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# KINEMATIC WAVE PARAMETERS FOR PARABOLIC STREAM CHANNELS

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SUMMARY: The application of kinematic wave theory to calculate the behaviour of unsteady storm rainfall-runoff hydraulics, first postulated in the mid-fifties (Ref 4) and considerably developed in the mid-sixties (Refs 3,7), is now widely accepted (Refs 1,2,5,6). Many recently developed computer programs, some with moisture accounting facilities for pervious areas, and some without, take advantage of the simplifying algorithm proposed by Alley et al. (Ref 1) which requires that every channel section must be represented by a simple power function relating the flow rate to the sectional area. Accurate power functions (which implicitly relate wetted perimeter to sectional area) have already been developed for overland flows, and for certain triangular cross-sections such as curb-side channels. Useful approximate relationships have also been developed for (shallow) rectangular channels and circular pipes. This paper presents the derivation of a similar power function for parabolic channels. The flow rate is shown to be proportional to the 13/9 power of the sectional area, provided that the central flow depth does not greatly exceed the height of the focal point above the vertex. Many natural stream channels are likely to be conveniently modelled by a parabolic shape without violating this depth restriction.

## 1. INTRODUCTION TO KINEMATIC WAVE THEORY

Most storm runoff events involve fairly slow rates of spatial and temporal change in momentum at all locations except drop structures, spillways, etc.. When the various acceleration terms in the one-dimensional St Venant momentum equation (Ref 7) are all very small in relation to the terms involving stream bed slope and friction slope, conditions may be described as quasi-steady, and these two slope terms must then roughly balance one another. This balance may be represented by a simple friction relationship such as the familiar Manning equation:

$$v = (1/n)*r^{2/3}*S^{1/2} \qquad (1)$$

where v is the mean sectional velocity, r is the hydraulic radius, and S is the stream bed slope.

Kinematic wave theory then couples this approximation with the unsteady continuity equation:

$$dA/dt = q - dQ/dx \qquad (2)$$

where A and Q are the sectional area and flowrate, q is the tributary inflow per unit length, x, and the differential terms are both partial derivatives. These equations may be integrated to estimate the behaviour of slowly changing flood waves, provided that the hydraulic gradient does not become so flat that backwater effects could be important.

Woolhiser & Liggett (Ref 7) have shown that, for overland flow down an inclined plane of length X and slope S, the kinematic wave approximation approaches the exact solution (to the St Venant equations) as the following dimensionless parameter k approaches infinity:

$$k = S*X/(v^2/g) \qquad (3)$$

where g is the gravitational acceleration, and v is the velocity at the downstream end of the plane segment. [Note that Eqn (3) is the writer's rearrangement of the original definition of k which involved the Froude number, rather than the velocity.] Woolhiser & Liggett showed that kinematic wave theory gives a good approximation to reality as long as k exceeds about 10.

If Eqn (3) is viewed in reciprocal form, a useful interpretation emerges:

$$(1/k) = (v^2/2*g)/(S*X/2) \qquad (4)$$

The right side here expresses the ratio of the kinetic energy of the downstream flow to the mean potential energy (above the exit level) of the flow on the plane. As k approaches infinity, this ratio approaches zero. When this is generalised to include flows in channel segments as well as overland flows, it means that the kinematic wave approximation is likely to be reliable as long as the kinetic energy at the downstream end of each segment is always a reasonably small fraction (say, less than 10%) of the mean potential energy just lost in traversing that segment.

Alley, Dawdy & Schaake (Ref 1) have shown that a very convenient algorithm for computer integration of Eqn (2) may be implemented if Eqn (1) can be rearranged in the form:

$$Q = e*A^f \qquad (5)$$

where e and f correspond to ALPHA and BETA in the notation of Ref 1.

Multiplying Eqn 1 by A, and inserting r=A/W, gives:

$$Q = (S^{1/2}/n)*A^{5/3}*W^{-2/3} \qquad (6)$$

If a power function relates A and W of the form:

$$W = c*A^d \qquad (7)$$

then substitution of Eqn (7) in Eqn (6), and comparison with Eqn (5), yields:

$$e = (S^{1/2}/n)*c^{-2/3} \qquad (8)$$

and $$f = 5/3 - d*2/3 \qquad (9)$$

Power functions like Eqn (7) have already been developed for overland flow, and for triangular channels (Ref 1). Approximate power functions can be worked out for wide rectangular sections, and even for circular pipes. However, no such power function to represent a parabolic section has yet been published.

