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Damping and Stability of Orifice Plate Surge Tanks by Approximate Analytical Technique

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SUMMARY A general criterion of stability for orifice plate surge tanks is developed by analytical integration of the equations of motion using the method of Krylov-Bogoliubov. Predicted maximum amplitude of oscillation is compared with experimental and numerical data.

1 INTRODUCTION

It is possible to improve the performance of a surge tank by fitting a restricting orifice at its entrance. The orifice increases the retardation of the flow in the tunnel in the case of a load rejection and accelerates the flow for a load acceptance. The dimension of the orifice depends among other things on the allowable pressure at the junction of the surge tank and the tunnel and on the maximum value of the pressure transmitted past the surge tank into the tunnel, the coefficient of transmission being inversely related to the size of the orifice. If an orifice surge tank is considered at the design stage of a hydropower station, then it is necessary to examine the effect of the orifice on the stability of operation and to assess the influence of the restriction on the damping of the surge tank. Both problems, stability of operation and damping have been previously studied for the case of a simple surge tank, under a variety of operational conditions. The major contributions are due to Thoma (1911) for small oscillations, more recently by Jaeger (1943) and Ruus (1969) for large oscillations. In these studies a number of basic premises were adopted, namely that :

i) The governor regulating the discharge in the penstock acts to keep the total power constant.
ii) The governor is assumed to react instantaneously to load fluctuations.
iii) The turbines have practically constant efficiency.
iv) The power plant is isolated from the grid.

Although it is not possible to assume that these conditions are wholly met in practice, the analysis based on these assumptions has led to useful design rules, namely Thoma's criterion for the stability of simple surge tanks under small oscillations.

Only a limited amount of work has been devoted to the problem of stability of orifice surge tanks. The most important contributions have been those of Escande (1952), Zicman (1953) and Zienkewicz (1956). They solved the problem of integrating the equations of motion by further assuming that the pressure differential across the orifice is proportional to the first power of the velocity, thus reducing the equation of motion to a linear second order differential equation. It is possible to solve the problem of stability without this additional simplifying assumption by attempting a purely numerical solution, as advocated by Mosonyi (1964) and Forster (1962). The numerical solution has the disadvantage that no general criterion of stability can be deduced from particular solutions.

In this paper the problems of stability and damping of the orifice surge tank are solved by the approximate analytical technique of Krylov-Bogoliubov, a technique developed in the 1930's for the solution of second order non linear differential equations of mechanics and electricity (Krylov-Bogoliubov 1943). In using this technique there is no need to introduce the linearizarion of orifice head loss , as done by Escande, Zicman and Zienkewicz. Thus it should provide a more realistic stability criterion than those obtained by these authors. It is found analytically that orifice surge tanks have much wider margins of stability than simple surge tanks. This prediction is compared with the results of "exact" numerical integrations of the equations of motion, and with the only reported case of experimental tests under the set of basic operating conditions i) to iv).

2 ANALYSIS

1-1 .- Consider the case of a surge tank isstallation as shown in Fig. 1 . The equations of motion for this system are the equations of conservation of mass and momentum, which are for the tunnel :

$$\frac{dU}{dt} = \frac{g}{L}\left(H_o - \frac{P}{\gamma}\right) - \frac{f_t}{2D} U|U| \qquad (1)$$

$$V = \frac{dz}{dt} = U\frac{A}{A_s} - \frac{Q}{A_s} \qquad (2)$$

(1) and (2) are supplemented by the momentum equation for the surge tank

$$\frac{dV}{dt} = \frac{g}{z}\left(\frac{P_1}{\gamma} - z\right) - \frac{f_s}{2D_s} V|V| \qquad (3)$$

together with the relation expressing the pressure differential across the orifice

$$\frac{P - P_1}{\gamma} = \frac{1}{c_D^2}\left(\frac{A_s}{A_o}\right)^2 \frac{V|V|}{2g} \qquad (4)$$

and by the discharge equation obtained from the condition of governing at constant power

$$Q = \frac{\zeta_o Q_o H_n}{\zeta H} \simeq \frac{Q_o H_n}{H} = \frac{Q_o H_n}{H_1 + P/\gamma} \qquad (5)$$

