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Aquifers with Very High Horizontal Permeability

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SUMMARY A perturbation method is applied to the problem of anisotropic aquifers in which the ratio of horizontal to vertical permeability is very large. Although the ratio is assumed constant here, the permeability can be a slowly-varying function of depth. An 'outer' expansion is used to represent the pressure field in the interior of the permeable medium, while an 'inner' expansion represents the pressure boundary layer close to the ground surface. Use is made of strained coordinates in discussing the question of uniformity in time. The first term of the outer expansion provides a good fit to pressure measurements taken at deep wells in a geothermal field undergoing drawdown.

1 INTRODUCTION

In an earlier paper (Wooding, 1979), a detailed analysis was made of pressure data taken in the Tauhara geothermal field, over a period of about 14 years, during the process of drawdown at the neighbouring Wairakei field. The five observation wells were located at distances of about 2 through 7 km from the main withdrawal area.

A standard aquifer model with uniform properties in the horizontal plane and simple boundary conditions - impermeable upper and lower boundaries, and a straight boundary impermeable to horizontal flow located to the south-east of the field - provided a surprisingly effective 'fit' to the pressure data.

Additional requirements on the model were an initial steady horizontal outflow from the Wairakei field which could be identified with the natural convective flow, and a vertical pressure gradient which would induce a distributed downflow of groundwater if finite vertical permeability existed. This vertical gradient is present in the production zone corresponding to the main aquifer of the model, and appears to be a consequence of the exploitation at Wairakei; such a gradient could be maintained provided that the field is undergoing sustained drawdown, and provided that the fluid mobility (defined as the ratio of permeability to dynamic viscosity) for horizontal flow increases with depth, so that the rate of horizontal propagation of pressure drawdown increases with depth also. A relatively low mobility for vertical flow would also be necessary to account for the observed magnitude of the vertical pressure gradient; a multi-layered field structure which exhibits this property 'in the large scale' has been discussed by Wooding (1978).

In this paper, an approximate theory is developed to treat the case where the ratio of vertical to horizontal fluid mobility is very small, using the method of matched asymptotic expansions.

2 THEORY

The system to be analysed comprises a semi-infinite porous medium ($0 \le z < \infty$), where z is measured vertically downwards in a cylindrical coordinate system (r , z) with origin situated in the surface. A vertical well of diameter 2a and unlimited depth is located at r = 0. (Subsequently $a \to 0$ will be taken so that the well becomes an idealised line source or sink.) It will be assumed that the medium is horizontally stratified, isotropic only in horizontal planes so that the vertical permeability differs from the horizontal, and both are functions of z; let the permeabilities be $k_1(z)$, $k_2(z)$ in the r- and z-directions respectively. Although the depth of the well is unlimited, it may be desirable to limit the flow field in the vertical, either by assuming that $k_1(z)$, $k_2(z)$ tend to zero appropriately as $z \to \infty$, or by letting $k_1 = k_2 = 0$ below some given depth. In either way, a finite well flow rate Q can be specified.

Now, if the dynamic viscosity is $\mu(z)$, the fluid mobility has components $\lambda_1(z) = k_1(z)/\mu(z)$ and $\lambda_2(z) = k_2(z)/\mu(z)$ respectively. From Darcy's law, neglecting gravity,

$$u = -\lambda_1(z)P_r , \quad w = -\lambda_2(z)P_z \qquad (1a,b)$$

where u, w are flow rates in the r- and z-directions, P is the pressure and $(\ )_r \equiv \partial/\partial r$, etc. If the compressibility γ of the porous medium is taken to be constant, the equation of continuity gives

$$\gamma P_t + u_r + u/r + w_z = 0 \qquad (2)$$

in which P is the pressure and t denotes time.

A convenient simplification, justifiable in the absence of specific information to the contrary, is to take $k_1(z)/k_2(z) = \lambda_1(z)/\lambda_2(z) = \varepsilon^2$, say - a constant. That is, the horizontal and vertical permeabilities depend upon only one function of z. Put $\lambda_1(z)/\gamma = D(z)$, $\lambda_2(z)/\gamma = \varepsilon^2 D(z)$. Then elimination of u, w between (1) and (2) leads to

$$P_t = D(z)\Delta_r P + \varepsilon^2\{D(z)P_z\}_z \qquad (3)$$

for the pressure, where D(z) is analogous to a variable diffusivity. Here $\Delta_r \equiv \partial^2/\partial r^2 + r^{-1}\partial/\partial r$.

2.1 Initial and Boundary Conditions

If a flux boundary condition is desired at the well (r = a), this can be applied conveniently as a total flux or flow rate Q(t), into or out of the well. The actual flow rate in the field will be distributed with depth according to the distribution of fluid mobility.