2. PERIMETER-AREA RELATIONSHIP FOR PARABOLIC CHANNEL

In order to apply the general methodology of Alley et al. to the particular case of a parabolic channel, a power function is sought of the form of Eqn (7).

A representative equation for the shape of a parabolic channel section with its axis of symmetry vertical (see Fig 1) is:

$$h = y^2/4b \qquad (10)$$

where h is the height above the channel invert level, y is the semi-width at height h, and b is the parabola's focal height (i.e. h = b when y = 2b).

It is well known from basic mathematics that the area on the concave side of a parabolic segment (one end at the vertex) is two-thirds of the area of the circumscribing rectangle, i.e.

$$A = y^3/3b \qquad (11)$$

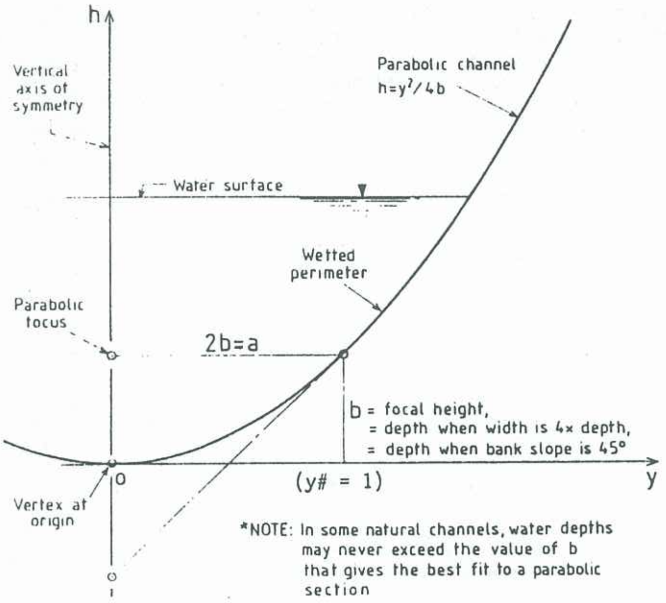


FIGURE 1. Parabolic Channel Shape Definition Sketch

## 2.1 Derivation of Arc-Length of Parabola

The arc length, L, of a parabola is derived by integration thus:

$$L = I[dy^2 + dh^2]^{0.5}$$

$$= I[1 + (dh/dy)^2]^{0.5}*dy \qquad (12)$$

where I represents the mathematical integration symbol.

Differentiating Eqn (10) with respect to y yields:

$$dh/dy = y/2b \qquad (13)$$

Inserting this in Eqn (12) gives:

$$L = I[1 + y^2/4b^2]^{0.5}*dy$$

$$= (1/2b)*I[4b^2 + y^2]^{0.5}*dy \qquad (14)$$

A standard table of integral functions gives:

$$I[a^2 + y^2]^{0.5}*dy$$

$$= 0.5*y*(a^2 + y^2)^{0.5}$$

$$+ 0.5*a^2*\sinh^{-1}(y/a) \qquad (15)$$

where a = 2b, in this case. Converting the inverse trigonometric term to its logarithmic equivalent involves:

$$\sinh^{-1}(y\#) = \ln[y\# + z\#] \qquad (16)$$

where $$z\# = [1+y\#^2]^{0.5} \qquad (17)$$

and where y# = y/a, i.e. y# is the dimensionless ratio of flow semi-width to focal semi-width.

The dimensional version of Eqn (17) is:

$$z = [a^2+y^2]^{0.5} \qquad (18)$$

Inserting this and Eqn (16) in Eqn (15) yields:

$$I(z)*dy = 0.5*y*z + 0.5*a^2*\ln[y\#+z\#] \qquad (19)$$

The parabolic arc length [Eqn (14)] now becomes:

$$L = (1/a)*[0.5*y*z + 0.5*a^2*\ln(y\#+z\#)] \qquad (20)$$

## 2.2 Conversion of Arc-Length into Wetted Perimeter

The wetted perimeter (i.e. twice the arc length) is:

$$W = y*z/a + a*\ln[y\# + z\#] \qquad (21)$$

This may be rendered entirely dimensionless by dividing by a:

$$W\# = W/a = y\#*z\# + \ln[y\# + z\#] \qquad (22)$$

As a numerical check, for a case where b = 0.5 m (a = 1 m), if the depth is h = 0.5 m (y = 1 m):

$y\#=1$, $z=2^{0.5}=1.41$m, $z\#=1.41$,

$y\#+z\# = 2.41$, $\ln(2.41)=0.88$,

$W\# = W/a = 1.41+0.88 = 2.29$.

