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A SIMPLIFIED AND ACCURATE METHOD OF DESCRIBING NON-TURBULENT TRANSPORT OF MOMENTUM IN TURBULENT FLOW THEORY

D.R.H. BEATTIE AND W.J. GREEN

LUCAS HEIGHTS RESEARCH LABORATORIES

AUSTRALIAN ATOMIC ENERGY COMMISSION, LUCAS HEIGHTS, N.S.W. 2234 AUSTRALIA

SUMMARY Discontinuities encountered in models for turbulent and non-turbulent transport parameters at the laminar/turbulent interface, near the boundary wall, can be eliminated by placing the reference position for turbulence at the interface rather than the flow boundary. Comparisons with experimental data have been made of a calculated universal velocity profile and of various predicted turbulence parameters. In all instances, agreement was better than for simple mixing-length models.

NOTATION

- a,b arbitrary coefficients in relation between eddy diffusivities of heat and momentum (equation A1)
- f friction factor
- Nu Nusselt number
- Pe Peclet number
- Pr Prandtl number
- q heat flux
- r radial position
- R duct radius
- Re Reynolds number
- u local axial velocity
- u* shear velocity, $\sqrt{\tau_w/\rho}$
- u^+ dimensionless velocity, u/u*
- u' fluctuating component of axial velocity
- v' fluctuating component of radial velocity
- y distance from wall
- y^+ dimensionless wall distance, $y\,u^*\rho/\mu$
- ε eddy diffusivity
- κ von Karman coefficient
- μ dynamic viscosity
- ρ density
- τ shear stress

Subscripts

- H heat
- m momentum
- o sublayer value
- t turbulent value
- w wall value

1 INTRODUCTION

Because of its simplicity and accuracy of prediction, the turbulent flow theory presented by Prandtl (1925) is often used in preference to more complex concepts. The simplicity of the Prandtl model arises in part from the treatment of viscosity: turbulent transport is neglected in the 'laminar sublayer' and non-turbulent transport is neglected in the 'turbulent core'. Although this treatment is reasonable when applied very close to and far from the wall, it produces inaccurate profile descriptions near the supposed sublayer/core interface where molecular and eddy transport processes are of comparable magnitude.

These weaknesses have been overcome by several authors such as Van Driest (1956) who have modified Prandtl's original theory to include more complex transport models. Proposed eddy diffusivity profiles are thus more complex and predictions are improved at the expense of model simplicity.

The present work offers another adaptation of earlier turbulence theory, but one in which improved flow description is obtained without unduly complicating the flow structure model. This is achieved by retaining the simple linear form for eddy viscosity but placing the reference position for turbulence at the laminar/sublayer core interface rather than at the wall. Discontinuities within earlier models arise from the contradiction between a laminar sublayer and a turbulent 'mixing length' which is proportional to the distance from the wall. However, a shift of reference position and allowance for the presence of non-turbulent transport away from the sublayer can remove this discontinuity.

For clarity, the analysis is confined to round tube geometries. In comparing the theory with experimental data, emphasis is placed on 'buffer' region data for which early theory predictions are poor.

2 NON-DIMENSIONAL VELOCITY PROFILE

The velocity u in fully developed, smooth round pipe flow is a function of the distance from the duct wall y, the fluid viscosity μ, the wall shear stress τ_w, and the fluid density ρ, i.e.

$$u = u(\tau_w, \rho, \mu, y) \quad . \tag{1}$$

From dimensional analysis

$$u^+ = f(y^+) \tag{2}$$

where $u^+ = u/u^* = u/\sqrt{\tau_w/\rho}$ and $y^+ = y\,u^*\rho/\mu$.

From the definition of viscosity and the assumption that shear stress is a linear function of distance from the wall,

$$\tau = (\mu + \mu_t)du/dy = \tau_w r/R \tag{3}$$

where μ and μ_t are, respectively, the laminar and eddy viscosities. In view of the dimensionless groups of equation (2) it is convenient to use equation (3) to obtain

$$u^+ = \int_o^y \frac{yu^*\rho r/R}{\mu + \mu_t}\,\frac{dy}{y} \quad . \tag{4}$$

As in other turbulence theories, it is assumed that a thin laminar sublayer extends from the wall to y_o. For this region $\mu_t = 0$. If it is assumed that (i) $y_o^+ = 11.8$, and (ii) the dimensionless group $yu^*\rho r/(\mu_t R)$ is constant and equal to $1/\kappa$ for $y^+ > 11.8$ (κ is the von Karman coefficient), then equation (4) reduces to

$$\left.\begin{aligned} u^+ &= y^+ , & y^+ \leqslant 11.8 \\ u^+ &= 5.66 + 5.66 \log_{10} y^+ , & y^+ > 11.8 . \end{aligned}\right\} \quad (5)$$

Apart from slightly different coefficients, this is the result obtained by Prandtl. The different coefficients arise from choices of κ and y_o^+ values which differ from those usually quoted. The present values ensure compatibility of equation (5) with the Nikuradse friction factor equation.