1-2 .- Change of Variables : As we are interested in the stability of operation of the system around the steady state situation, the vertical origin is moved to the steady state water level, and a new vertical ordinate is defined by :

$$S = z - (H_o - y_o) \qquad (6)$$

Combining now (4) and (3) and introducing the expression for Q from (5) into the continuity equation (2) , it is found that :

$$\frac{P}{\gamma} = \left(S + (H_o - y_o)\right)\left[1 + g\frac{dV}{dt} + \frac{f_s}{2D_s}\frac{V|V|}{g}\right] + \frac{1}{c_D^2}\left(\frac{A_s}{A_o}\right)^2 \frac{V|V|}{2g} \qquad (7)$$

and :

$$V = \frac{dS}{dt} = U\frac{A}{A_s} - \frac{Q_o H_n}{A_s H} \qquad (8)$$

1-3 .- Introduction of Dimensionless Variables: It is convenient to rewrite the system of equations

(1),(7) and (8) in a non dimensional basis, using as reference quantities:
Velocity Scale U = Steady State Tunnel Velocity
Time Scale T = $(LA_s/gA)^{1/2}$
Length Scale $Z_* = U_0 (LA/gA_s)^{1/2}$
The following dimensionless variables are thus defined

$$u = \frac{U}{U_0} \;;\; v = \frac{V}{U_0 A/A_s} \;;\; \eta = \frac{S}{Z_*} \;;\; \tau = \frac{t}{T} \;;\; q = \frac{H}{H_n}$$

It can be shown that under this scaling, the magnitude of the surge tank's inertia and frictional terms in (7) is very small, so that (7) can be written as:

$$\frac{P}{\gamma Z_*} = \left(\eta - \frac{H_0 - y_0}{Z_*}\right) + r_0\, v|v| \qquad (9)$$

where:

$$r_0 = \left(\frac{A}{A_0}\right)^2 \frac{1}{C_D^2} \frac{U_0^2}{2gZ_*} \qquad (10)$$

The dimensionless discharge q is now written in terms of the gross head H_G and the new dimensionless variables as :

$$q = \left(1 + \left(\frac{Z_*}{H_G - y_0}\right)\left[\eta - r_0 v|v|\right]\right)^{-1} \qquad (11)$$

It can be noticed from (9) and (11) that the characteristics of the system are defined in terms of three length parameters: Z ,H ,y .They may be combined into two dimensionless coefficients :

$$\alpha = \frac{y_0}{H_G} \quad \text{and} \quad \beta = \frac{Z_*}{H_G} \qquad (12a,b)$$

Typical values for these coefficients are, for α a range from 0 to 0.12,for β a range from .15 to .5 . Introducing these definitions into the equations of motion , one obtains :

$$\frac{du}{d\tau} = \frac{\alpha}{\beta}(1 - u|u|) - \eta - r_0 \frac{d\eta}{d\tau}\left|\frac{d\eta}{d\tau}\right| \qquad (13)$$

$$\frac{d\eta}{d\tau} = u - \left(1 + \frac{\beta}{1-\alpha}\left(\eta - r_0 \frac{d\eta}{d\tau}\left|\frac{d\eta}{d\tau}\right|\right)\right)^{-1} \qquad (14)$$

1-4 .- Initial conditions of the system : For the study of the stability of the system it is assumed that for $t \leq 0$ the system is under steady state conditions, that is, has values of :

$$u = 1 \qquad \eta = 0 \qquad (15)$$
$$\frac{du}{d\tau} = 0 \qquad \frac{d\eta}{d\tau} = 0$$

At t=0 a perturbation is introduced, consisting in a change in the rated power to a fraction ϕ of the initial power. This variation is described by :

$$\text{Power}(\tau) = \text{Power}(0)\left[1 - (1-\phi)f(\tau)\right] \qquad (16)$$

where $f(\tau)$ is an arbitrary function of time, but such that $f(0)=0$ and $f(\tau \geq \tau_1) = 1$. Here τ_1 is the time assigned for the change in load .

1-5 .- The Stability Problem : In order to test the stability of the system, a value of $\phi \neq 1$ is introduced in (16), together with an appropriate function $f(\tau)$. The solution of the system (13) and (14) with the initial conditions (15) and (16) will answer the questions of the magnitude of u and η with time and whether these values return to steady state levels under the new load regime .There is no difficulty in obtaining a numerical solution of the system of equations, which will be used as a check for the analytical solution herein discussed .

