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Computer Simulation of Water Waves and the Natural Frequencies of a Submerged Cylinder

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1 INTRODUCTION

The vertical force on a cylinder submerged in a uniform stream was first studied by Havelock [1] in which a doublet is used to represent the cylinder. It is found that this vertical force is directed downward when the doublet is near to the free surface; this force changes to upward direction beyond a certain depth of submersion before it vanishes at greater depth. In other words, there is a range of submersion where the rate change of this vertical force with respect to the depth is positive. Thus one would respect the doublet (representing a cylinder) to possess a natural frequency of oscillation in the vertical direction.

Tuck [2] extends the solution for doublet waves by considering the non-linear effects. He uses a series of images of the doublet which is generated by successive reflections across the free surface in order to obtain a closer approximation for the flow around the cylinder. He discovered that no closed streamline can be formed by a finite number of images generated in this way. As each additional image also introduces its wave system, a closed circular streamline will contain all the images as singularities within it and a complete free surface wave system. It is essential that all these singularities and their waves system be considered when evaluating the forces on the cylinder. An evident example is in the evaluation of forces by the Blasius formula depends on these singularities. Further, for non-steady flow, for example [3], the Blasius formula will only be meaningful if the perimeter of the cylinder is a streamline. These requirements can be met by numerical mean and will be discussed in the next section.

The frequency at which the cylinder will oscillate will also depend on the wave radiated by its oscillation. However, if one is interested in the frequency of the cylinder oscillating with very small amplitude, then the radiating waves system is negligible.

This frequency f is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}} \qquad (1)$$

where k is the first derivative of the vertical force with respect to the depth of submersion, and M is the effective mass of the cylinder

2 FORMULATION AND NUMERICAL SCHEME

2.1 The Free Surface Equations

Consider that the amplitude of the free surface wave is small compared with both its wavelength and the depth in which the cylinder is submerged. Thus the condition on the waves can be extrapolated from that along the calm free surface. Let the x-axis be the horizontal axis and u(x) and v(x) are the horizontal and vertical wave perturbed velocity components on this calm free surface. The Poisson Integral relates u(x) and v(x) as follows :-

$$v(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(x^*)}{x-x^*}\,dx^* \qquad (2)$$

where the Cauchy Principal Value of this is to be considered.

A circle of radius 'a' representing the cylinder has its centre at the origin of the cartesian coordinate system. The calm free surface is parallel to the x-axis and situated along the line whose ordinate is 'd' which is the depth of submersion. When u(x) is known along the calm free surface, the use of equation (2) implies a bounded vortex sheet with vorticity distribution 2u(x). Let U be the velocity of the free stream and denoting the complex variable z = x + iy, the complex velocity vector q of the flow. After making use of the Miln-Thomson circle theorem, the conjugate complex velocity $\overline{q}$ is given by :

$$\overline{q} = U - \frac{Ua^2}{z^2} + \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{u(x^*)dx^*}{z - z^*} \qquad (3)$$

$$- \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{u(x^*)dx^*}{z - \frac{a^2}{z^*}} + \frac{1}{\pi i\, z}\int_{-\infty}^{\infty} u(x^*)dx^*$$

where z* = x* + i d.

It can be easily verified that the perimeter of the circle is a streamline and there is no net circulation within this circle.

On the free surface z = x + id, let u'(x) − i v'(x) represent the conjugate complex velocity which arrives from the image of the free surface :

$$u'(x) - i\,v'(x) = -\frac{1}{\pi i}\cdot\int_{-\infty}^{\infty}\frac{u(x^*)\,dx^*}{x+id - \frac{a^2}{z^*}}$$

$$+ \frac{1}{\pi i}\frac{1}{x+id}\int_{-\infty}^{\infty} u(x^*)\,dx^* \qquad (4)$$

and for the velocity components due to the doublet term :