Note that if $y<<a$, $z \simeq a$, $y\#+z\# \simeq y\#+1$,

$\ln(1+y\#) \simeq y\#$ [because $y\# << 1$],

thus $W\# \simeq 2*y\#$, (see Eqn (30) below).

## 2.3. Relationship between Area and Wetted Perimeter

Now it is required to eliminate y between Eqs (11) and (22) to yield a power function like Eqn (7). The dimensionless version of Eqn (11) is:

$$A\# = A/a^2 = y^3/1.5*a^3 = y\#^3/1.5 \qquad (23)$$

This may be rearranged to yield y# explicitly in terms of area:

$$y\# = [1.5*A\#]^{(1/3)} = 1.145*A\#^{(1/3)} \qquad (24)$$

When this is inserted in Eqn (17), the result is:

$$z\# = [1 + 1.310*A\#^{(2/3)}]^{0.5} \qquad (25)$$

Combining Eqs (16), (22), (24) and (25) yields:

$$W\# = k1*A\#^t*[1+k2*A\#^{2*t}]^{0.5}$$

$$+ \sinh^{-1}[k1*A\#^t] \qquad (26)$$

where k1=1.45, k2=1.310 and t=0.333.

## 2.4 Power Function Approximation by Series Expansion

A power function like Eqn (7) is now sought via series expansions of the binomial and inverse trigonometric expressions in Eqn (26).

Remembering that $y\# = k1*A\#^{t}$,

and that $y\#^{2} = k2*A\#^{2*t}$

[from Eqs (17) and (25)], the following series expansions are relevant:

(a) Binomial Series

$$[1 + y\#^{2}]^{0.5} = 1 + (1/2)*y\#^{2}$$

$$- (1/8)*y\#^{4} + (1/16)*y\#^{6} - \ldots \quad (27)$$

(b) Inverse Trigonometric Series

$$\sinh^{-1}[y\#] = y\# - (1/6)*y\#^{3}$$

$$+ (3/40)*y\#^{5} - (5/112)*y\#^{7} + \ldots \quad (28)$$

Each of these series is valid only if y# (being essentially positive) does not exceed unity. This means that both series become invalid if the water surface in a parabolic channel rises above the level of the focus (i.e. if the surface width becomes less than 4 times the central depth).

Inserting Eqs (27) and (28) in Eqn (26) yields:

$$W\# = y\# + (1/2)*y\#^{3}$$

$$- (1/8)*y\#^{5} + (1/16)*y\#^{7} - \ldots$$

$$+ y\# - (1/6)y\#^{3}$$

$$+ (3/40)*y\#^{5} - (5/112)*y\#^{7} + \ldots$$

which, after accounting for partial cancellation of high order coefficients, simplifies to:

$$W\# = 2*y\# + (1/3)*y\#^{3}$$

$$- (1/20)*y\#^{5} + (1/56)*y\#^{7} - \ldots \quad (29)$$

If y#=0.5, the second term involving $y\#^{3}$ contributes only 4% to this series, while the third subtracts less than 0.2%. For values of y# < 0.5, only the first term is significant:

$$W\# \simeq 2*y\# = 2*1.145*A\#^{(1/3)}$$

$$= 2.29*A\#^{(1/3)} \quad (30)$$

Remembering that W#=W/a, $A\#=A/a^{2}$, and a=2b, this gives:

$$W = 2.29*a^{(1/3)}*A^{(1/3)}$$

$$= 2.89*b^{(1/3)}*A^{(1/3)} \quad (31)$$

For all practical purposes, since the neglected terms sum to a small positive increment, the coefficients in Eqn (31) may be taken as 2.3 and 2.9 respectively.

Incorporating the simplification suggested above, Eqn (31) represents an approximate power function relating the wetted perimeter of a parabolic channel to the flow section area. The exponent of this power function is d = 1/3.

The coefficient is $c = 2.9*b^{(1/3)}$, where b is the parabola's focal depth.

## 2.5 Verification of Approximate Power Function

Eqs (22), (23) and (25) may be used to calculate a range of numerical values for A# and W# to check the validity of the approximate power function represented by Eqn (31). Normally, one would expect values of y# to lie in a range between about 0.1 and about 1. Table 1 sets out calculations of A# and W# over this range, and compares the exact values of W# from Eqn (22) with the aproximations from modified Eqn (31).