Let the total flux Q be a constant, after being started at $t = 0$. The initial field pressure P can be taken as zero. As gravity is neglected, P is always independent of z at the well.

The other boundary condition of importance is $P = 0$ on the plane $z = 0$, for $t \geq 0$. Other boundaries could be introduced at z = constant > 0, and the conditions there should be amenable to treatment similar to that used for the condition at $z = 0$.

2.2 Integral Transforms

If the Laplace transform of P with respect to time is defined as

$$\overline{P}(r,z,s) = \int_0^\infty P(r,z,t)\, e^{-st}\, dt \qquad (4)$$

then equation (3) gives

$$s\overline{P} = D(z)\Delta_r\overline{P} + \varepsilon^2\{D(z)\overline{P}_z\}_z \qquad (5)$$

The Bessel transform is introduced in equation (25).

2.3 Outer Expansion

Now assume that P, and its integral transforms, can be represented by regular perturbation expansions in powers of ε; for the Laplace transform,

$$\overline{P} = \overline{P}_0 + \varepsilon^2\overline{P}_2 + \ldots \qquad (6)$$

Here a term in $O(\varepsilon)$ could have been included, but it would have obeyed the same differential equation as $\overline{P}_0$. (In some cases its inclusion might still be necessary.) Substituting (6) into (5) gives

$$O(1): \qquad q^2\overline{P}_0 = \Delta_r\overline{P}_0 \qquad (7a)$$

$$O(\varepsilon^2): \qquad q^2\overline{P}_2 = \Delta_r\overline{P}_2 + \{D(z)\overline{P}_{0z}\}_z / D(z) \qquad (7b)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

where $q^2 \equiv q^2(z,s) = s/D(z)$.

A comparison with (5) indicates that each equation of (7) has lost its derivatives of the dependent variable with respect to z. That is, z enters each equation only as a parameter, so that it is more readily solved than the original equation (5). In general, however, equations (7a,b,...) cannot satisfy boundary conditions specified at any given z; (6) has the character of an outer expansion. Unless this expansion satisfies the boundary condition $P = 0$ fortuitously at $z = 0$, the solution must assume a boundary-layer character at small z. This boundary-layer can be represented by an inner expansion with a suitably scaled z-variable.

The coefficients in the outer expansion (6) are calculated successively:

$O(1)$: Let $x(r,z,s) = q(z,s)r$. Then, in the notation introduced at equation (3), (7a) becomes $(\Delta_x - 1)\overline{P}_0 = 0$. Since $P \to 0$ as $r \to \infty$ (from the initial condition $P = 0$), the solution is of the form

$$\overline{P}_0(x,z,s) = A_0(z,s)\, K_0(x) \qquad (8)$$

where K_0 is the modified Bessel function of order zero which is singular at the origin. But P is independent of z on $r = a$, giving $\overline{P}_0 = \overline{P}_0(x_a,s) = \overline{P}_{0a}(s)$, say, and

$$A_0(z,s) = \overline{P}_{0a}(s) / K_0(x_a) \qquad (9)$$

where $x_a \equiv x_a(z,s) = q(z,s)a$.

From Darcy's law, the flux per unit height is given by $F = -2\pi a\lambda_1(z)\, \partial P_0/\partial r|_{r=a}$, and its Laplace transform can be written as

$$\overline{F} = -2\pi a\gamma D(z) q(z,s)\, \overline{P}_{0a}(s)\, K_0'(x_a)/K_0(x_a) \qquad (10)$$

using (9) and a differentiated form of (8). Then the Laplace transform of the total flux Q is

$$\overline{Q} = Q/s = \int_0^\infty \overline{F}\, dz \quad .$$

Combined with (10), this gives an expression for $\overline{P}_{0a}(s)$, the transform of the leading term for well pressure. From (8) ff.,

$$\overline{P}_0(x,z,s) = C_a(z,s)\, K_0(x)/s \qquad (11a)$$

if

$$C_a(z,s) = Q/\{2\pi a K_0(x_a)\gamma\int_0^\infty dz D(z) q(z,s) K_1(x_a)/K_0(x_a)\} \qquad (11b)$$

