$

In the first "indeterminate" case it is found that $\underset{\sim}{k}(t) = \{0, \alpha_2, \alpha_3\}$ and $\underset{\sim}{B}(t) = 0$. Thus

$$\underset{\sim}{z}(t) = e^{-t\underset{\sim}{\Gamma}}\,\underset{\sim}{v}^o$$

and hence

$$\underset{\sim}{y}(t) = e^{-\nu(\alpha_2^2 + \alpha_3^2)t}\begin{pmatrix}1 & -Et & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\underset{\sim}{v}^o . \tag{16}$$

If $\alpha_2^2 + \alpha_3^2 \neq 0$ the solution decays exponentially to zero as $t \to \infty$. On the other hand, if $\underset{\sim}{\alpha} \equiv 0$, the "rigid motion" initial disturbance simply convects with the fluid and grows linearly in time just as an element of fluid in the base flow would.

$0 < \lambda \leqslant 1, \quad (-1 + \sqrt{\lambda})\alpha_1 + (1 + \sqrt{\lambda})\alpha_2 = 0$

In this case an analytic solution to (15) is not possible. However asymptotic behaviour of $\underset{\sim}{z}(t)$ can be determined with the aid of a theorem by Levinson (1948). From this theorem one finds that

$$\|\underset{\sim}{y}(t)\| \sim O\left\{\exp\left(E\sqrt{\lambda} - \nu\alpha_3^2\right)t\right\} \tag{17}$$

Thus, in this case $\|\underset{\sim}{y}\| \to 0$ as $t \to \infty$ if and only if $\alpha_3 > \left(\frac{E\sqrt{\lambda}}{\nu}\right)^{\frac{1}{2}}$.

To sum up one can draw the following conclusions:

In the case of simple shear flow ($\lambda = 0$) all spatially periodic initial disturbances with wave vectors $\underset{\sim}{\alpha} \neq 0$ decay to zero at large time.

For "strong" flows where $0 < \lambda \leqslant 1$ all spatially periodic initial disturbances for which $(-1 + \sqrt{\lambda})\alpha_1 + (1 + \sqrt{\lambda})\alpha_2 \neq 0$ decay to zero at large time. A disturbance with $(-1 + \sqrt{\lambda})\alpha_1 + (1 + \sqrt{\lambda})\alpha_2 = 0$ will decay to zero if $\alpha_3 > \left(\frac{E\sqrt{\lambda}}{\nu}\right)^{\frac{1}{2}}$ and grows exponentially in time if $\alpha_3 < \left(\frac{E\sqrt{\lambda}}{\nu}\right)^{\frac{1}{2}}$.

INTERPRETATION OF INSTABILITY

The physical mechanism for instability can be described by appealing to the case where $\lambda = 1$ (pure extensional flow) in which an exact analytical solution is possible. It is then inferred that the vortex line stretching along the principal axis of extensional strain provides the source of instability. This exponential growth of vorticity due to vortex line stretching cannot overcome the stabilizing effect of convection enhanced diffusion if an initial periodic disturbance is not invariant in the direction of the inlet streamline: Any periodicity along the inlet streamline leads to a faster than exponential rate of decay due to the viscous diffusion terms. If an initial periodic disturbance does not have any periodicity along the inlet streamline, the stabilizing effect of diffusion along the inlet streamline is absent and the disturbance vorticity grows as $\exp(E\sqrt{\lambda}t)$ which can exceed diffusion decay of $\exp(-\nu\alpha_3^2 t)$ if $\alpha_3 < (E\sqrt{\lambda}/\nu)^{\frac{1}{2}}$. For simple shear flow ($\lambda = 0$) the vorticity disturbances can only grow linearly with time as a result of the weaker vortex line stretching. Viscous diffusion of disturbance vorticity still grows at an exponential rate for an initial periodic disturbance. Thus simple shear flow is stable with respect to all disturbances (provided that the wave vector $\underset{\sim}{\alpha}$ is not identically zero).

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