2-1 .- Analytical Solution of the Equations of Motion
Although no exact solution to the equations of motion is presently available, it is possible to derive expressions for the amplitude of the oscillation by means of the method of Krylov-Bogoliubov (K-B)[1]. The system of equations (13) and (14) is first reduced to a standard form by eliminating u and expanding the fraction in (14), resulting in :

$$\frac{d^2\eta}{d\tau^2} + \left(1 - \frac{2\alpha\phi^2}{1-\alpha}\right)\eta + \Big\{\left(\frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)\frac{d\eta}{d\tau} + \Big(\left(1 - \frac{2\alpha\phi^2}{1-\alpha}\right) r_0 \operatorname{sgn}\dot\eta$$

$$+ \frac{\alpha}{\beta}\Big)\left(\frac{d\eta}{d\tau}\right)^2 + \left[\frac{\alpha}{\beta} - \frac{\beta}{1-\alpha}\right]\left(- \frac{2\beta\phi r_0}{1-\alpha}\right)\left(\frac{d\eta}{d\tau}\right)^3 \operatorname{sgn}\dot\eta +$$

$$\left[- \frac{3\alpha\beta r_0}{(1-\alpha)^2}\phi^2\right]\left(\frac{d\eta}{d\tau}\right)^4 + \cdots \Big\} = \frac{\alpha}{\beta}(1-\phi^2) \qquad (17)$$

an equation that may be written as :

$$\frac{d^2\eta}{d\tau^2} + \omega^2\eta + F(\eta,\dot\eta,\ddot\eta) = \frac{\alpha}{\beta}(1-\phi^2) \qquad (18)$$

the function $F(\eta,\dot\eta,\ddot\eta)$ standing for the terms within curly brackets in (17). The K-B method assumes a solution of the type:

$$\eta = a(\tau)\sin(\omega\tau + \delta(\tau)) + \frac{\alpha}{\beta}(1-\phi^2) \qquad (19a)$$

together with:

$$\frac{d\eta}{d\tau} = a(\tau)\,\omega\cos(\omega\tau + \delta(\tau)) = a\omega\cos\psi \qquad (19b)$$

$a(\tau)$ is a time dependent amplitude and $\delta(\tau)$ is a time depemdent phase angle. These terms can be approximated by :

$$\frac{da}{d\tau} = -\frac{1}{2\pi\omega}\int_0^{2\pi} F(a\sin\psi, a\omega\cos\psi)\cos\psi\, d\psi \qquad (20)$$

$$\frac{d\delta}{d\tau} = -\frac{1}{2\pi a}\int_0^{2\pi} F(a\sin\psi, a\omega\cos\psi)\sin\psi\, d\psi \qquad (21)$$

The integration indicated in (20) is now carried out, introducing the assumed form for η and $\dot\eta$ into the definition of $F(\eta,\dot\eta,\ddot\eta)$, at the same time respecting the assumption inherent in the K-B method that both a and δ remain constant within the integral sign .The integrals of all terms of order below $(\beta/(1-\alpha))^3$ are equal to zero, with the exception of those noted below :

$$\frac{da}{d\tau} = -\frac{\phi}{2}\left(\frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)a - \frac{4}{3\pi}\omega^3 r_0 a^2 - \frac{3\omega^2 r_0 \beta}{4(1-\alpha)}\left(\frac{\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)a^3 \qquad (22)$$

This equation defines the amplitude of the oscillation and will be considered for these two cases:

2-2 .- The case of the Simple Surge Tank : In this case r_0 =0 and (22) reduces to:

$$\frac{da}{d\tau} = -\frac{\phi}{2}\left(\frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)a \qquad (23)$$

which integrates to : $a = a_0 \exp\left(-\frac{\phi}{2}\left(\frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)\tau\right)$ (24)
(24) shows that the oscillation will be damped for $\tau \to \infty$, provided that the term

$$\epsilon = \frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}$$

is positive. This is equivalent to stating the condition that:

$$A_s/A \geq D/f H_n \qquad (25)$$

which is the well known Thoma condition.

