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Admissibility Requirements and the Least Squares Finite Element Solution for Potential Flow

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SUMMARY It is shown that continuity of the first derivatives of the velocity components is required in the least squares finite element formulation of potential flow. The admissibility requirements are thus one order higher than commonly supposed. The solution procedure is illustrated using both fifth-order and first-order trial functions for the components of velocity and the results compared.

1 INTRODUCTION

The starting point for the variational finite element method is the requirement that some integral be minimized. Such an integral is generally obtained using the calculus of variations, but alternatively can be obtained from the least square criterion. Where a governing equation can be written in the usual form

$$Au = f \tag{A1}$$

the residual R is defined for an approximate solution $\hat{u}$ by

$$R = A\hat{u} - f \tag{A2}$$

and the least squares criterion is that $\int_D WR^2 dD$ shall be a minimum, where W is a positive weighting function. If there is more than one governing equation, it is the sum of their independently-formed weighted integrals that is minimized (Finlayson and Scriven, 1965). The finite element method based on the minimization of such integrals has the advantage of yielding a symmetric and positive definite system K matrix.

Boundary conditions in least squares formulations can be handled in a fashion similar to other residual techniques (Finlayson (1972), Norrie and de Vries (1973)). Two commonly used procedures are to require the trial function to satisfy the boundary conditions or to impose the least squares criterion on the residuals from the boundary equations. In the former case, the boundary conditions can be incorporated into the element or system matrix equations as so-called 'equivalent coupled Dirichlet conditions' (Norrie and de Vries, 1978).

The least squares finite element method has been used successfully for a variety of linear and nonlinear problems by Hinton and Irons (1968), Akin (1973), Lynn and Arya (1973,1974), Zienkiewicz et al. (1974), Lynn (1974), Lee (1974), Rossow (1975), Blackburn (1976), Steven (1976), Balasubramanian et al. (1977), Milthorpe and Steven (1978), Tuomala and Pramila (1979). Potential flow would seem to be a straightforward application, but as the following indicates, requires the proper consideration of the admissibility conditions i.e. the continuity requirements for the trial function.

Early least squares solutions of potential flow used trial functions with C^0 continuity. The results obtained were generally of low accuracy, indeed, less than would be expected. As is shown subsequently, the continuity requirements are, in fact, more stringent (C^1) and if satisfied lead to much improved solutions. This suggests that in other least squares formulations, the admissibility conditions may deserve more consideration than they are sometimes given.

2 ANALYSIS

For two-dimensional, incompressible, irrotational flow, the governing equations can be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ in D} , \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \text{ in D} , \tag{1, 2}$$

where u and v are the x and y components of the velocity vector $\bar{q}$. Only two boundary conditions will be considered here. The first is that the velocity $\bar{q}$ is prescribed on a portion S_1 of the boundary of D, that is

$$\bar{q} = \bar{Q} \text{ on } S_1 , \tag{3}$$

where $\bar{Q}$ is a prescribed function of velocity along S_1. The second boundary condition is that there is no flow across a fixed rigid boundary, that is

$$\bar{q} \cdot \bar{n} = 0 \text{ on } S_2 , \tag{4}$$

where S_2 is a solid boundary and $\bar{n}$ is the unit outward normal to S_2. For the problem to be considered, S_1 and S_2 comprise the total boundary S enclosing the domain D, hence

$$S = S_1 + S_2 . \tag{5}$$

Those functions u(x,y) and v(x,y) which satisfy the governing field equations, Eqs. (1) and (2), subject to the boundary conditions, Eqs. (3) and (4), will now be sought.

