

Influence of Channel Slope on Numerical Stability in Unsteady Flow Simulation

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SUMMARY The problem of numerical stability in the simulation of unsteady open channel flow using implicit schemes often arises when channel slope is large and space-step size ΔX is not properly chosen. This paper reports results of a theoretical analysis and numerical experiments which show that ΔX should vary inversely with channel slope (S_0) when S_0 is greater than a critical value. Within the range of present study, the relationship can be approximated by $\Delta X = 0.3 Y_0 / (S_0 - 0.0005)$ indicating the critical value of channel slope to be 0.0005. Here Y_0 is the initial flow depth. It is recommended that $\Delta X = 0.3 Y_0 / S_0$ be used for practical application.

1 INTRODUCTION

Simulation of unsteady flow in open channels using numerical techniques to solve governing equations of motion and continuity has been developing very rapidly in recent years. Many investigators have shown in theory as well as in practice that implicit schemes, such as Preissmann's, for discretization of governing equations are unconditionally stable provided that the weighting factor for the advanced time-points is greater than 0.5 (Abbott and Ionescu, 1967; Cunge 1966). However, this criterion has failed to hold when channel slope (S_0) is large and the grid size (ΔX) is not properly chosen (Yen, Wang and Yu, 1977). By theoretical analysis and numerical experiment, the present study investigates the influence of S_0 on the maximum ΔX under which numerical stability can still be maintained.

2 THEORETICAL ANALYSIS

The equations of motion and continuity for open channel flow are discretized by Preissmann's scheme (Mahmood and Yevjevich, 1975) to obtain a set of finite difference equations. Solutions to these finite difference equations contain numerical errors. If the errors grow continuously, the solution will be unstable and eventually destroyed. To examine the nature of this instability, the solution, in the form of Fourier series, to the linearized differential equations of motion and continuity are substituted into the finite difference equations to derive the amplification factor from which the stability criterion is obtained as follows

$$\left| 1 + \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right| \leq 1 \quad (1)$$

where

$$\left. \begin{aligned} A &= 1 + 4a^2(-1 + 1/F^2)\theta^2 + 4a\theta i \\ B &= 8a^2(-1 + 1/F^2)\theta + 2b + 4a(1 + 5b\theta/3)i \\ C &= 4a^2(-1 + 1/F^2) + 20abi/3 \end{aligned} \right\} \quad (2)$$

and $i = \sqrt{-1}$, $a = U_0(\Delta t/\Delta X)\tan(\alpha\Delta X/2)$, $b = gS_0\Delta t/U_0$, $F = U_0/\sqrt{gY_0}$, U_0 = initial velocity, Y_0 = initial depth, Δt = time-step size, α = wave number of

Fourier component, and g = gravitational acceleration. From Eqs.(1) and(2) the regions of stability and instability are determined as a function of a , b , F and θ . The results are plotted in the form of a vs. b with F and θ as parameters. Figure 1 shows one of these plots. Each line represents the division between stable and unstable regions for the Froude number indicated. The region above the line is stable and the one below is unstable. Each of these lines can be approximated by

$$U_0 \frac{\Delta t}{\Delta X} \tan\left(\frac{\alpha\Delta X}{2}\right) = k_1 \left(\frac{gS_0\Delta t}{U_0} - 1 \right) \quad (3)$$

in which k_1 , being the slope of the line it represents, is a function of F and θ . Assuming that the α value of the wave component that causes instability has a special relation to ΔX so that $\tan(\alpha\Delta X/2) = 1/k_2$ is a constant, then Eq.(3), after multiplying by Y_0/U_0 on both sides of the equal sign, can be rewritten as

$$\frac{Y_0}{\Delta X} = \frac{k_1 k_2}{F^2} (S_0 - \frac{U_0}{g\Delta t}) \quad (4)$$

If $\frac{1}{\Delta t} \ll \frac{gS_0}{U_0}$, Eq.(4) can be approximated by

$$\frac{Y_0}{\Delta X} = k_1 k_2 S_0 / F^2 \quad (5)$$

It can further be shown that the group of $k_1 k_2 / F^2$ is a rather weak function of F (Yen et al, 1979).

The above results are based on the analysis performed on linearized governing equations. The linearization assumes that unsteady flow is a perturbation of small amplitude to steady flow. The validity of these results when the perturbation is no longer small as in the case of actual flood flow should be verified by numerical experiments.