2-3 .- The case of the Orifice Surge Tank : It may be noticed from equation (22) that as long as ϵ remains positive, the RHS of (22) is negative, for the typical values of α and β quoted before,thus the amplitude of the oscillation decays to zero for long times. The decay factor contains now a term proportional to the orifice head loss coefficient r_0 , which accelerates the damping as compared with the simple surge tank. As was intuitively expected (and heuristically stated by Gardel(1957) this additional damping disappears for very small oscillations. If the surge tank cross section is smaller than the limit given by the Thoma condition (25), ϵ becomes negative and the amplitude of the oscillation grows until it reaches a value equal to the positive root of the quadratic equation formed by setting $\frac{da}{d\tau}$ =0 in (22). This root is :

$$a_1 = -\frac{A_1}{2A_2}\left(\left(1 - 4A_2/A_1^2\right)^{0.5} - 1\right) \qquad (26a)$$

also : $a_2 = (A_1/2A_2)\left(\left(1 - 4A_2/A_1^2\right)^{0.5} + 1\right)$

where :

$$A_0 = -\frac{\phi}{2}\left(\frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)$$

$$A_1 = -\frac{8}{3\pi}\frac{\omega^3 r_0}{A_0} \qquad (26b)$$

$$A_2 = -\frac{3}{4}\frac{\beta r_0}{(1-\alpha)A_0}\left(\frac{\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)$$

The roots of the equation remain real provided that

$$4\,A_2/A_1^2 \leq 1$$

equivalent to the coefficient r_0 being greater than :

$$r_c = 8.327\,\frac{\phi^2}{\omega^4}\left(\frac{\beta}{1-\alpha}\right)\left(\frac{\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)\left(\frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}\right) \quad (27)$$

As this value of r_0 is too small to be found in any practical situation, only real roots will be considered. An approximate expression for the positive root in equation (22) may be deduced by developing in series the square root in (26) and retaining only the first terms, a procedure which leads to :

$$a_1 \cong -\frac{1}{A_1} = 1.178\,\frac{\phi\left(\frac{2\alpha}{\beta} - \frac{\beta}{1-\alpha}\right)}{\omega^3 r_0} \quad (28)$$

As the maximum amplitude of the oscillation is proportional to the factor ϕ , it follows that the maintained amplitude a_1 is larger for small load changes than for larger ones, a somewhat surprising conclusion. In summary, the stability criteria derived from the analytical integration show that , if $A_S > A_{TH}$ henceforth called the Thoma area, then the system is stable. If $A_S < A_{TH}$, but $r_0 > r_c$ in (27), then a tends to a_1 , as defined in (28), the oscillations being maintained at this amplitude. Finally, if $A_S < A_{TH}$ and $r_0 < r_c$ the system is unstable .
If maintained oscillations of a certain amplitude are not ruled out by design considerations, then an orifice tank system with a cross section well below that given by Thoma's criterion is feasible . The magnitude of the maximum amplitude may be reduced by adjustments to the parameter r_0 .

2-4 .-Damping of the Oscillation: The magnitude of the amplitude as a function of time follows from the integration of the differential equation (22):

$$\frac{a}{1 + A_1 a + A_2 a^2}\left(\frac{a_1 - a}{a - a_2}\right)^{\Delta^{-0.5}} = \exp\left(2A_0(\tau + C_0)\right) \quad (29)$$

where A_0, A_1, A_2 are defined in (26b), C_0 is a constant of integration and Δ is the discriminant of (26a). This solution is restricted to the practical case $\Delta > 0$ .Because (29) is linear in τ , it is possible to find the values of a for a given value of τ .To compute the constant of integration, it can be shown that, to order β^3 , the initial amplitude a_0 and the initial phase angle δ_0 are the solutions of the system :

$$a_0 \cos\delta_0 = 1 - \left(1 - \frac{4(1-\phi)}{1-\alpha}\beta\phi\right)^{0.5}$$

$$a_0 \sin\delta_0 = -\frac{\alpha}{\beta}\left(1 - \phi^2\right) \quad (30)$$

3-1 .- Verification of the Analytical Solution :The analytical solution was verified by means of a numerical integration of the equations (13) and (14). Then both numerical and analytical methods were compared with the experimental evidence provided by a surge tank model supplied with a constant power discharge control.