It is shown in Appendix A that among all trial functions $\hat{u}(x,y)$ and $\hat{v}(x,y)$ which are admissible, those functions which minimize the functional

$$\chi = \frac{1}{2}\iint_D [(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y})^2 + (\frac{\partial \hat{u}}{\partial y} - \frac{\partial \hat{v}}{\partial x})^2]dxdy , \tag{6}$$

also satisfy the governing field equations provided the trial functions $\hat{u}(x,y)$ and $\hat{v}(x,y)$ satisfy the boundary conditions

$$\hat{\bar{q}} \cdot \bar{n} = \bar{Q} \cdot \bar{n} \text{ on } S_1 , \quad \hat{\bar{q}} \cdot \bar{n} = 0 \text{ on } S_2 . \qquad (7), (8)$$

For the trial functions to be admissible, they must be continuous, must have continuous first derivatives and piecewise continuous second derivatives in D (see Appendix A).

It will now be shown that although there is a difference between the boundary conditions given in Eqs. (3) and (7), the solution to Eqs. (1), (2), (3) and (4) is identical to the solution to Eqs. (1), (2), (7) and (8).

Let the solution to Eqs. (1), (2), (3) and (4) be denoted by

$$\bar{q}_1 = u_1\bar{i} + v_1\bar{j} , \qquad (9)$$

and the solution to Eqs. (1), (2), (7) and (8) by

$$\bar{q}_2 = u_2\bar{i} + v_2\bar{j} . \qquad (10)$$

Since the velocity field $\bar{q}_1$ satisfies the boundary condition, Eq. (3), it must necessarily also satisfy

$$\bar{q}_1 \cdot \bar{n} = \bar{Q} \cdot \bar{n} \text{ on } S_1 . \qquad (11)$$

The difference q_3 between the two velocity fields is given by

$$\bar{q}_3 = \bar{q}_1 - \bar{q}_2 = (u_1 - u_2)\bar{i} + (v_1 - v_2)\bar{j} = u_3\bar{i} + v_3\bar{j}, \qquad (12)$$

where u_3, v_3 are the x,y components of $\bar{q}_3$. Because $\bar{q}_1$ and $\bar{q}_2$ satisfy Eqs. (1), (2), (7) and (8), the new velocity field q_3 satisfies

$$\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y} = 0 \text{ in D} , \quad \frac{\partial u_3}{\partial y} - \frac{\partial v_3}{\partial x} = 0 \text{ in D} , \qquad (13), (14)$$

subject to the boundary conditions

$$\bar{q}_3 \cdot \bar{n} = 0 \text{ on } S_1 , \qquad (15)$$

by virtue of Eqs. (11), (7), (12) and

$$\bar{q}_3 \cdot \bar{n} = 0 \text{ on } S_2 , \qquad (16)$$

because of Eqs. (4) and (8). The problem posed in Eqs. (13), (14), (15) and (16) is that of incompressible, irrotational flow between two rigid walls S_1 and S_2. On account of the well-known theorem (Milne-Thomson, 1962), *Acyclic irrotational motion is impossible in a liquid bounded entirely by fixed rigid walls,* it can be concluded that

$$\bar{q}_3 = \bar{0} \text{ in D} , \qquad (17)$$

and the two solutions $\bar{q}_1$ and q_2 must be *identical.*

In summary, if the trial functions $\hat{u}(x,y)$ and $\hat{v}(x,y)$ satisfy the boundary conditions, given in Eqs. (7) and (8), the solution given by the variational procedure is *identical* to the solution to the problem posed originally in Eqs. (1), (2), (3) and (4). The admissibility conditions on the trial functions are those earlier stated, which, it should be noted, require *continuity of the first derivatives.*

To try to assess whether satisfying these admissibility conditions gives an improved accuracy compared with satisfying only continuity of the trial function itself, the flow around a cylinder was computed using both linear and fifth-order triangular elements. For the fifth-order element (Cowper et al, 1968) the nodal parameters comprised the function value and its first and second derivatives at each of the vertex nodes. The (constrained) fifth-order trial functions $\hat{u}$ and $\hat{v}$ were substituted in (6) and the functional minimized in the usual way with respect to the nodal parameters. The boundary conditions were inserted at the element level as 'equivalent coupled Dirichlet conditions' (Norrie and de Vries, 1978). Further details may be found in de Vries et al. (1976).