Now, let $a \to 0$: $\quad C_a(z,s) \to C_0 = Q/(2\pi\Lambda H) \qquad (12)$

where $\Lambda H = \gamma\int_0^\infty D(z)\, dz = \int_0^\infty \lambda_1(z)\, dz$ is the mobility-thickness product. Then, in the limit $a \to 0$,

$$\overline{P}_0(x,s) = C_0 K_0(x)/s \qquad (13)$$

and $\quad P_0(r,z,t) = \tfrac{1}{2}C_0\int_\eta^\infty e^{-v}dv/v = \tfrac{1}{2}C_0 E_1(\eta) \qquad (14)$

is its inverse in terms of the exponential integral (Abramowitz and Stegun, 1965). In (14),

$$\eta = r^2/\{4D(z)t\}\, .$$

In the standard aquifer model, which assumes 'top-hat' permeability profiles, H is the aquifer thickness and Λ is the fluid mobility. If D(z) is constant, equation (14) reduces to the solution for an aquifer of that type, with impermeable upper and lower boundaries. However, although ΛH is readily determined from field observations, it is difficult to establish the separate factors with precision (Grant, 1977a,b; Wooding, 1979).

In the steady state, the solution reduces to

$$P_0(r,z) = \tfrac{1}{2}C_0\log(r_1^2/r^2) + P_{01} \qquad (15)$$

where P_{01} is a reference pressure given at $r = r_1$. This is indistinguishable from the solution for the standard model, although the distribution of flow rate differs in the two cases.

$O(\varepsilon^2)$: In terms of the variable $x(r,z,s) = q(z,s)r$, if it is assumed that $a \to 0$, equation (7b) becomes

$$(\Delta_x - 1)\overline{P}_2 = -\{D(z)\overline{P}_{0z}\}_z/\{D(z)q^2(z,s)\}$$

$$= \frac{-C_0}{4sq^2}\{(D'/D)'2xK_1(x) + (D'/D)^2x^2K_2(x)\} \qquad (16)$$

after substituting for $\overline{P}_0$ from (13) and expanding the righthand side. Here $(\ )' \equiv d/dz$ and K_n signifies a modified Bessel function of order n, singular at the origin.

With the aid of the identity

$$(\Delta_x - 1)\{x^nK_n(x)/2n\} = -\, x^{n-1}\, K_{n-1}(x) \qquad (17)$$

the particular integral of (16) can be written down at once:

$$\overline{P}_2(x,z,s) = \frac{C_0}{24sq^2}\{(D'/D)'3x^2K_2(x) + (D'/D)^2x^3K_3(x)\} \tag{18}$$

The general solution involves a term $A_2(z,s)K_0(x)$ which tends to zero as $x \to \infty$. However, since the total well flux has been specified at the well, it is necessary that

$$\lim_{a \to 0} \int_0^\infty D(z)q(z,s)\,\overline{P}_{2x}(x_a,z,s)\,dz = 0 \tag{19}$$

This condition is satisfied identically if $\overline{P}_{2x}$ is calculated from (18), so that $A_2(z,s) = 0$ in this limiting case. (Note that, in the general case for which $r = a$ is finite, A_2 does not vanish.) Then the $0(\varepsilon^2)$ correction to the pressure is given, by inversion of (18), as

$$P_2(r,z,t) = \frac{1}{24}C_0\{(D'/D)'\,6Dt\,e^{-\eta} + (D'/D)^2\frac{r^3}{(4\pi Dt)^{\frac{1}{2}}}e^{-\frac{1}{2}\eta}K_{\frac{3}{2}}(\tfrac{1}{2}\eta)\} \tag{20}$$

2.4 Inner Expansion

From the form of equation (3), or of its Laplace transform (5), it is evident that the term involving diffusion in the z-direction is significant near the boundary $z = 0$ at distances of $0(\varepsilon)$; therefore, let

$$Z = z/\varepsilon \tag{21}$$

be an inner variable so that (6), for example, becomes

$$s\overline{P} = D(\varepsilon Z)\Delta_r\overline{P} + \{D(\varepsilon Z)\,\overline{P}_Z\}_Z \tag{22}$$

It will be assumed that D possesses a Taylor expansion at $z = 0$. Now, assume that $\overline{P}$, for example, has the inner expansion

$$\overline{P} = \overline{P}^{(0)} + \varepsilon\overline{P}^{(1)} + \ldots \tag{23}$$

Then, if D is expanded in powers of ε, (22) gives

$$0(1):\quad q_0^2\overline{P}^{(0)} = \Delta_r\overline{P}^{(0)} + \overline{P}^{(0)}_{ZZ} \tag{24a}$$

$$0(\varepsilon):\quad q_0^2\overline{P}^{(1)} = \Delta_r\overline{P}^{(1)} + \overline{P}^{(1)}_{ZZ} + (D_0'/D_0)\{Z\Delta_r\overline{P}^{(0)} + (Z\overline{P}^{(0)}_Z)_Z\} \tag{24b}$$

. . . .

where $q_0 \equiv q_0(s) = s/D(0) = s/D_0$.