3 DIMENSIONAL ANALYSIS

Just as in the case of experiments in hydraulic laboratory, dimensional analysis must be performed, before the experiments are run, in order to have a good grasp of parameters controlling the physical phenomenon involved so that the investigation can proceed systematically. For a wide rectangular channel, the maximum space-step, ΔX , that will

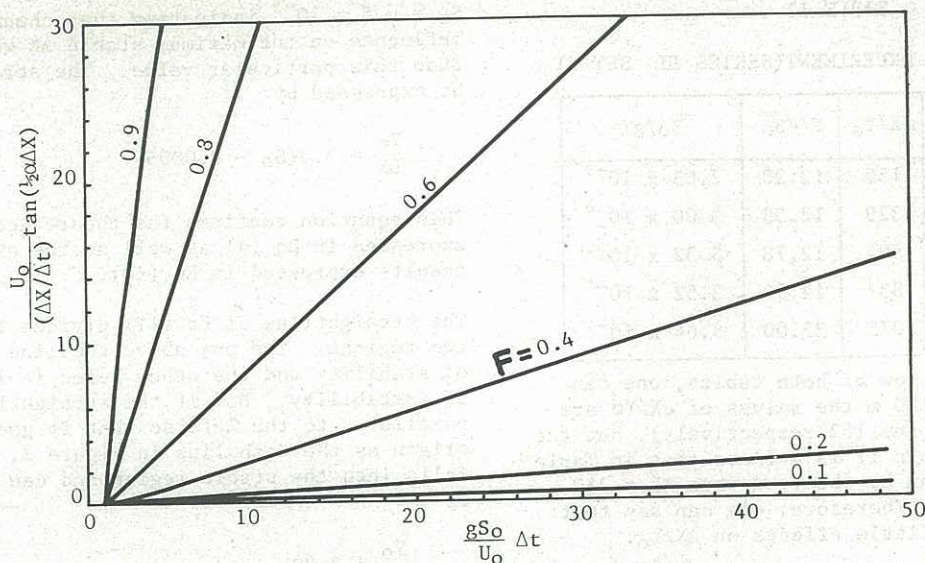


Figure 1. Regions of Stability and Instability for Various Froude Numbers ($\theta = 1$)

produce a stable numerical solution can be expressed as follows:

$$\Delta X = f(S_0, Y_0, q_0, q_{pi}, g, t, \theta, T) \quad (6)$$

in which S_0 = channel bottom slope, Y_0 = initial depth, q_0 = initial discharge per unit width, q_{pi} = peak inflow discharge per unit width, θ = weighting factor for the advanced time-points, and T = equivalent duration defined as:

$$T = \frac{1}{q_{pi}} \int_0^{T_0} q \, dt \quad (7)$$

where T_0 = duration of inflow hydrograph, and q = discharge per unit width at any time t . Eq.(7) simply transfers the inflow hydrograph into a rectangular hydrograph having discharge q_{pi} and duration T .

There are eight independent variables and one dependent variable in Eq.(7), containing two basic dimensions. Therefore they can be organized into seven dimensionless parameters as the following:

$$\frac{\Delta X}{Y_0} = f_1\left(S_0, \frac{F}{\sqrt{S_0}}, \frac{q_{pi}}{q_0}, \frac{Y_0}{gT}, \frac{\Delta t}{T}, \theta\right) \quad (8)$$

in which $F = q_0 / \sqrt{gY_0^3}$ is the Froude number.

4 NUMERICAL EXPERIMENTS

Based on Preissmann's scheme, the numerical solution was sought by using Gaussian elimination method. A computer program was constructed for this purpose. The numerical experiments were designed in accordance with Eq.(8) so that the influence of the parameters on numerical stability can be examined one by one. In each experiment, a sine-curve shaped hydrograph was used as upstream boundary condition and a stage-discharge rating curve as downstream boundary condition. The form of the rating curve was arbitrarily chosen so that its depth at the initial discharge did not confirm with uniform flow condition in the channel reach considered. There were three series of experiments carried out. In all three series θ is equal to 1.

In Series I, the variable is q_{pi}/q_0 . Other condi-

tions are: $\Delta t = 1$ hr, $T = 12$ hrs, $q_0 = 1$ cms/m and $\Delta X = 250$ m; The peak inflow discharge q_{pi} was varied so as to make q_{pi}/q_0 a variable. All other parameters are constant. In each run of experiment, a value of q_{pi}/q_0 was assigned and S_0 increased in steps from small to large values until instability appeared. Then interpolation was employed to locate the critical value of slope where instability actually occurred. Five runs of experiments have been carried out in this series with values of q_{pi}/q_0 ranging from 2.43 to 29.57. The results of this series have shown that the ratio q_{pi}/q_0 has no effects on $\Delta X/Y_0$.