3 RESULTS AND DISCUSSION

For the fifth-order solution, the region around the cylinder was initially discretized as shown in Figure 1. The final solution also took advantage of symmetry and used only one quadrant of the region shown. Because of the proximity of the outer boundary to the cylinder, the theoretically correct boundary conditions for flow around a cylinder in an otherwise uniform (left to right) stream was imposed on this outer boundary. In all cases, the results obtained were very close to the theoretical values. Figure 2 shows the values of u and v obtained along the cylinder surface, for the grid of Figure 1, with the percentage variation from the theoretical indicated.

The velocity at the cylinder shoulder (point A) is shown plotted in Figure 3, against the total number of nodal parameters in the domain, for both the fifth-order and linear elements. Similar results were obtained for points elsewhere in the region.

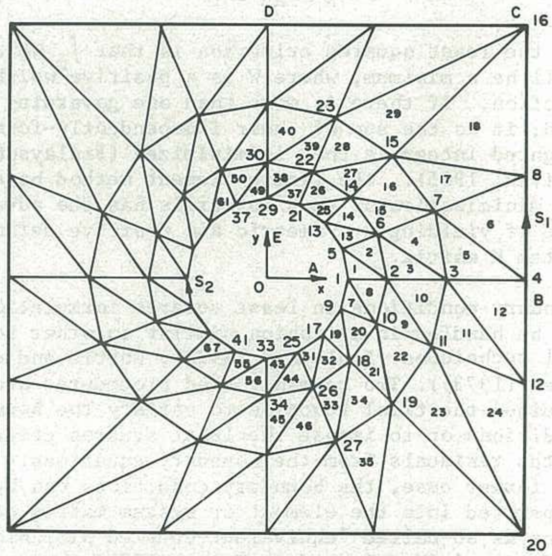

Figure 1 Discretization of Whole Region

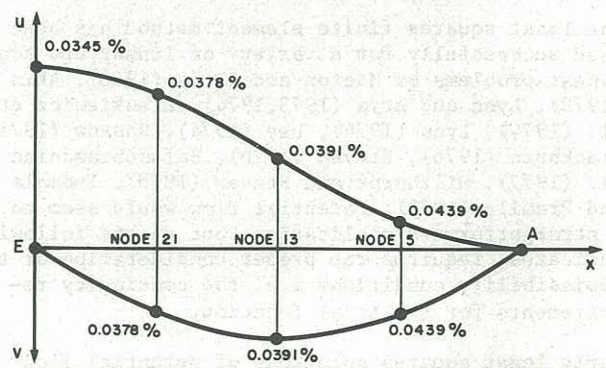

Figure 2 Profile of u and v Along the Cylindrical Surface EA

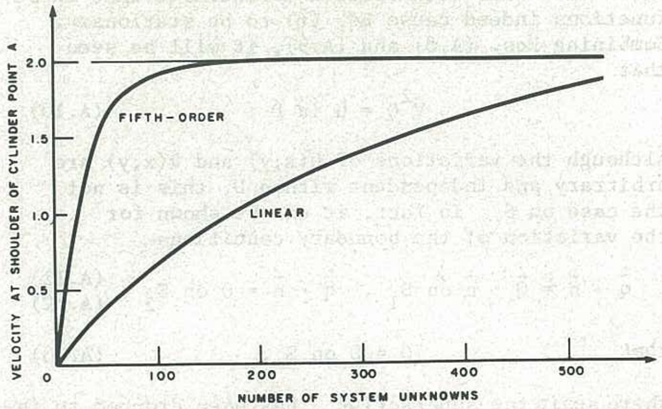

Figure 3 Comparison of Solutions for First- and Fifth-Order Polynomials

The improvement in accuracy for the fifth-order element with a given number of system nodal parameters, as indicated in Figure 3 could, of course, be attributed simply to the higher-order element having a greater 'efficiency' i.e. being better able to fit the true solution. It is known, however, that use of a trial function with a lower order of continuity than admissibility would require is generally equivalent to neglecting interelement integrals in the analysis, resulting in an increase in solution error in most cases. This suggests that at least some of the deviation between the two sets of results shown in Figure 3, and possibly a major portion, is due to the admissibility violation in the case of the linear element.