0(1): Assume that $a \to 0$, and introduce the zero-order Bessel transform

$$\tilde{\overline{P}}^{(0)}(\xi,Z,s) = \int_0^\infty \overline{P}^{(0)}(r,Z,s)\,J_0(\xi r)\,r\,dr \tag{25}$$

Then

$$\widetilde{\Delta_r\overline{P}}^{(0)} = -\lim_{r \to 0}(r\overline{P}^{(0)}_r) - \xi^2\tilde{\overline{P}}^{(0)} \tag{26}$$

Now, $-\lim_{r \to 0}(r\overline{P}^{(0)}_r) = -\lim_{r \to 0, z \to 0}(r\overline{P}_{0r}) = C_0/s$ from (13), and the general solution for the Bessel transform of (24a) is

$$\tilde{\overline{P}}^{(0)}(\xi,Z,s) = A\,e^{\chi Z} + B\,e^{-\chi Z} + C_0/(s\chi^2) \tag{27a}$$

where

$$\chi^2 = q_0^2 + \xi^2 \tag{27b}$$

This solution must be matched to the Bessel transform of the first term of the outer expansion; from (13),

$$\tilde{\overline{P}}_0(\xi,z,s) = (C_0/s)\{q^2(z,s) + \xi^2\}^{-1} \tag{28a}$$

$$= (C_0/s\chi^2)\{1 - 2\varepsilon q_0q_0'Z/\chi^2 + 0(\varepsilon^2)\} \tag{28b}$$

after substituting the inner variable Z and expanding in powers of ε. Therefore, $\tilde{\overline{P}}$ is matched to 0(1) by the particular integral in (27), and $A = 0$.

For the given boundary condition $P = 0$ at $z = 0$, (27) gives

$$\tilde{\overline{P}}^{(0)}(\xi,Z,s) = (C_0/s)(1 - e^{-\chi Z})/\chi^2 \tag{29}$$

and the inverse Bessel transform is

$$\begin{aligned}\overline{P}^{(0)}(r,Z,s) &= (C_0/s)[K_0(q_0r) - \int_Z^\infty d\zeta\,\exp\{-q_0(r^2+\zeta^2)^{\frac{1}{2}}\}/(r^2+\zeta^2)^{\frac{1}{2}}] \\ &= (C_0/s)\int_0^{\sinh^{-1}(Z/r)}\exp(-q_0r\cosh\phi)\,d\phi\end{aligned} \tag{30}$$

Since letting the upper limit of the final integral in (30) tend to infinity gives $K_0(q_0r)$, the integral behaves like an 'incomplete modified Bessel function'.

$0(\varepsilon)$: Taking the zero-order Bessel transform of equation (24b), and substituting for $\tilde{\overline{P}}^{(0)}$ from (29), leads to the equation

$$q_0^2\tilde{\overline{P}}^{(1)} = -\lim_{r\to 0}(r\overline{P}_r^{(1)}) - \xi^2\tilde{\overline{P}}^{(1)} + \tilde{\overline{P}}^{(1)}_{ZZ} + (C_0/s\chi^2)(D_0'/D_0)\{q_0^2Z(1 - e^{-\chi Z}) + \chi e^{-\chi Z}\} \tag{31}$$

Here $-\lim_{r\to 0}(r\overline{P}^{(1)}_r) = -\lim_{r\to 0, x\to 0}(r\overline{P}_{1r})$, where $\overline{P}_{1r}$ is derived from the $0(\varepsilon)$ term in (28b). This limit is zero. Then the solution appropriate to the boundary condition $P = 0$ at $z = 0$ is

$$\tilde{\overline{P}}^{(1)}(\xi,Z,s) = (C_0/s\chi^2)(D_0'/D_0)\{q_0^2Z/\chi^2 + \frac{Z}{4}(2 - q_0^2/\chi^2 - q_0^2Z/\chi)e^{-\chi Z}\} \tag{32}$$

The first term on the righthand side matches the $0(\varepsilon)$ term in the expansion of $\tilde{\overline{P}}_0$ and so, in a uniformly-valid composite expansion, would represent a common term to be dropped. The remaining terms tend to zero exponentially as $Z \to \infty$.