Neglecting μ_t in the laminar sublayer and assuming $yu^*\rho r/(\mu_t R)$ is constant beyond the sublayer results in an unrealistic discontinuity in the μ_t profile at the laminar/turbulent interface. This difficulty can be overcome by assuming

$$(y - y_o)\, u^*\rho r/(\mu_t R) = 1/\kappa , \quad y > y_o ,$$

which can also be written as

$$\mu_t/\mu = \kappa(y^+ - y_o^+)r/R \qquad y > y_o . \qquad (6)$$

This approximates the earlier assumption at high y^+ where it is most accurate and moreover is continuous with the sublayer expression $\mu_t = 0$ at y_o.

In order that u^+ be independent of r/R, equation (6) needs to be redefined by

$$\left.\begin{aligned} \mu_t/\mu &= \kappa(y^+ - y_o^+)r/R - y/R , & y > y_o \\ \mu_t/\mu &= 0 , & y \leqslant y_o . \end{aligned}\right\} \quad (7)$$

Near the duct centre, equation (7) is unrealistic. This is not significant because errors in velocity profile at the duct centre are of little importance when assessing overall flow characteristics.

Substitution of equation (7) into equation (4) produces the integral

$$\int_{y_o}^{y^+} \frac{dy^+}{(1 + \kappa(y^+ - y_o^+))} .$$

The unit value in the denominator arises from the retention of μ in the region $y^+ > y_o^+$; it does not appear in similar analyses.

Evaluation of equation (4) with μ_t given by equation (7) leads to

$$\left.\begin{aligned} u^+ &= y^+ , \qquad y^+ \leqslant y_o^+ , \\ u^+ &= y_o^+ - (\ell n\, \kappa)/\kappa + \kappa^{-1}\, \ell n(y^+ - y_o^+ + \kappa^{-1}) , \\ & \qquad y^+ > y_o^+ . \end{aligned}\right\} \quad (8)$$

It should be noted that both u^+ and du^+/dy^+ are continuous at y_o^+.

To ensure that equation (8) asymptotically approaches the values of u^+ given by equation (5) at large values of y^+, values of κ and y_o^+ are chosen to be 0.407 and 7.87 respectively. The non-dimensional velocity profile can then be stated as

$$\left.\begin{aligned} u^+ &= y^+ , & y^+ \leqslant 7.87 , \\ u^+ &= 5.66(1 + \log_{10}(y^+ - 5.41)) , & y^+ > 7.87 . \end{aligned}\right\} \quad (9)$$

3 SHEAR, TURBULENCE PRODUCTION AND VISCOUS PROFILES

In the above theory, a postulated eddy viscosity profile was used to derive a non-dimensional velocity profile. From an experimental viewpoint, the velocity profile can be confirmed directly, but the eddy viscosity profile cannot. It can be inferred from measurements of turbulent shear $\rho\,\overline{u'v'}$ via the relation $\rho\,\overline{u'v'} = \mu_t(du/dy)$. Alternatively, if the turbulent energy production per unit volume, $(\mu\,\overline{u'v'}/u^{*2})(du/dy)$, or the energy dissipated per unit volume $\rho^{-1}[(\mu/u^*)(du/dy)]^2$ are considered for the region near the wall (i.e. where $r/R \simeq 1$) and made non-dimensional by dividing by τ_w, they can be shown to be related to the ratio μ_t/μ in the following manner:

dimensionless shear

$$\left.\frac{\overline{u'v'}}{(u^*)^2}\right|_{r \simeq R} = (1 + \mu/\mu_t)^{-1} , \qquad (10)$$

dimensionless turbulent energy

$$\left.\frac{\mu}{\rho}\frac{\overline{u'v'}}{(u^*)^4}\frac{du}{dy}\right|_{r \simeq R} = (2 + \mu_t/\mu + \mu/\mu_t)^{-1} \qquad (11)$$

and dimensionless energy dissipation

$$\left.\left[\frac{\mu}{\rho(u^*)^2}\frac{du}{dy}\right]^2\right|_{r \simeq R} = (1 + \mu_t/\mu)^{-2} . \qquad (12)$$

These quantities can be used to investigate indirectly the validity of the assumed eddy viscosity profile of equation (7).