Of course, it is known that (Finlayson and Scriven, 1967) non self adjoint problems are not in general amenable to a *classical* variational formulation, and in such cases, admissibility requirements in the strict calculus of variations sense would not exist. The basic requirement for continuity of the trial function in a least squares formulation then stems from the necessity for existence across the domain of such trial functior derivatives as occur in the least squares 'functional'. In the self-adjoint problem, as the present example has shown, a higher level of continuity than this may be imposed by the variational admissibility requirements.

As is seen from Figure 3, both solutions converge as the element size decreases towards zero. It is by no means unknown for convergence to be obtained when the admissibility requirements are violated, as the non-conforming elements demonstrate. Indeed, sometimes the convergence rate can be significantly higher under these conditions, although this is the exception rather than the general rule. Disregard of admissibility conditions can, in general, be expected to increase solution error for the reason indicated earlier, but convergence will still be obtained if the interelement errors reduce at an appropriate rate with a decrease in element size.

The results obtained for potential flow suggest that the admissibility requirements for least squares formulations may be worth more consideration than sometimes given.

4 ACKNOWLEDGEMENTS

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6 APPENDIX A

For the functional, given by Eq. (6), to be stationary, it is required that the first variation of χ vanishes (Forray, 1968). Introducing the notation

$$\hat{P} = \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\ , \quad \hat{Q} = \frac{\partial \hat{u}}{\partial y} - \frac{\partial \hat{v}}{\partial x} \qquad \begin{matrix}(A.1)\\(A.2)\end{matrix}$$

and the identity

$$\hat{P}[\frac{\partial}{\partial x}(\delta\hat{u})] = \frac{\partial}{\partial x}(\hat{P}\,\delta\hat{u}) - \delta\hat{u}\,\frac{\partial\hat{P}}{\partial x}\ , \qquad (A.3)$$

allows the variation of Eq. (6) to be written as

$$\delta\chi = -\iint_D [\delta\hat{u}\,(\frac{\partial\hat{P}}{\partial x} + \frac{\partial\hat{Q}}{\partial y}) + \delta\hat{v}\,(\frac{\partial\hat{P}}{\partial y} - \frac{\partial\hat{Q}}{\partial x})]\,dxdy$$

$$+ \iint_D [\frac{\partial}{\partial x}(\hat{P}\delta\hat{u} - \hat{Q}\delta\hat{v}) + \frac{\partial}{\partial y}(\hat{P}\delta\hat{v} + \hat{Q}\delta\hat{u})]\,dxdy = 0. \qquad (A.4)$$

Gauss's Theorem may be applied to the second term of Eq. (A.4) *provided that* <u>*both*</u> *$\hat{u}(x,y)$* <u>*and*</u> *$\hat{v}(x,y)$ are continuous in D+S and have* <u>*piecewise*</u> <u>*continuous*</u> <u>*first*</u> <u>*derivatives*</u> *in D* (Sternberg and Smith, 1946). If the trial functions are so restricted, then Eq. (A.4) may be rewritten as

$$\delta\chi = -\iint_D [\delta\hat{u}\,(\frac{\partial\hat{P}}{\partial x} + \frac{\partial\hat{Q}}{\partial y}) + \delta\hat{v}(\frac{\partial\hat{P}}{\partial y} - \frac{\partial\hat{Q}}{\partial x})]\,dxdy$$

$$+ \int_S [\hat{P}\delta(\hat{\bar{q}}\cdot\bar{n}) + \hat{Q}(\delta\hat{u}\,n_y - \delta\hat{v}\,n_x)]\,dS = 0\ . \qquad (A.5)$$