In Series II, except that $\Delta t = 0.5$ hrs, all the other conditions are the same as those in Series I. Also five runs of experiments have been carried out. Although the value of $\Delta t/T = 0.0417$ in this series is only one-half of that in Series I, the results are identical. Therefore, one can say that the parameter $\Delta t/T$ has practically no effects on $\Delta X/Y_0$.

In Series III, two sets of experiments have been carried out with the conditions as follows:

	1st set	2nd set
Δt (hr)	0.5	2
T (hr)	12	48
q_{pi} (cms/m)	15.29	15.29
q_0 (cms/m)	1	1

The results for these two sets are listed in Tables I and II below

TABLE I

RESULTS OF NUMERICAL EXPERIMENT (SERIES III, SET 1)

ΔX (m)	S_0	Y_0 (m)	$\Delta X/Y_0$	$F/\sqrt{S_0}$	Y_0/gT
100	1/415	0.656	152	12.12	0.64×10^{-7}
250	1/675	0.759	329	12.55	0.75×10^{-7}
500	1/932	0.836	593	12.77	0.82×10^{-7}
750	1/1118	0.883	850	12.88	0.87×10^{-7}
1000	1/1262	0.916	1092	13.00	0.90×10^{-7}

TABLE II

RESULTS OF NUMERICAL EXPERIMENT (SERIES III, SET 2)

ΔX (m)	S_0	Y_0 (m)	$\Delta X/Y_0$	$F/\sqrt{S_0}$	Y_0/gT
100	1/435	0.670	150	12.20	2.63×10^{-7}
250	1/680	0.761	329	12.58	3.00×10^{-7}
500	1/960	0.846	591	12.78	3.32×10^{-7}
750	1/1180	0.896	837	12.94	3.52×10^{-7}
1000	1/1230	0.930	1075	33.00	3.64×10^{-7}

Comparing the second row of both tables, one can see that when $\Delta X = 100$ m the values of $\Delta X/Y_0$ are nearly the same (150 and 152 respectively), but the value of Y_0/gT in Table II is 4 times that in Table I. Again the same can be observed when $\Delta X = 250$, 500, 750 and 1000 m. Therefore, one can say that parameter Y_0/gT has little effects on $\Delta X/Y_0$.

Now further examination of Columns 2, 4 and 5 in Tables I and II will reveal that the value of $\Delta X/Y_0$ varies with S_0 and $F/\sqrt{S_0}$ indicating a close relationship among these three parameters. Because in these experiments only Manning's coefficient of resistance was kept constant ($n = 0.0247$) and Darcy-Weisbach coefficient of resistance, f , was not fixed. Consequently $F/\sqrt{S_0} = \sqrt{8g}/f$ is not a constant. The variations of $\Delta X/Y_0$ shown in these tables are therefore considered to be under the influence of S_0 and $F/\sqrt{S_0}$. Since other parameters have been shown to have little or no effects on $\Delta X/Y_0$, Eq.(8) is then reduced to

$$\frac{\Delta X}{Y_0} = f(S_0, F/\sqrt{S_0}) \quad (9)$$

For the present study, values of $F/\sqrt{S_0}$ being in a small range (12.12 - 13.00), Eq.(10) can be further reduced to

$$\frac{\Delta X}{Y_0} = f(S_0) \quad (10)$$

5 INFLUENCE OF CHANNEL SLOPE

The results of numerical experiments are plotted in the form of $Y_0/\Delta X$ vs. S_0 as shown in Figure 2 where one can see that a straightline can be drawn through the data points implying that ΔX is inversely proportional to S_0 . This straightline intersects with the S_0 - axis at approximately

$S_0 = 0.5 \times 10^{-3}$ indicating that channel slope has no influence on the maximum stable ΔX when S_0 is less than this particular value. The straightline can be expressed by

$$\frac{Y_0}{\Delta X} = 3.3(S_0 - 0.0005) \quad (11)$$

This equation confirms the theoretical results expressed in Eq.(4) as well as the experimental results expressed in Eq.(10).