$$u''(x) - iv''(x) = \frac{U a^2}{(x+id)^2} \qquad (5)$$

Let $\phi(x,t)$ be the velocity potential on the free surface excluding the doublet term and let $h(x,t)$ be the wave height. The linearized force surface condition and kinematic condition are given by :

$$\phi_t = - g h - U[u(x) + u'(x) + u''(x)] \qquad (6)$$

and

$$h_t = v(x) + v'(x) + v''(x) - U h_x \qquad (7)$$

from the above definition of ϕ we have

$$\phi_x = u(x) + u'(x) \qquad (8)$$

2.2 Numerical Scheme

Equations (6) and (7) can be solved numerically as an initial value problem. The numerical approximation for the singular integral as shown in (2) is carried out by discretizing the free surface into a finite element of equal length Δx. Let x_j be the mid points of the j^{th} segment for $j = 1,2, \ldots n$. These n segments cover a sufficiently large interval such that the contributions of the integral beyond this interval are not significant. Thus,

$$V(x_j) = \frac{1}{\pi} \sum_{k=1}^{n} \int_{x_k - \frac{1}{2}\Delta x}^{x_k + \frac{1}{2}\Delta x} \frac{u(x^*)}{x_j - x^*} dx^* \qquad (9)$$

Within each segment, from $x_k - \frac{1}{2}\Delta x$ to $x_k + \frac{1}{2}\Delta x$, the value of $u(x)$ is approximately

$$u(x) = u_k + \left[\frac{\partial u}{\partial x}\right]_{x_k} (x - x_k) \qquad (10)$$

Hence each term in the summation of equation (9) can be expressed as :

$$\int_{x_k - \frac{1}{2}\Delta x}^{x_k + \frac{1}{2}\Delta x} (\ldots.)\, dx^* = - \left[\frac{\partial u}{\partial x}\right]_{x_k} \Delta x$$

$$- (1 - \delta_{jk}) \left\{ u_k + \left[\frac{\partial u}{\partial x}\right]_{x_k} (x_j - x_k) \right\}$$

$$. \ell n \left[\frac{x_j - x_k - \frac{1}{2}\Delta x}{x_j - x_k + \frac{1}{2}\Delta x} \right] \qquad (11)$$

where δ_{jk} is the Kronecker delta.

Note that the values of $\left[\partial u/\partial x\right]_{xk}$ in (11), ϕ_x in (8) and h_x in (7) are derived by numerical differentiations. A fourth order Runge-Kutta method is used for the time wise integration of equations (6) and (7). Equation (8) has to be solved by an iteration process whenever $u(x)$ and $u'(x)$ are to be separated.

2.3 The Blasius Formula

The horizontal force X and the vertical force Y can be evaluated from the Blasius Formula :

$$X - i Y = i \rho \oint \frac{\partial w}{\partial t} dz - \frac{i \rho}{2} \oint q^2 dz \qquad (12)$$

where $\frac{\partial w}{\partial t}$ is the time derivative of the complex velocity potential given by,

$$\frac{\partial w}{\partial t} = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x^*)\, \ell n(z - z^*)\, dx^*$$

$$- \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x^*)\, \ell n \left(\frac{a^2}{z} - \bar{z}^* \right) dx^* \qquad (13)$$

The free surface condition (6) is used to express $\frac{\partial}{\partial t} u$ in terms of other variables, thus

$$\frac{\partial}{\partial t} u(x) = - g h_x - U \frac{\partial}{\partial x} [u + u' + u'']$$

$$- \frac{\partial}{\partial t} u' \qquad (14)$$

By means of equation (4) to replace u' and $\frac{\partial}{\partial t} u'$, a monotonic decreasing series can be generated.

