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COMPUTER GENERATED FLOW-NETS

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SUMMARY Flow-nets offer a compact, efficient and widely accepted form of presentation of the results from seepage and groundwater analyses. Since finite element solutions are generally obtained for h, values of ψ must also be computed before the flow-net can be produced. Two methods for computing ψ are given and techniques for obtaining the necessary boundary conditions are discussed. Examples of computer generated flow-nets are presented.

Notation

h	piezometric head
k	hydraulic conductivity
n,s	directions normal and tangent to a boundary
q	two-dimensional discharge
r	equivalent nodal flow
T	transmissivity
v	Darcy seepage velocity
x,y	Cartesian coordinates
ψ	stream function

1 INTRODUCTION

The finite element method is frequently used for the analysis of seepage and groundwater flows for which the governing differential equation is

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right) = 0 \qquad (1)$$

Subject to boundary conditions of h or $\partial h/\partial n$ known for all boundary segments.

The finite element formulation leads to a set of simultaneous linear equations

$$[C]\{h\} = \{r\} \qquad (2)$$

where $\{r\}$ contains the 'equivalent nodal flows'. Values for $\{r\}$ are known for nodes along boundary segments where $\partial h/\partial n$ is specified.

After solution of the equations, the Darcy velocity components

$$v_x = -k_x \frac{\partial h}{\partial x}, \qquad v_y = -k_y \frac{\partial h}{\partial y} \qquad (3)$$

may be computed element by element, and any unknown 'equivalent nodal flows' evaluated.

Equations (1) and (3) are written for two-dimensional steady seepage but also apply to two-dimensional steady groundwater flows (without recharge) if k is replaced by T and v by q.

The finite element analysis results in large amounts of tabulated data which are difficult to check and assimilate. Computerized plotting of the h contours is often used to give a compact presentation of the results. If facilities are available for a plot of h contours, there are major advantages and small extra costs in plotting the flow-net.

2 ADVANTAGE OF FLOW-NETS

Reasons for using flow-nets include presentation, communication and vertification. The flow-net is a concise, visually attractive form of presentation. If facilities for printing text, labelling contours, drawing borders, etc were included in the plotting routines, the figures produced would be suitable for immediate inclusion in a report or paper. Because practicing engineers are familiar with flow-nets, they provide an excellent means of communication between the analyst and the user. Furthermore, the user can subject the flow-net to simple, independent checks for correctness (see Cedergren, 1967) and experienced engineers can often detect anomalies by visual examination of the flow-net. All computed results should be subjected to independent checks as a safeguard against data errors. A single data error may remain undetected amongst the massive amount of numerical data but its presence is usually obvious when the flow-net is produced (see Figure 1).

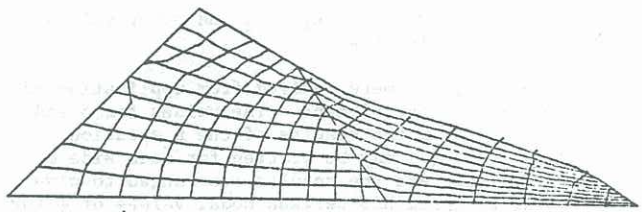

Figure 1 Error detection by flow-net

3 METHODS FOR COMPUTING

3.1 Method 1

Since

$$\frac{1}{k_x}\frac{\partial \psi}{\partial y} = -\frac{\partial h}{\partial x} \text{ and } \frac{1}{k_y}\frac{\partial \psi}{\partial x} = \frac{\partial h}{\partial y} \qquad (4)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{k_y}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{k_x}\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 h}{\partial x\,\partial y} - \frac{\partial^2 h}{\partial x\,\partial y} = 0 \qquad (5)$$

This equation for ψ has the same form as that used for h (Equation (1)) with k_x replaced by $1/k_y$ and k_y by $1/k_x$. Therefore the solution for ψ can be obtained with the same program used for h provided ψ or $\partial\psi/\partial n$ can be specified on all segments of the boundary. The boundary values can be obtained from the solution for h (see below). This method has been used by Christian (1980).

3.2 Method 2

This method is novel and has some advantages over Method 1 for the user. It is based on the application of Equation (4) to the velocity components calculated from the h solution. Details are presented for the 6-node triangular element (see Figure 2) but the method can be applied to other elements commonly used in groundwater and seepage analyses.