The straightline of Eq.(11) divides the plane into two regions. The one above the line is the region of stability and the other below it is the region of instability. Now if the straightline is shifted parallel to the left so that it goes through the origin as the dash line in Figure 2. The whole line falls into the stable region and can be expressed by

$$\frac{Y_0}{\Delta X} = 3.3S_0 \quad (12)$$

or

$$\Delta X = 0.3Y_0/S_0 \quad (12a)$$

The maximum stable ΔX determined by Eq.(12a) is smaller than that by Eq.(11) and therefore it is on the conservative side. Here it is interesting to note that in water surface profile computation, McBean and Perkin (1975) have shown that the maximum stable $\Delta X = 0.6 Y_0/S_0$. This is twice as large as that determined by Eq.(12a) for unsteady flow.

6 CONCLUSIONS

Theoretical analysis has shown that the maximum space-step size ΔX for unsteady flow simulation using Preissmann's scheme can be expressed by

$$\frac{Y_0}{\Delta X} = \frac{k_1 k_2}{F^2} (S_0 - \frac{U_0}{g\Delta t})$$

where Y_0 = initial depth, S_0 = channel bottom slope, U_0 = initial velocity, g = gravitational acceleration, Δt = time-step size, $F^2 = U^2/(gY_0)$ the Froude number, and $k_1 k_2$ = coefficients. This form of relationship has also been confirmed by numerical experiments which result in

$$\frac{Y_0}{\Delta X} = 3.3(S_0 - 0.0005)$$

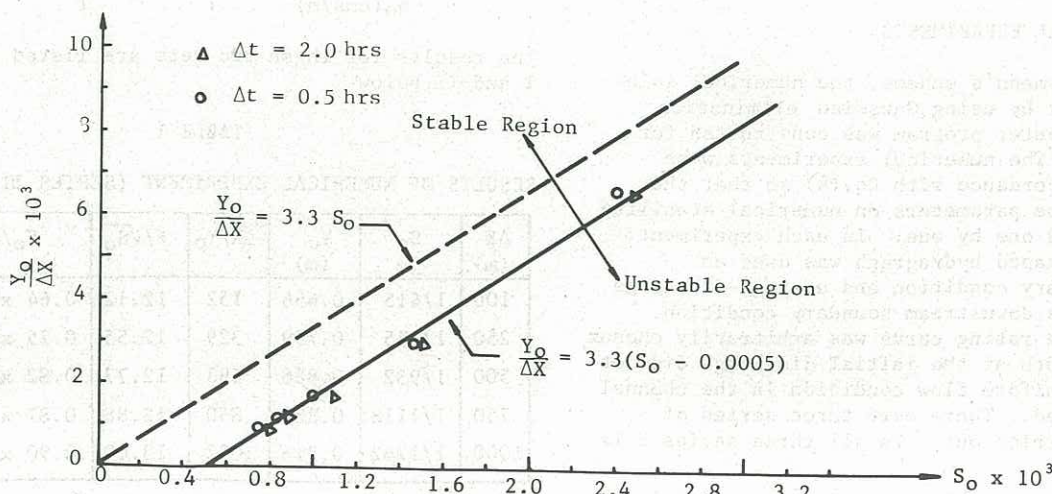


Figure 2 Results of Numerical Experiments

for the range of present experiments. This expression can be further approximately by

$$\frac{y_0}{\Delta x} = 3.3S_0$$

which yields conservative values of Δx and therefore can be safely adopted for application.

7 REFERENCES

- ABBOTT, M.B., and IONESCU, F. (1967), On the numerical computation of nearly horizontal flows, Jour. of Hydraulic Research, Vol.5, No.2, pp.97-117.
- CUNGE, J.A. (1966), Etude d'un schéma de différences finies appliqué à l'intégration numérique d'un certain type d'équation hyperbolique d'écoulement, Thesis Presented to the Faculty of Science, Grenoble University.
- MAHMOOD, K. and YEVJEVICH, V., ed. (1975), Unsteady flow in open channels, Vol.1, pp. 111-115, Water Resources Publication, Inc., Fort Collins.
- MC BEAN, E. and PERKIN, F. (1975), Numerical errors inwater profile computation, Jour. of Hyd. Div., ASCE, Vol.101, No.HY11.
- YEN, C.L., HSU, N.S. and HSU, M.C. (1979), Influence of channel slope on stability of numerical solution of unsteady flow, Research Report No.HY 6807, Dept. of Civil Engineering, National Taiwan University, Taipei (in Chinese).
- YEN, C.L., WANG, T.W. and YU, P.S. (1977), Mathematical model of flood flow for Cho-Shui River, Research Report, No. HY 6601, Dept. of Civil Engineering, National Taiwan University, Taipei (in Chinese).