In general, the integration of the time derivative term of the Blasius Formula [(12) and (13)] will depend on the choice of the cut in the complex plane. However, as no net circulation is found within the cylinder, a unique value is obtained. Let this term be denoted by $X_1 - i Y_1$ and $X_2 - i Y_2$ represents the remainder of (12), we have,

$$X_1 - i Y_1 = 2 i \rho a^2 \int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial t} u(x^*) dx^*}{z^*} \qquad (15)$$

and

$$X_2 - i Y_2 = i \rho \quad 2 U a^2 \int_{-\infty}^{\infty} \frac{u(x^*) dx^*}{z^{*2}}$$

$$- \frac{2}{\pi i} \cdot \int_{-\infty}^{\infty} u(x^*) dx^* \cdot \int_{-\infty}^{\infty} \frac{u(x^*) dx^*}{z^*}$$

$$- \frac{2}{\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{u(x^*) u(x^{**})\, dx^{**}}{\frac{a^2}{z^{**}} - z^*} \qquad (16)$$

Numerical treatments of the above integral are consistant with that used for equation (2).

3 RESULTS AND DISCUSSION

The waves generated by a cylinder of radius $a = 0.3\ U^2/g$ with depth of submersion ranging from $d = 1.0\ U^2/g$ to $2.5\ U^2/g$ are computed. The initial conditions are chosen to be $u(x) = 0$ on the free surface. A sample result is shown in figure 1. As the computation proceeds, waves begin to develop with a wavefront travelling downstream at the group velocity $0.5\,U$. A dominant sinusoidal wave form is established very quickly which has the wavelength and amplitude of that steady state solution [4]. These are :

$$\lambda_s = 2\pi U^2/g \qquad (17)$$

$$A_s = 4\pi \frac{g}{U^2} a^2 \exp(-g\,d/U^2) \qquad (18)$$

There is another wave train with a wavelength twice the value of λ_s and an amplitude just under $0.2\ A_s$. This amplitude diminished slowly with time. This transient waves system is found to be dependent on the initial conditions. As one observes downstream of the cylinder, the amplitude of the waves increases to $1.3\ A_s$ and then reduces gradually to zero towards the wave front.

The forces are evaluated from equation (15) and (16). The plots of horizontal force $X(= X_1 + X_2)$ and vertical force $Y(= Y_1 + Y_2)$ with respect to time are shown in figures 2 and 3. Due to the existence of unsteady components in the forces, these lines oscillate in phase for different values of depth d. The amplitude reduces as depth increases. For depth equal to $2.5\ U^2/g$, the mean value of horizontal force is $2.72 \times 10^3\ \rho\ U^4/g$ and is 26% higher than the wave resistance predicted by linearized steady doublet waves system. This discrepancy increases to 46% when the depth is $1.0\ U^2/g$. For vertical force, Y, averaged over time, the comparison with Havelock's result [1] for linearized steady doublet waves system shows that the magnitude of the present result is larger by 12% at depth equal $2.5\ U^2/g$ and 23% at $1.0\ U^2/g$.

The average value of the vertical force Y and its derivative with respect to depth k are plotted in figure 4. The value of k is used as a first approximation for the estimation of the frequency in vertical oscillation. Thus in the absence of damping and external forces, the equation of motion of the cylinder is :

$$M\ddot{y} + k\,y = 0 \qquad (19)$$

where y is the vertical displacement of the cylinder from the origin and M its effective mass. When the depth is greater than $2\ U^2/g$, the value of k becomes negative and hence the cylinder has no natural frequency beyond this depth but the force tends to move it towards the free surface. As the depth decreases from $2\ U^2/g$ to $1.25\ U^2/g$, the value of k increases from zero to $0.04\ \rho\ U^2$. As an example, a neutrally buoyant cylinder under this unsteady condition will have an effective mass close to twice the value of its added mass, i.e. $2\pi\rho a^2$. As $d = 0.3\ U^2/g$ and the natural frequency f for this case is given by