4 FRICTION FACTOR

The Nikuradse friction factor equation is derived from Prandtl's mixing length theory by averaging the logarithmic component of equation (5) over the total flow section. Errors introduced by inadequate treatment of the near-wall velocity profile are negligible.

The same friction factor equation results from the present theory if the logarithmic component of equation (9) is averaged over its range of real values, i.e. over the region $y^+ > 5.41$. Exclusion of the region $0 < y^+ < 5.41$ and neglect of the relationship $u^+ = y^+$ that applies for $y^+ \leqslant 7.87$ have an insignificant influence on calculated friction factors.

Thus, friction factors from the present theory closely approximate those from Prandtl's mixing length theory.

5 COMPARISON OF PREDICTED PERFORMANCE WITH EXPERIMENTAL DATA

The proposed model involves the same variable parameters introduced by Prandtl into his mixing length theory, namely the von Karman coefficient κ and the thickness of the laminar sublayer y_o^+. The values of these parameters have been selected to ensure that values of u^+ at large y^+ are in close agreement with those determined from Prandtl's theory. This results in (a) a value of y_o^+ that is different from that used by Prandtl, and (b) essentially the same value for κ.

Because of the procedure used to determine values of y_o^+ and κ, using the proposed theory to predict profiles and associated parameters at low values of y^+ does not require curve fitting of experimental data in these regions. Comparison of predicted and experimental parameters at low values of y^+ is therefore critical for testing the validity of the model.

Predictions have been compared with experimental data from various sources for radial distributions of eddy diffusivity, velocity, turbulent shear, and turbulence production and dissipation (see Figs. 1 to 5). For comparative purposes, predictions arising from Prandtl's theory are also shown on these figures.

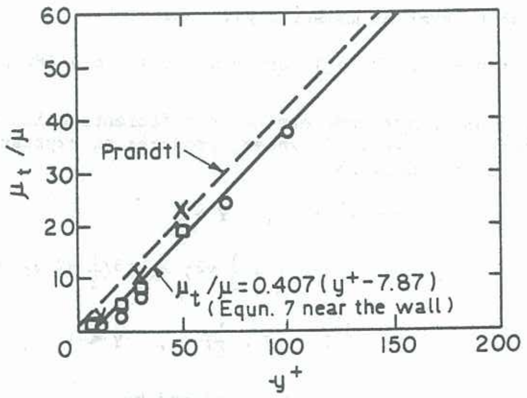

Figure 1 Comparison of predicted and experimental eddy viscosity distributions. X, O, □, data of Laufer (1954), Schubauer (1954) and Abbrecht and Churchill (1960) respectively.

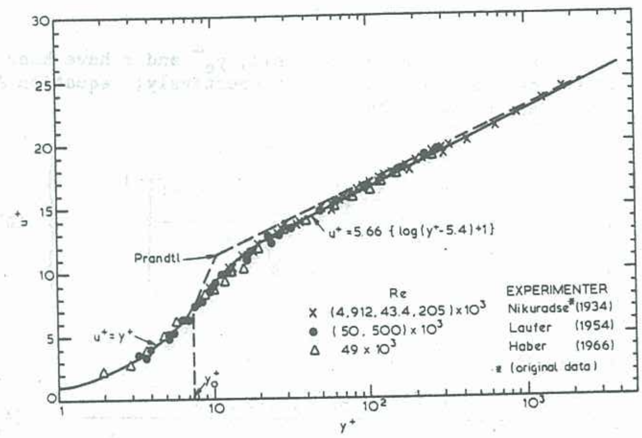

Figure 2 Comparison of predicted and experimental non-dimensional velocity profile