3-2 .- Numerical Verification: The equations of motion were integrated by means of a third order Adams-Bashforth predictor corrector method. The perturbation consisted in a sudden load rejection of $\frac{1}{4}$ of the initial load, from a power coefficient ϕ =1 to ϕ =.75 . The new steady state condition consisted in a value of u as given by the useful root of the cubic:

$$(1 - \alpha u^2)\,u = \phi(1-\alpha)$$

derived from the constant power equation (5),together with the corresponding value of η equal to:

$$\eta = \frac{\alpha}{\beta}(1 - u^2)$$

This process of numerical verification was applied to the three preliminary conclusions drawn in 2-3. The first, that the amplitude of the oscillation is damped if $A_S > A_{TH}$ was checked by reference to the model tests of Mc Caig and Jonker (1959). In the model tests, the stability of operation for several design alternatives for the surge tank of the Bersimis N° 1 power plant in Canada was analysed. Some of the proposed designs incorporated an orifice control, some were simple surge tanks. The experimental data is shown in Fig. 2, which exhibits the variation of the maintained amplitude a_m/Z_* of the oscillation as a function of the ratio A_S/A_{TH}. The experimental trend is in reasonable agreement with both the numerical and analytical predictions which are in themselves quite similar .The two curves shown in Fig. 2 assume instantaneous rejection of 5 and 25 % of the initial load .Both experimental and analytical results share the rapid increase in the amplitude of the oscillation which results when the ratio A_S/A_{TH} descends below unity.

3-3 .- Numerical Verification for Different Values of r_0 : As the example analysed in 3-2 had a value of r_0 close to unity, a test with a wider range of values of this parameter was conceived, in which r_0 adopted values between .1 and 3, whilst α ranged between .02 and .12 . The value of β was then modified until the system entered into a state of maintained oscillations of constant amplitude. The numerical results are compared with the analytical predictions in Table I. This table contains also the calculated values for the critical r_0 in (27). In all tested cases, instability resulted when r_0 descended below the critical value.The comparison between numerical and analytical results shows that the agreement is better when the value of the parameter α is low, near the lower limit of .01 . This is in agreement with the basic premise of the K-B method, which assumes that the non linear terms(in this case the frictional losses) are modified by a coefficient small with respect to unity. But even in cases of the larger values for α the analytical results give an upper bound that may be useful in the initial specification for governor design .

3-4 .- Comparison between the present results and those of earlier investigators: As mentioned in the Introduction, Escande ,Zicman and Zienkewicz considered only the case of small oscillation with linear losses through the orifice plate. On this basis Escande obtained as the magnitude of the maintained oscillation (for $\phi \cong 1$):

$$a_m = 1.18\,\frac{\beta^2 - 2\alpha(1-\alpha)}{\beta r_0(1-3\alpha)} \quad (31)$$

Zicman proposed a very complex expression, which, for small α reduces to:

$$a_m = \frac{2.55\left(\beta^2 - 2\alpha(1-\alpha)\right)}{\left(\beta r_0(1-3\alpha) + \alpha(1-\alpha)\right)} \quad (32)$$

The present results (18) , are seen to be quite close to those of Escande, when $\phi \cong 1$. In that case (18) reduces to:

$$a_m = 1.178\,\frac{\beta^2 - 2\alpha(1-\alpha)}{\beta r_0(1-3\alpha)}\left(\frac{1-\alpha}{1-3\alpha}\right)^{0.5} \quad (33)$$

Zicman's expression gives values about twice as large as Escande's or the present ones.
Finally, it may be worth quoting the very extensive numerical experiments of Forster, who found that from the point of view of stability, it is small load changes around rated power that are most dangerous, in concordance with the statement in 2-3.

3 CONCLUSIONS

The analytical solution of the equations of motion of the orifice surge tank by the method of Krylov-Bogoliubov indicates that an orifice surge tank will behave stably when its area is larger than the Thoma area, that for cross sectional areas below the Thoma area, the amplitude of the oscillation upon a load change does not grow indefinitely, but has an upper limit which depends inversely on the parameter τ_o. It appears that there is a minimum acceptable value of the parameter τ_o which will assure stable operation with areas less than the Thoma area. This has been labelled the critical value of τ_o. The relation between the amplitude of the oscillation and time has been obtained as a rather complex exponential expression.

