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Use of the Method of Lines for Choking Flow in a Nozzle

C.K. SOMASUNDARAM
Research Fellow, School of Mechanical and Industrial Engineering, The University of New South Wales
and
R.D. ARCHER
Associate Professor, School of Mechanical and Industrial Engineering, The University of New South Wales

SUMMARY Under the constraint of choking mass flow, and considering only an inviscid flow field, the Method of Lines (MOL) has been used to solve for the transonic flow field in axisymmetric nozzles of arbitrary profile. The numerical application is very fast and requires only a good first estimate based on one dimensional flow. No axial or centreline condition other than symmetry is required or imposed. A wall tangency condition is the only other constraint. A solution is illustrated for a throat radius of curvature of 4.

A full flow field solution for the whole nozzle including a length of upstream inlet pipe has also been obtained. Although requiring significantly longer computer time, details of the zones of adverse and favourable pressure gradients are predicted.

Comparisons with some experimental results for both kinds of solution are available.

1 INTRODUCTION

The Method of Lines (MOL) is one means of reducing the gasdynamic equations of motion from simultaneous partial differential equations to an approximating numerical procedure.

The MOL (Ref. 1) has been used elsewhere for the solution of transonic flow around circular and elliptic cylinders without lift (lifting airfoils) and cones (Ref. 2), and the supersonic blunt body (Ref. 5), and is also treated in the general context of numerical methods by Belotserkovskii (Ref. 3) and Holt (Ref. 4).

2 THEORY

Consider the arbitrarily shaped nozzle shown in Figure 1 in which the nozzle wall is a succession of straight lines and circles of arbitrary radii. Note that the problem, as posed, includes an initial length of straight duct or pipe. We write the equation in two dimensional general form to include both plane and axisymmetric cases. The general inviscid, isentropic gasdynamic equations normalised on stagnation or maximum values

$$\nabla \cdot \rho \vec{q} = 0$$

$$K\frac{\nabla p}{\rho} + \nabla\frac{\vec{q}^{\,2}}{2} + \vec{q} \times \nabla \times \vec{q} = 0$$

$$\nabla \times \vec{q} = 0$$

where $p, \rho, \vec{q}$ are pressure, density and velocity and $K = (\gamma-1)/2\gamma$, become for plane $(i=0)$ or axisymmetric $(i=1)$ flow in x, y co-ordinates (Fig. 1):

$$\frac{\partial}{\partial x}\rho u y^i + \frac{\partial}{\partial y}\rho v y^i = 0$$

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{K}{\rho}\frac{\partial p}{\partial x} &= 0 \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{K}{\rho}\frac{\partial p}{\partial y} &= 0 \end{aligned}\right\} \text{ and } \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

The nozzle flow field is evidently bounded at the upstream end by an assumed uniform flow, along the outer wall by a tangency condition and at the centreline by a symmetry condition equation. The location and shape of the downstream sonic line $(q = c^*)$ is initially unknown as also is the choking mass flow, all of which are to be found as part of the solution.

3 NUMERICAL PROCEDURE

Form a rectangular map in ξ, η (Figure 2) using

$$\xi = \frac{x-l_u}{l_d-l_u} \qquad \eta = \frac{y}{B(x)}$$

Thus

$$\frac{\partial\xi}{\partial x} = \frac{1}{l_d-l_u} = X_x \qquad \frac{\partial\xi}{\partial y} = 0 = X_y$$

$$\frac{\partial\eta}{\partial x} = -\eta B'(x)(E_y) = (E_x) \qquad \frac{\partial\eta}{\partial y} = \frac{1}{B(x)} = (E_y)$$

Now the gasdynamic equations can be written as

$$A\frac{\partial V}{\partial\eta} + C = 0$$

where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ $V = \begin{pmatrix} u \\ v \end{pmatrix}$ $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

a, b, c are functions of $u, v, \frac{\partial u}{\partial\xi}, \frac{\partial v}{\partial\xi}, X_x, X_y, E_x, E_y$. The latter two contain B and B'.