$$0.281\ \frac{g}{U^2}\sqrt{\frac{k}{\rho}}\ .$$

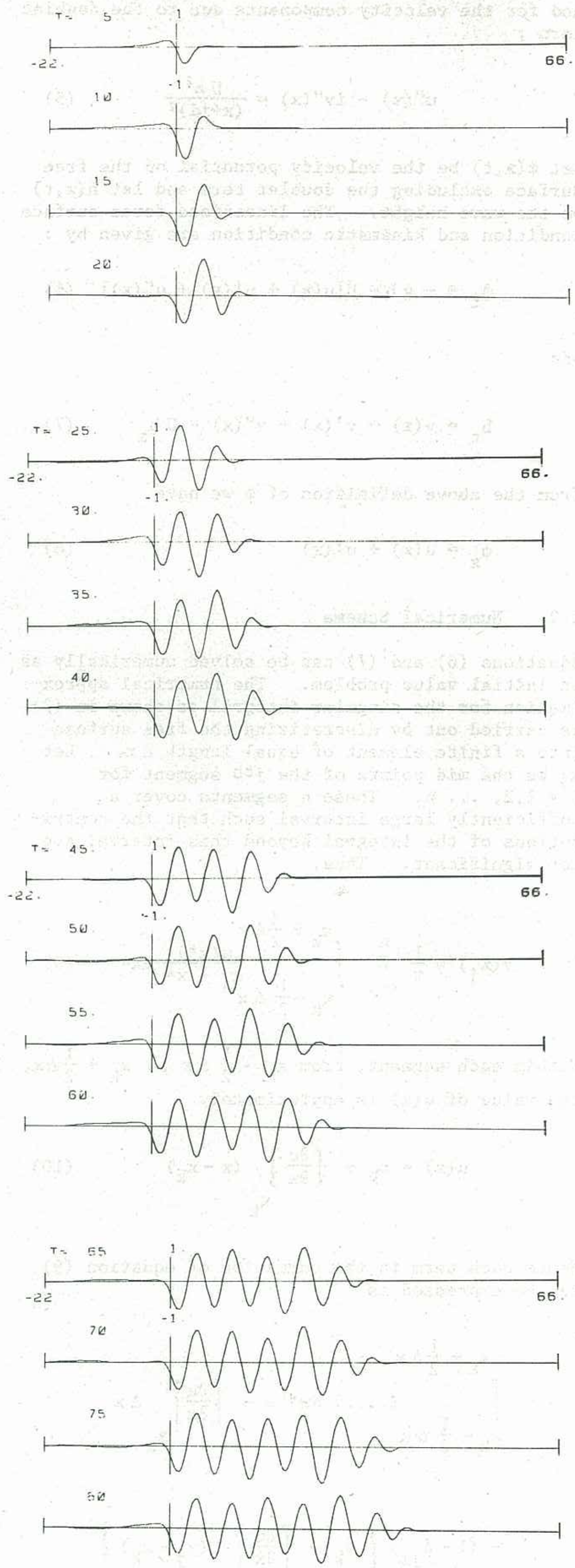

Figure 1. Waves generated by cylinder at various time $T = t\ g/U$. The abscissa xg/U^2 is the calm free surface. The ordinates h/A_s is the non-dimensionalized wave height. Radius of cylinder $ag/U^2 = 0.3$, depth of submersion $dg/U^2 = 2.5$.

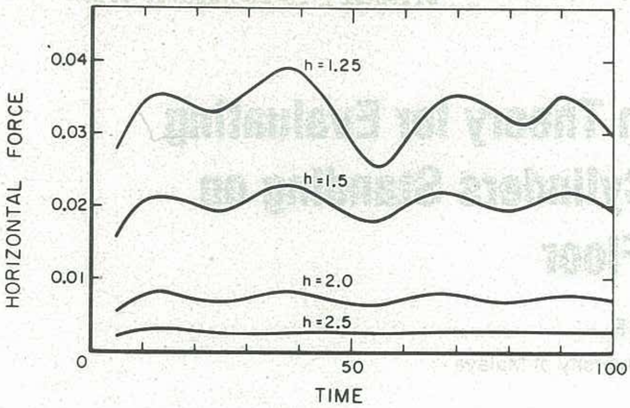

Figure 2. Plots of horizontal forces X g/(ρU^4) against time t g/U for depth dg/U^2 = 1.25, 1.5, 2.0, 2.5. Radius of cylinder ag/U^2 = 0.3 .