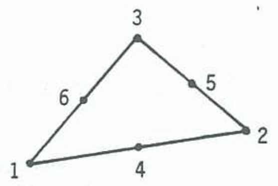

Figure 2 6-node Triangular Element

For this element the velocity components vary linearly and can be expressed uniquely in terms of their values at the apex nodes. Define δ and γ by

$$\delta_1 = \psi_3 - \psi_2 = \frac{-1}{2}\left\{\left(v_{x3} + v_{x2}\right) b_1 + \left(v_{y3} + v_{y2}\right) c_1\right\} \qquad (6)$$

$$\gamma_1 = 2\psi_5 - \psi_2 - \psi_3 = \frac{1}{4}\left\{\left(v_{x3} - v_{x2}\right) b_1 + \left(v_{y3} - v_{y2}\right) c_1\right\} \qquad (7)$$

where $b_1 = y_2 - y_3$, $c_1 = x_3 - x_2$ and other values are obtained by cyclic order.

Equations (6), (7) were derived from application of Equation (3) to the element. The values for δ and γ are computed from the results of the h solution. Equations (6), (7) may be written for each side of the element in turn and the results rearranged to give the following relationship between nodal values of ψ for the element.

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} = \begin{Bmatrix} \delta_1 - \delta_3 \\ \delta_3 - \delta_1 \\ \delta_1 - \delta_2 \\ \gamma_3 \\ \gamma_1 \\ \gamma_2 \end{Bmatrix} \qquad (8)$$

A global set of equations relating all nodal values of ψ may be assembled from the contributions of all elements. After the equations have been modified for known values of ψ they can be solved to obtain all unknown nodal values.

The coefficient matrix in Equation (8) shows that the apex nodal values are independent of the nodal values at the mid-side nodes and, if only the equations for the apex nodal values are assembled, the global coefficient matrix will be symmetric. Therefore, it is computationally efficient to solve initially for the apex nodal values and then to compute the mid-side nodal values.

An important advantage of this method is that ψ need be specified at only one apex node for a solution to be obtained. The solution is improved when more nodal values of ψ are specified. Values along ψ = constant boundaries can be easily found (see below) and are sufficient for an accurate solution. The user does not have to evaluate ψ or $\partial\psi/\partial n$ along the remaining boundary segments.

After solutions for h and ψ have been obtained, the flow-net is produced by plotting contours of h and ψ. A "square" flow-net will be produced for an homogeneous isotropic media if the contour intervals, $\Delta\psi$ and Δh, satisfy $\Delta\psi = k\Delta h$.

4 BOUNDARY CONDITIONS FOR STREAM FUNCTION

The solution for h is used to determine boundary values of ψ or $\partial\psi/\partial n$. In general, it is easier to calculate and specify boundary values for ψ and only this boundary condition is considered here.

If ψ is known at node i on the boundary, it can be evaluated theoretically at any other node j by integration of the normal velocity along the boundary.

$$\psi_j - \psi_i = \int_i^j v_n \, ds = \int_i^j v_x \, dy - \int_i^j v_y \, dx \qquad (9)$$

However, since the velocity components are obtained by numerical differentiation they are only approximate and their use in Equation (9) leads to approximate values for ψ which may not be sufficiently accurate. If k and j are nodes on ψ = constant boundaries, a more accurate result is obtained from the relationship.

$$\psi_j - \psi_i = \sum_i^j \text{"equivalent nodal flows"} \qquad (10)$$

because the "equivalent nodal flows" are evaluated directly from the equations for h and satisfy overall continuity. Equation (10) can be used to establish accurate values for ψ along ψ = constant boundaries and these are sufficient boundary values for Method 2. If Method 1 is used, ψ must be specified along the other boundary segments also. Equation (10) cannot be used directly for intermediate nodes on these segments because the "equivalent nodal flow" at a node receives contributions from the element segments on each side of the node. A set of satisfactory values for ψ at these intermediate nodes is obtained if values initially calculated from Equation (9) are scaled to satisfy overall continuity as expressed by Equation (10).