It may be concluded from examination of the last two columns of Table 1 that it is only when y# exceeds about 0.6 (i.e. when the stream width becomes less than about 7 times the central depth) that the approximation given by Eqn (31) becomes quite unsatisfactory.

TABLE 1 : Calculations of dimensionless areas and wetted perimeters for parabolic sections

| y# | A# Eqn(23) | z# (25) | y#*z# | [y#+z#] | ln[..] | Exact W# (22) | Aprox W# [**] |
|---|---|---|---|---|---|---|---|
| 0.05 | .000083 | 1.0013 | 0.0501 | 1.0513 | 0.0500 | 0.100 | 0.100 |
| 0.1 | .00066 | 1.005 | 0.101 | 1.105 | 0.0999 | 0.200 | 0.201 |
| 0.2 | .00533 | 1.020 | 0.204 | 1.220 | 0.199 | 0.403 | 0.402 |
| 0.4 | .04267 | 1.077 | 0.431 | 1.477 | 0.390 | 0.821 | 0.804 |
| 0.6 | .1440 | 1.166 | 0.700 | 1.766 | 0.569 | 1.27 | 1.21 |
| 0.8 | .3413 | 1.220 | 0.976 | 2.020 | 0.703 | 1.68 | 1.61 |
| 1.0 | .6667 | 1.414 | 1.414 | 2.414 | 0.881 | 2.30 | 2.01 |

[**] Approximated by $W = 2.3*a^{(1/3)}A^{(1/3)}$

$=2.9*b^{(1/3)}*A^{(1/3)}$, like Eqn (31).

## 3. CONCLUSIONS

A fresh interpretation of the main criterion which determines whether or not kinematic wave theory is applicable for analysis (or prediction) of a particular surface runoff event is quite amenable to computerisation. It involves checking regularly at the downstream end of each channel and overland flow segment to ensure that the mean kinetic energy of the flow (expressed in metres of water) is not significant in relation to the mean elevation of that segment above its downstream exit level.

Most artificial storm drainage channels have sectional shapes which are, for reasons largely involving ease of construction, either simply straight-sided (e.g. rectangular, or triangular) or circular (e.g. pipes). With certain restrictions, these channels have been able to be incorporated into kinematic wave models that utilise the algorithm of Alley et al. (Ref 1). Nevertheless, some artificial channels (such as trapezoidal, or unformed earth reinforced with erosion-resistant fabric), and many natural channels, have shapes which have hitherto been difficult to accommodate in such models on account of the lack of a suitable power function relating flowrate to sectional area.

The potential for accommodating these awkward shaped channels into such models is now sparked by the development of an approximate relationship between wetted perimeter and sectional area for a parabolic channel. This relationship leads to a convenient flow-area power function, which indicates that the flowrate is proportional to the 13/9 power of the parabolic sectional area, provided that a reasonably tolerable depth restriction is not violated.

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## LIST OF SYMBOLS

| Symbol | Meaning | Units |
|---|---|---|
| A | Sectional area of flow | ($m^2$) |
| A# | Dimensionless area ($A/a^2$) | (-) |
| a | Semi-width at focal depth (=2*b) | (m) |
| b | Parabolic focal height above vertex | (m) |
| c | Coefficient of perimeter-area power function | ($m^{1-2d}$) |
| d | Exponent of perimeter-area function | (-) |
| e | Coefficient of flowrate-area power function | ($s^{-1}m^{3-2f}$) |
| f | Exponent of flowrate-area function | (-) |
| h | Height above vertex at semi-width y | (m) |
| I | Mathematical integration function | (-) |
| k1 | Cube root of 1.5 | (-) |
| k2 | Cube root of (1.5 squared) | (-) |
| L | Arc length of parabola from vertex | (m) |
| n | Manning roughness coefficient | ($s\,m^{-1/3}$) |
| Q | Flowrate | ($m^3/s$) |
| q | Tributary inflow per unit length | ($m^2/s$) |
| r | Hydraulic radius (=A/W) | (m) |
| S | Channel hydraulic gradient | (-) |
| t | One third (1/3) | (-) |
| v | Sectional mean velocity (=Q/A) | (m/s) |
| W | Wetted perimeter (=2*L) | (m) |
| W# | Dimensionless wetted perimeter (=W/a) | (-) |
| x | Longitudinal downstream distance | (m) |
| y | Semi-width of parabolic section | (m) |
| y# | Dimensionless semi-width (=y/a) | (-) |
| z | Square root of (a*a + y*y) | (m) |
| z# | Dimensionless value of z (=z/a) | (-) |