Formally, the inverse with respect to the Bessel transform in expression (32) can be written down as a sum of integrals

$$\overline{P}^{(1)}(r,Z,s) = (C_0/s)(D_0'/D_0)[\tfrac{1}{2}q_0ZrK_1(q_0r) + \tfrac{1}{4}Z\{2F_1 - q_0^2F_3 - q_0^2Z\,F_2\}] \tag{33}$$

where

$$\begin{aligned}F_n \equiv F_n(r,Z,q_0) &= \int_0^\infty \chi^{-n-1}e^{-\chi Z}J_0(\xi r)\,\xi\,d\xi \\ &= \{(n-1)!\}^{-1}\int_Z^\infty(Z-\zeta)^{n-1}d\zeta\cdot\exp\{-q_0(r^2+\zeta^2)^{\frac{1}{2}}\}/(r^2+\zeta^2)^{\frac{1}{2}}\end{aligned} \tag{34}$$

Inversion of the Laplace transforms (30) for $\overline{P}^{(0)}$ and (33) for $\overline{P}^{(1)}$ has not been attempted by analytical means. Several efficient algorithms for numerical inversion are known, and previous experience using the Gaver-Stehfest algorithm (Stehfest, 1970; Sandal et al., 1979) suggest that this method should be fast and accurate. Generally, the algorithm is suited to monotonic functions which include the class of well functions considered here.

2.5 Strained Coordinates

In the outer expansion, the leading term (14) has a logarithmic singularity for $t \to \infty$, while in equation (20) for the second term the righthand side grows asymptotically as t, so that the expansion is not uniformly valid in time. Lighthill's technique may be applied to render the series $P_0 + \varepsilon^2 P_2 + \ldots$ uniform. For example, straining the t-coordinate, let

$$t = \tau + \varepsilon^2 t_2(z,\tau) + \ldots \tag{35}$$

and substitute into (14) and (20) (Pritulo, 1962). However, the simplest form obtainable for t_2 is

$$t_2(z,\tau) = -\{\tfrac{1}{2}(D'/D)' + \tfrac{2}{3}(D'/D)^2\}D\tau^2 \tag{36}$$

which limits t to a finite range of values. This result could be expected on physical grounds since, as the pressure disturbance spreads laterally, the influence of vertical flow becomes increasingly important. Similar considerations would apply to the inner expansion.

3 RELATIONSHIP TO DATA

The analysis described by Wooding (1979) for pressure data from the Tauhara geothermal field can be adapted readily to use the present theory. However, the boundary-layer zone is not within the scope of the theory because two-phase effects are encountered at shallow depths. (In any case, the present observations do not cover that range.) Also, the influence of higher perturbation quantities is undetectable due to scatter in the measurements. Hence only zero-order terms of form (14) and/or (15) appear to be usable.

Let the zero-order pressure-field model be

$$P = AP_0(r,z,t) + \{Q_0/(4\pi\Lambda H)\}\log(r_1^2/r^2) + B \tag{37}$$

where P_0 is given by (14). This term involves D(z), here assumed to be given by

$$\gamma D(z) = \Lambda H\{1 + C(z - \overline{z}_m)\} \tag{38}$$

where $\overline{z}_m$ is the mean value of the z-observations. Then A, B, Q and C are regression coefficients, where C involves a non-linear regression.

Parameter values are taken from Table 2 of Wooding (1979), and the notation is identical except for the mobility $\Lambda = \overline{\lambda}$ and the different use of parameter C. The variance-ratio F as defined in that paper is used to test significance (Beyer, 1966). The field model, which is described briefly in the Introduction, corresponds to model 2 of Wooding (1979) with an impermeable boundary to the southeast of the Tauhara field.

Table 1 shows results from a hierarchy of regression models. Evidently the parameter C, corresponding to a linear variation of fluid mobility with depth, is highly significant. The fitted value is $C = 2.95 \times 10^{-4}$ per metre depth. Permitting the initial well pressures to vary arbitrarily between wells (A, B_i) and the drawdown rates to vary (A_i, B_i) does not significantly improve the goodness of fit.

TABLE 1
SIGNIFICANCE TESTS ON 72 OBSERVATIONS FROM 5 TAUHARA WELLS

Regression Coefficients	Degrees of Freedom	Sum of Squares	F	Significance
A,B	70	40.78		
A,B,Q_0	69	24.28	68.7	$\ll 10^{-3}$
A,B,Q_0,C	68	16.00	34.9	$\ll 10^{-3}$
A,B_i	66	15.69	0.75	not sig.
A_i,B_i	62	14.90	0.82	not sig.

Well number $i = 1,\ldots,5$.

Other fitted coefficients are A = -0.982, giving a permeability thickness of about 35 Darcy-metres (1 Darcy = $10^{-12}m^2$), B = 78.8 bars and Q_0 = 0.45 tonnes/sec. (natural steady source strength).

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