The stability predictions were verified against a numerical solution of the same equations, for a wide range of values of the parameters α and τ_o. Both methods are in reasonable agreement, being specially close for the case of low values of α, that is for systems with low frictional losses. Numerical and analytical predictions were in turn compared with the experiments of Mac Caig and Jonker on a model of the Bersimis N^{o}1 power plant. The empirical data substantiates the theoretical predictions.

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NOMENCLATURE:

All the geometrical terms are defined in Fig 1. In addition:

U_o = mean velocity in tunnel at rated power
p = pressure at surge tank junction, below orifice
p_1 = pressure immediately above orifice
V = vertical velocity in surge tank
Q = penstock discharge
Q = ditto at rated power
f_e = effective friction loss coefficient, including entrance and exit losses
f_s = surge tank friction coefficient
y_o = friction losses at rated power

TABLE I

COMPARISON OF ANALYTICAL AND NUMERICAL VALUES FOR THE AMPLITUDE OF THE MAINTAINED OSCILLATION

τ_o	τ_c	α	β	a_m Analytical	a_m Numerical
.1	.006	.03	.256	.279	.231
.1	.008	.04	.292	.293	.235
.1	.015	.05	.324	.323	.237
.1	.018	.07	.378	.382	.241
.1	.039	.10	.446	.571	.239
1.0	.057	.01	.265	.175	.167
1.0	.073	.02	.310	.175	.163
1.0	.112	.04	.381	.183	.163
1.0	.158	.06	.437	.196	.164
1.0	.288	.10	.526	.239	.170
2.0	.163	.01	.350	.136	.126
2.0	.287	.05	.470	.142	.120
3.0	.812	.03	.580	.166	.134
3.0	.887	.05	.606	.166	.128

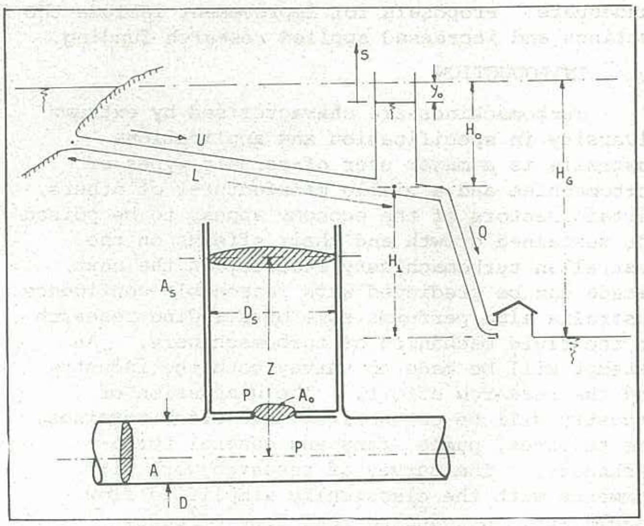

Figure 1 Surge Tank and System Geometry

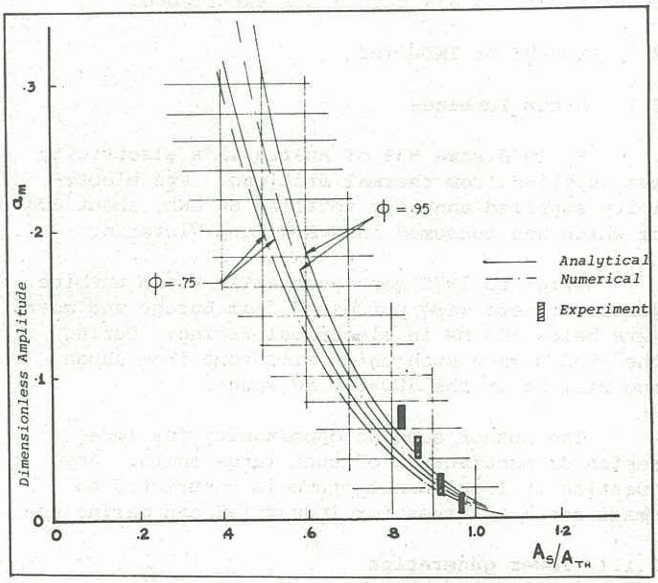

Figure 2 Comparison of theoretical data with experiments by McCaig and Jonker