We divide the flow field up by a small number of $m[O(10)]$ lines parallel to η (y axis). The flow equations are written along the $m+1$ lines, including the initial and final boundaries, forming the system of $m+1$ equations,

$$A_j\frac{dV_j}{d\eta} + C_j = 0 \qquad 1 \leqslant j \leqslant m+1$$

Next, finite difference approximations are written for the ξ derivatives yielding a system of simultaneous ordinary differential equations which must satisfy $2(m+1)$ sets of boundary conditions at each end of each line. If we start integration at one end of the lines, say at $\xi = 1$, we must adopt an optimisation method to minimise the numerical value of

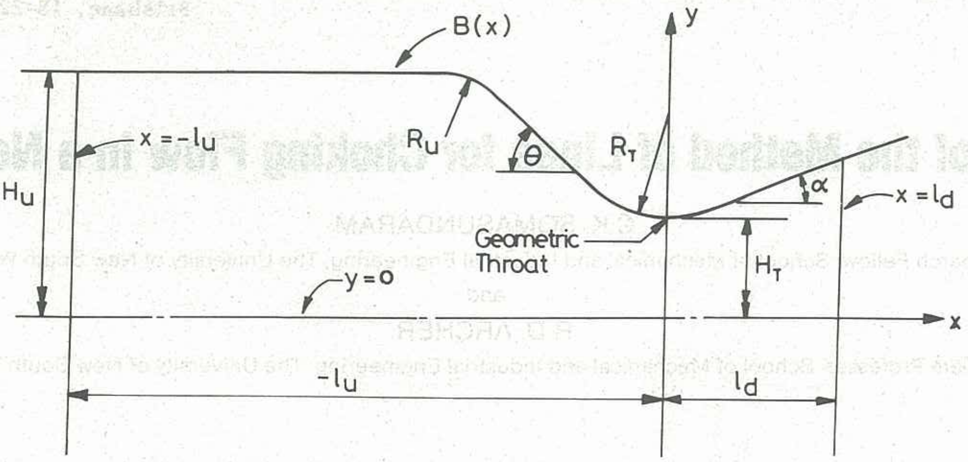

Figure 1 Co-ordinate system for nozzle flow

the residuals ε_k representing the difference between calculated and known boundary values at the other end $(\xi = 0)$ of the $m+1$ lines. The Newton-Raphson method can be used, for example,

$$\sum_{j=1}^{m+1} \Delta u_j \frac{\partial f_k}{\partial u_j} + \varepsilon_k = 0 \qquad (k = 1, \ldots, m+1)$$

to find corrections Δu_j to u_j to satisfy the boundary condition $f_k = 0$. This investigation has used lines parallel to η axis, various finite different schemes and variable line spacing.

4 DISCUSSION

A numerical solution for the throat region alone of the axisymmetric nozzle was obtained for a maximum of 11 lines before instability occurred. Fewer than 6 lines produced noticeable pressure differences but the most consistent mass flow stream function along the wall, when used as a test of the quality of the result, was obtained for 11 lines. Computer time on CYBER 72 was about 12 seconds involving 13 iterations. A numerical solution for the complete nozzle, including an upstream length 19.5 (all lengths are normalised on the throat radius) of constant diameter pipe required much longer computing time $(\times 30)$, the maximum number of lines for stable solution was 26 and the number of iterations required was 15. However, higher order accuracy was involved in order to obtain satisfactory uniformity of the stream function at the wall.

Comparison with MIR (Ref. 3) and an experiment for the same nozzle given above, is shown in Figure 3. A feature of this is the prediction by the MOL of the modification to the flow field in the approach to the inlet to the nozzle, or 'precursor effect', involving a length of positive unfavourable wall pressure gradient. The estimated accuracy in measurement is $\pm 4 \times 10^{-5}$ on nominated wall pressure. Agreement of the MOL solution with experiment is good over the whole range of flow to the sonic line (Fig. 3a). However, on the expanded scale of Fig. 3b, the measurement error is not enough to explain the difference between experiment and MOL, although the position, extent and nature of the local hump in the wall pressure distribution in the region of the inlet is well predicted. Mach number contours and streamlines for the full flow field up to and including the sonic line are given in Figure 4.