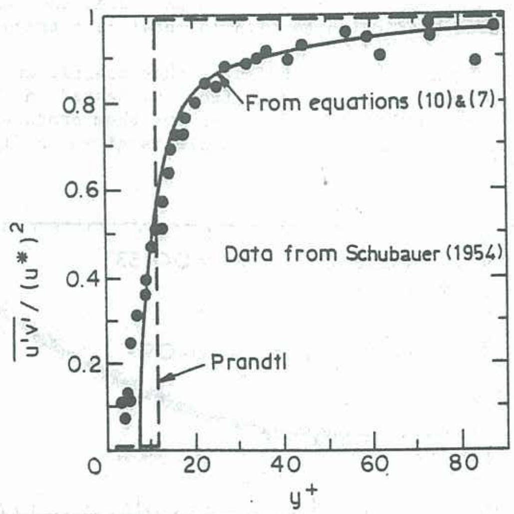

Figure 3 Comparison of predicted and experimental non-dimensionalised turbulent shear stress distribution

Although it is recognised that more sophisticated theories give better agreement with experimental data than does Prandtl's theory, comparing the present concept with that of Prandtl is reasonable on two

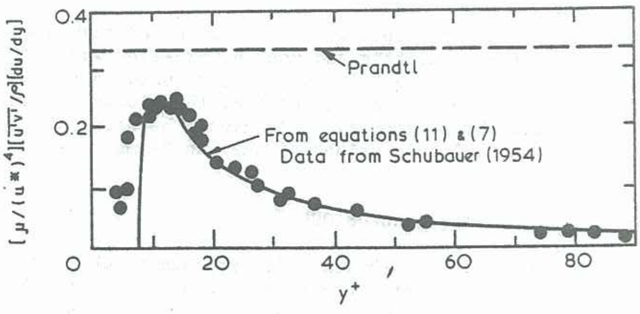

Figure 4 Comparison of predicted and experimental non-dimensionalised turbulence production distributions

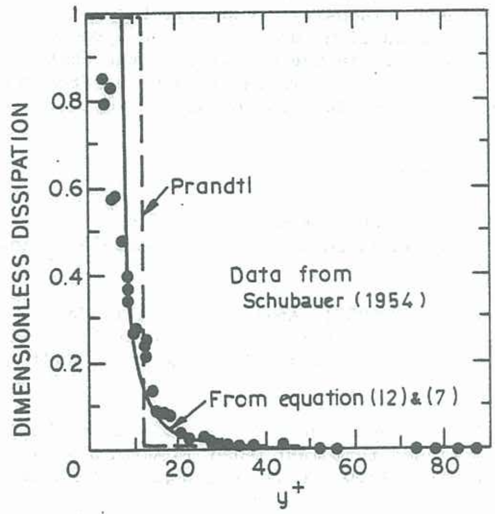

Figure 5 Comparison of predicted and experimental non-dimensional turbulence dissipation distributions

counts: (i) the present model is comparable with Prandtl's theory, and (ii) improved prediction has been achieved in other theories only by introducing multi-region models.

Figures 1 to 5 show that the proposed model predicts experimental data well but errors arising from a disregard of the turbulence in the viscous-dominated region $y^+ < 7.87$ are apparent (see figures 3 to 5). Slight discrepancies between prediction and experiment can also be seen in the region $10 < y^+ < 30$ (see figures 2 to 5).

Two aspects of figure 2 should be noted:

(a) As is known, Nikuradse (1932) modified his experimental values of y^+ by subtracting 7 (Miller 1949). This may have been done so that the experimental data would fit more closely the semi-logarithmic equation proposed by Prandtl, and it indicates that the original data should be described by Prandtl's equation, with y^+ being replaced by $y^+ - 7$, which approximates $y^+ - 5.41$ in equation (9).

(b) For the region $30 < y^+ < 1000$, equation (9), as shown on figure 2, can be approximated by a straight line of slightly greater slope than that arising from Prandtl's theory, even though essentially the same von Karman coefficient has been used in both equations. Indeed, for the range $50 < y^+ < 1000$, equation (9) is very close to the semi-logarithmic form $u^+ = 2.78 \ln y^+ + 3.8$ proposed by Diessler (1952). It is noted that the von Karman coefficient used by Diessler was 0.36, whereas a value of 0.407 has been used in the present work; this shows that the slopes of semi-logarithmic velocity plots cannot be used to obtain reliable values of the von Karman coefficient.

6 SUMMARY AND CONCLUSIONS

A simple model of turbulence is proposed in which (a) the laminar sublayer concept is retained, (b) continuity of eddy viscosity is achieved by assuming it to be proportional to the distance from the laminar sublayer instead of the wall, and (c) non-turbulent transport is allowed for in the turbulent core.