5 EXAMPLES

The first example (Figure 3) is the flow net for seepage through a two-zone, anisotropic earth dam. $k_x = 4$ and $k_y = 1$ in zone 1 and $k_x = 16$, $k_y = 4$ in zone 2. For an isotropic zone, a square flow-net is produced if contour intervals, $\Delta\psi$ and Δh, satisfy $\Delta\psi = k\Delta h$. $\Delta\psi$ and Δh were chosen so that with

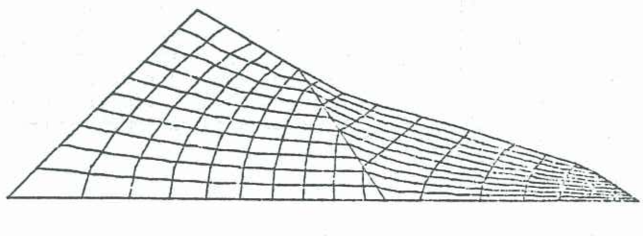

Figure 3 Flow-net for two-zone drain (natural scale)

$k = (k_x k_y)^{1/2}$ a square flow-net is produced in zone 1 on the transformed x', y' plane where x' = x, y' = 2y (Figure 4). It is worth noting that both methods for calculating ψ can handle correctly the transfer conditions across the interface between the two zones.

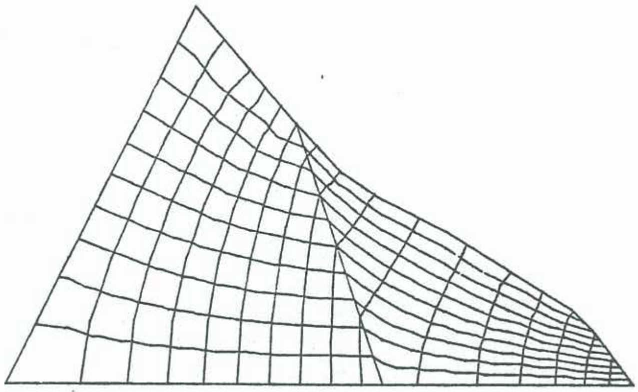

Figure 4 Flow-net for two-zone dam (transformed plane)

The second example (Figure 5) is a flow-net for groundwater flow through a non-homogeneous aquifer and shows the effect of variations in transmissivity on the flow-net pattern. In both examples, ψ was computed by Method 2 with boundary values specified only for nodes along ψ = constant boundaries.

6 CONTOUR PLOTTING PROGRAM

For most practical applications it is not necessary that the plotting program should reproduce the contours exactly as described by the interpolation polynomials appropriate to the chosen finite element. Satisfactory contours can be produced if each element is replaced by a number of sub-elements and linear interpolation (which is easily programmed) is used over each sub-element. The analyses for this paper used the 6-node triangular element (Figure 2) and the contours were plotted using linear interpolation over the four triangular sub-elements produced by joining the mid-side nodes. The same program is used to plot h and ψ contours.

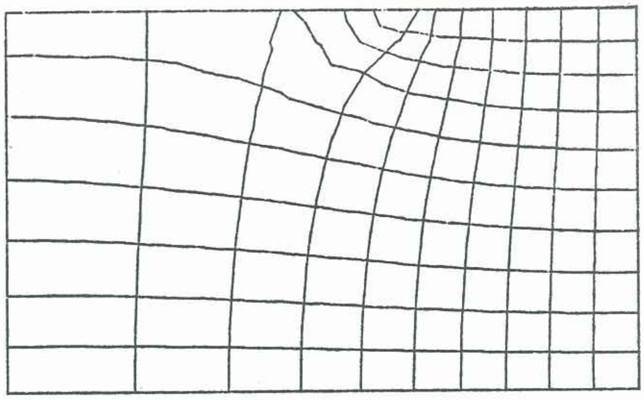

Figure 5 Flow-net for groundwater flow in non-homogeneous medium

7 CONCLUSION

This paper argues that flow-nets have an important role in this age of computer based analyses. The flow-net requires that ψ as well as h be evaluated and two methods for computing ψ, following the analysis for h have been presented. The first has the advantage that the program used for h can be used for ψ while the second has the advantage of simpler boundary conditions. From a user's point of view, Method 2 is certainly the better of the two.

8 REFERENCES

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