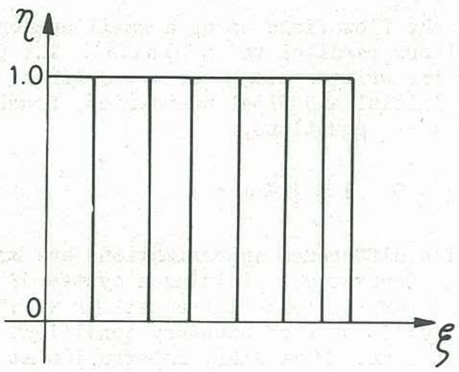

Figure 2 Transformed domain of flow field with lines of integration along η

5 CONCLUSION

The inviscid compressible flow equations have been solved by the Method of Lines for a choked axisymmetric nozzle. The numerical solution has succeeded in revealing essential features of the flow in a nozzle of throat radius of curvature of 4.

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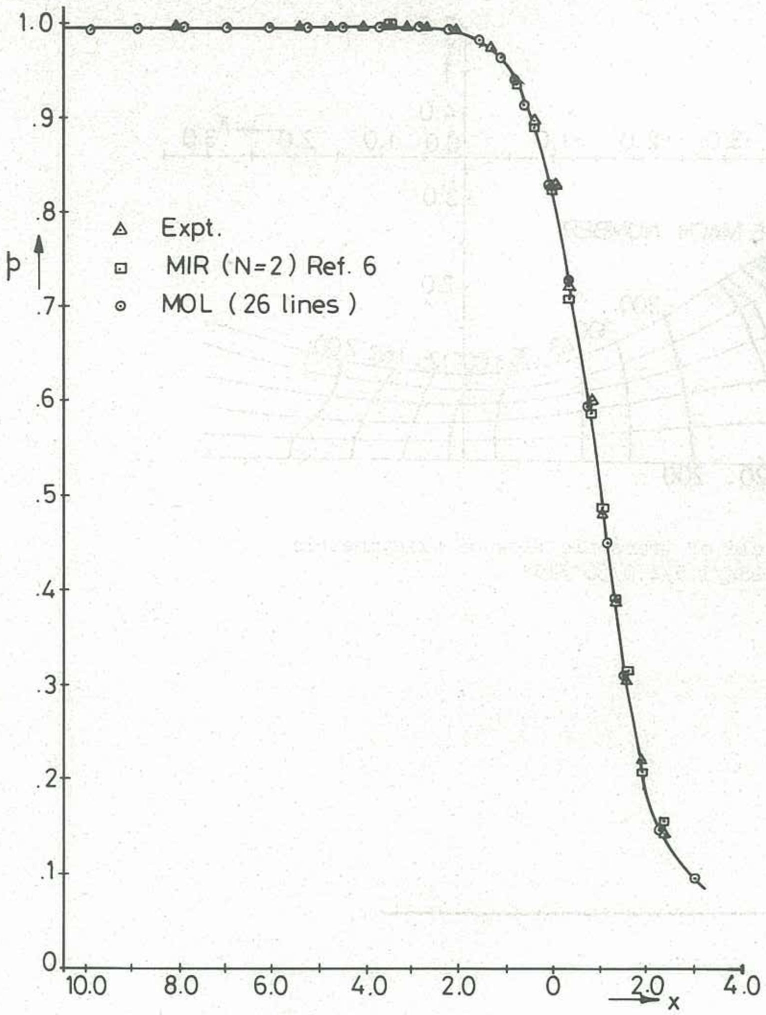
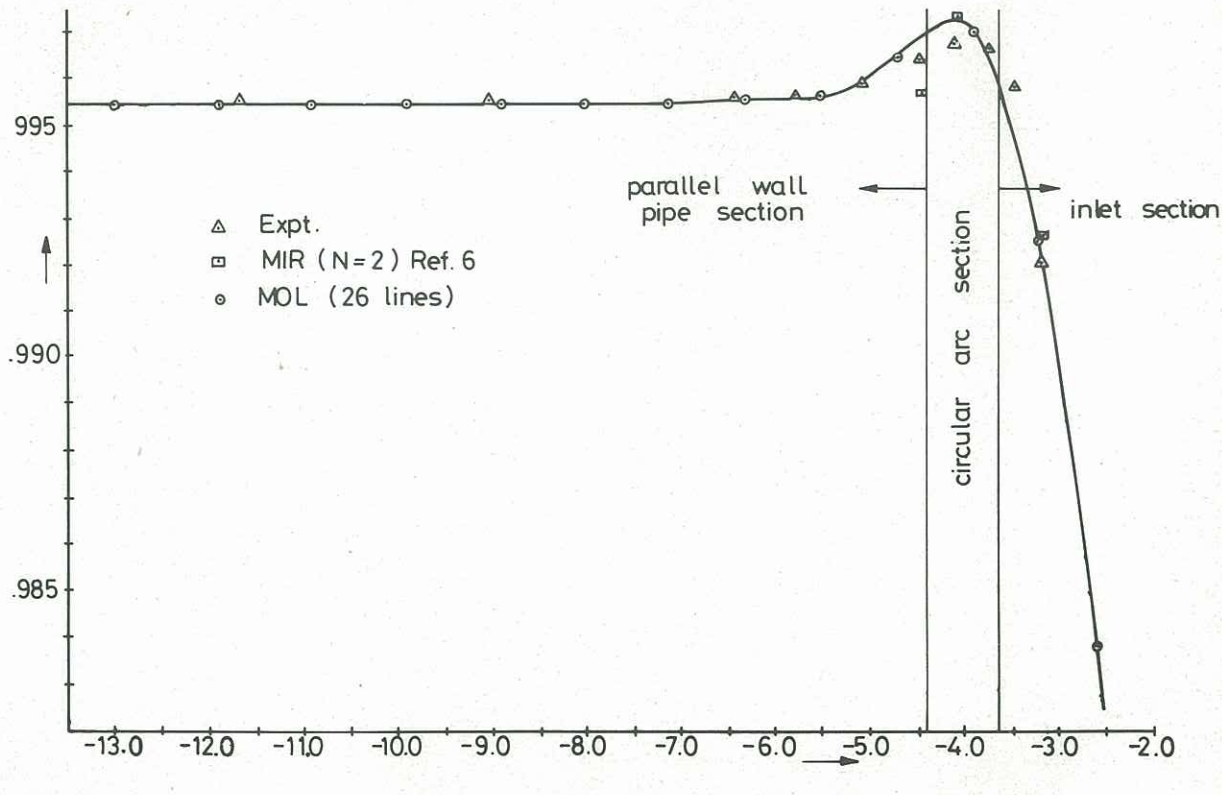