Choosing a suitable laminar sublayer thickness $y_o^+ = 7.87$ ensures that the model represents observed data at large wall distances. Moreover, equations developed to determine eddy viscosity, velocity, turbulent shear, turbulence production and dissipation agree well with experimental data in the vicinity of the 'buffer' region as well as data which are distant from the wall. The agreement with data is comparable to that of more advanced models.

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APPENDIX - APPLICATION OF THE MODEL TO HEAT TRANSPORT

Application of classical turbulence models to heat transport produces heat transfer equations which are poor when compared with liquid metal heat transfer data. This is because classical theory neglects molecular transport in the turbulent core, whereas molecular transport of heat remains significant throughout the turbulent core of liquid metals. The present model, which retains molecular transport processes in the turbulent core, should provide heat transfer equations which are more accurate for liquid metals than those of classical theory. A preliminary application of the concepts to heat transfer, for (a) local heat transfer given by $q/q_w = \tau/\tau_w$, and (b) ratio of eddy diffusivity of heat to that of momentum given by

$$\varepsilon_H/\varepsilon_m = a - b\,\mu/(Pr\,\mu_t) \quad \text{or} \quad = 0 \text{ for } a < b\,\mu/(Pr\,\mu_t) \quad ,$$

where a and b are undetermined coefficients which are independent of wall distances, produces an expression for Nusselt number as

$$\left.\begin{aligned} Nu &= 6 \quad , \quad Y > 1 \\ \text{and} \quad Nu^{-1} &= \tfrac{1}{2} Y(1 - Y + \tfrac{1}{3} Y^2) + X^{-3}\{\tfrac{1}{2} Z^2 \ln Z \\ &\quad - \tfrac{3}{4} Z^2 + Z - \tfrac{1}{4}\} \quad , \quad Y \leqslant 1 \quad , \end{aligned}\right\} \text{(A1)}$$

where X, Y, Z are parameters defined by

$$X = a\,\kappa\,Pr\,\frac{Re}{2}\sqrt{\frac{f}{2}} \quad ,$$

$$Y = (y_o^+ + \frac{b}{a\,\kappa\,Pr})\,\frac{2}{Re}\sqrt{\frac{2}{f}}$$

and

$$Z = 1 + X(1 - Y) \quad .$$

Since from the momentum analysis, y_o^+ and κ have been determined as 7.87 and 0.407 respectively; equation A1 can be approximated by

$$\left.\begin{aligned} Nu &= 6 \quad , \quad Y > 1 \\ Nu &= 6\left[1 - \frac{[a\,\kappa\,Pr(\frac{Re}{2}\sqrt{\frac{f}{2}} - y_o^+) - b\,]^4}{4(a\,\kappa\,Pr\,\frac{Re}{2}\sqrt{\frac{f}{2}})^3}\right]^{-1} \quad , \\ &\quad Y < 1 \end{aligned}\right\} \text{(A2)}$$

for small Peclet numbers, and by

$$Nu = \frac{a\,Pr\,Re\,f/2}{1 + y_o^+\sqrt{\frac{f}{2}}\,(a\,Pr-1) + \frac{1}{\kappa}\sqrt{\frac{f}{2}}\,(b - \ln\frac{1}{aPr})} \qquad \text{(A3)}$$

for large Peclet numbers.

Equation A3 is similar to equations presented in standard texts. The third denominator term accounts for non-turbulent transport in the core, and leads to improved prediction for liquid metal heat transfer.

Comparison with data indicates that coefficients a and b vary with axial distance from the heated inlet, and that suitable choice of values for them produces good agreement with data. An example is given in figure 6.

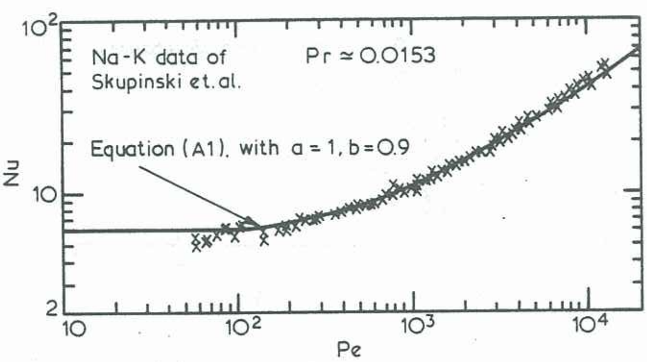

Figure 6 Comparison of theoretical and experimental heat transfer characteristics