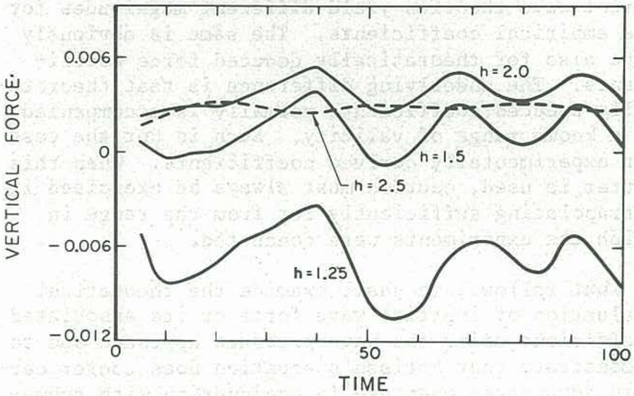

Figure 3. Plots of vertical force Y g/(ρU^4) against time t g/U for depth dg/U^2 = 1.25, 1.5, 2.0, 2.5: Radius of cylinder ag/U^2 = 0.3.

4 CONCLUDING REMARKS

It has been demonstrated that the boundary condition of a submerged cylinder is important and must be enforced in order to compute the force exerted by the waves on the cylinder. The estimation of the natural frequency of a submerged cylinder can only be considered as a first approximation. The numerical method presented here has more general application in the simulation of transient water waves. It can be applied to the case of an oscillating cylinder. The equation for oscillating doublet and the corresponding images will have to be introduced into equations (4) and (5). Finally, the

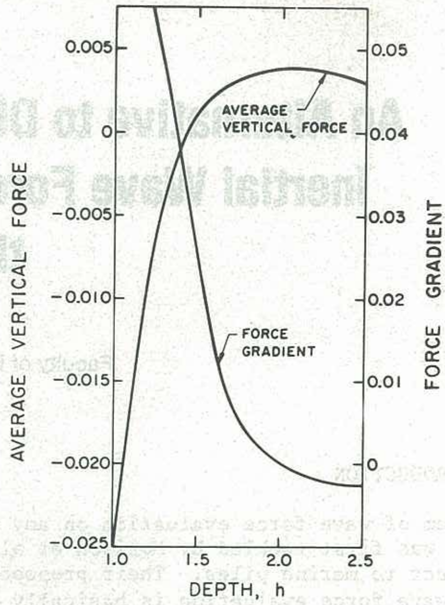

Figure 4. Plots of average vertical force Yg/(ρU^4) and its gradient k/(ρU^2) against time tg/U.

presence of vorticities shed by the cylinder is known to produce oscillating forces on the cylinder as shown in the work by Sarpkaya and Shoaff [5]. The extent in which the surface wave force is affected by these shed vorticity required further investigation.

REFERENCES

[1] HAVELOCK, T.H. (1929). The vertical force on a cylinder submerged in a uniform stream. Proceeding of the Royal Society, London, A, vol. 122, pp 297-303.

[2] TUCK, E.O. (1965). The effect of non-linearity at the free surface on flow past a submerged cylinder. Journal of Fluid Mechanics, vol. 22, part 2, pp 401-414.

[3] HAVELOCK, T.H. (1949). The resistance of a submerged cylinder in accelerated motion. Quarterly Journal of Mechanics and Applied Mathematics. vol. II, part 4, pp 545-553.

[4] LAMB, H. (1932). Theoretical hydrodynamics. 6th edition. Cambridge University Press, pp 410-412.

[5] SARPKAYA, T. and SHOAFF, R.L. An inviscid model of the two dimensional vortex shedding for transient and asymptotically-steady separaged flow over a cylinder. 17th Aerospace Science Meeting, New Orleans, La. January, 1979.