(a) Comparison with experimental results and MIR solutions along the wall of the axisymmetric nozzle



(b) Comparison with experimental and MIR solution along the wall in the inlet region of the axisymmetric nozzle

Figure 3 Predicted pressure distribution for transonic flow in nozzle 2.684/1.5/4.0/30°/15°

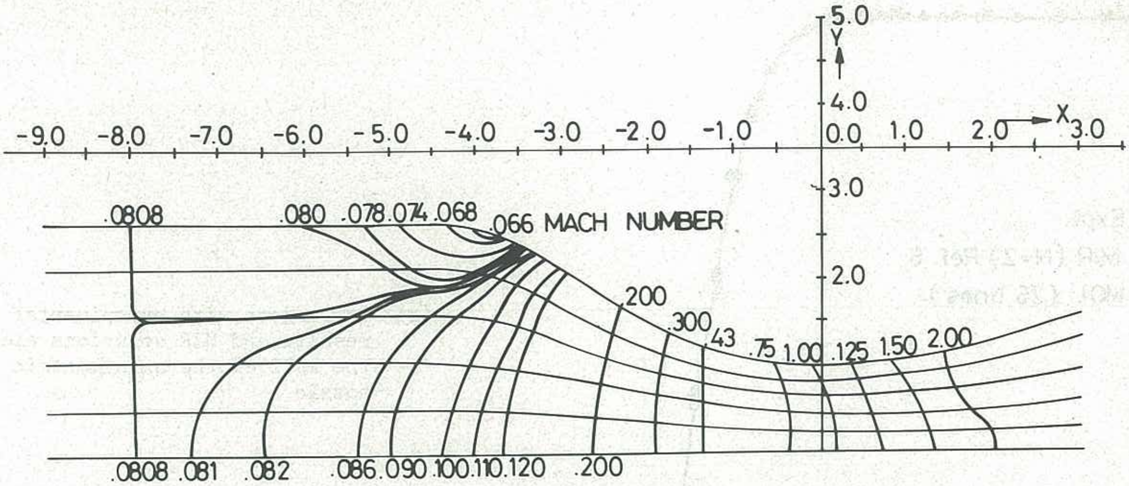


Figure 4 Predicted flow field of transonic flow of axisymmetric nozzle 2.684/1.5/4.0/30°/15°