Since the domain and surface integrals in Eq. (A.5) are independent, the necessary conditions for the functional of Eq. (6) to be stationary become

$$\iint_D [\delta\hat{u}\,(\frac{\partial\hat{P}}{\partial x} + \frac{\partial\hat{Q}}{\partial y}) + \delta\hat{v}\,(\frac{\partial\hat{P}}{\partial y} - \frac{\partial\hat{Q}}{\partial x})]\,dxdy = 0, \qquad (A.6)$$

and

$$\int_S [\hat{P}\delta(\hat{\bar{q}}\cdot\bar{n}) + \hat{Q}\,(\delta\hat{u}\,n_y - \delta\hat{v}\,n_x)]\,dS = 0\ . \qquad (A.7)$$

Because the variations of $\hat{u}(x,y)$ and $\hat{v}(x,y)$ are arbitrary and independent *within* D, it follows from Eq. (A.6) that both

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \text{ in D}\ , \quad \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 \text{ in D}\ . \qquad \begin{matrix}(A.8)\\(A.9)\end{matrix}$$

where the ^ has been deleted to indicate that these functions indeed cause Eq. (6) to be stationary. Combining Eqs. (A.8) and (A.9), it will be seen that

$$\nabla^2 Q = 0 \text{ in D}\ . \qquad (A.10)$$

Although the variations of $\hat{u}(x,y)$ and $\hat{v}(x,y)$ are arbitrary and independent within D, this is not the case on S. In fact, it can be shown for the variation of the boundary conditions

$$\hat{\bar{q}}\cdot\bar{n} = \bar{Q}\cdot\bar{n} \text{ on } S_1\ , \quad \hat{\bar{q}}\cdot\bar{n} = 0 \text{ on } S_2\ , \qquad \begin{matrix}(A.11)\\(A.12)\end{matrix}$$

that

$$Q = 0 \text{ on S}\ , \qquad (A.13)$$

where again the superscript ^ has been dropped to indicate that Q is obtained from those functions u(x,y) and v(x,y) which give the functional a stationary value.

From Eqs. (A.10) and (A.13), it can be further shown with the aid of Gauss's theorem that

$$Q \equiv 0\ , \text{ in D and on S}\ . \qquad (A.14)$$

And furthermore, it can be shown (de Vries et al., 1976) that P must also be identically equal to zero throughout D and on S, provided that both P and Q are continuous and have piecewise continuous first derivatives in D.

Since Q is required to be continuous in D+S, Eq. (A.2) shows that $\partial u/\partial y$ and $\partial v/\partial x$ must also be continuous in D+S. In general (excluding the special case of u(x,y) and v(x,y) with stepped slope discontinuities to the y and x axes respectively), this requires that $\partial u/\partial x$ and $\partial v/\partial y$ also be continuous in D+S.

To summarize, those functions u(x,y) and v(x,y) which cause the functional given in Eq. (6) to be stationary satisfy

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ in D}\ , \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \text{ in D}\ , \qquad \begin{matrix}(A.15)\\(A.16)\end{matrix}$$

as well as the boundary conditions

$$\bar{q}\cdot\bar{n} = \bar{Q}\cdot\bar{n} \text{ on } S_1\ , \quad \bar{q}\cdot\bar{n} = 0 \text{ on } S_2\ , \qquad \begin{matrix}(A.17)\\(A.18)\end{matrix}$$

and hence constitute the solution to the problem posed earlier.

The requirements of admissibility are that the trial functions $\hat{u}(x,y)$ and $\hat{v}(x,y)$ belong to that *class of functions which are continuous, have continuous first derivatives, and have piecewise continuous second derivatives in* D.

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