Boundary Element Methods for Free Surface Viscous Flows

J.F. MILTHORPE

Senior Research Assistant, Department of Mechanical Engineering, The University of Sydney

and

R.I. TANNER

Professor of Mechanical Engineering The University of Sydney

SUMMARY There are many problems in viscous fluid dynamics where a prime object is to find the shape of the free-surface. Numerical work already done by the authors uses the classical Galerkin-based finite-element method. After assuming a boundary shape the problem is solved, the boundary is checked to see if it satisfies all the necessary free-surface boundary conditions, and if not, the boundary shape is updated until convergence is obtained. During this process the entire internal fields for velocity and pressure are computed at each interation, even though they are not of interest. In the present paper we explore the use of the boundary-element method. Here only the surface values of stress and velocity are dealt with at each interation and after the final free-surface shape has been determined, the fields inside the fluid can be computed if needed. Some simple exact solutions are first solved and then the problem of extrusion is tackled. We conclude that the method is convenient for flow problems at low Reynolds numbers.

1   INTRODUCTION

In many problems of viscous fluid dynamics, a major aim is to find the shape of a free-surface. We instance problems of jets leaving a nozzle, extrusion of polymer from dies and melt-spinning processes, all of which are steady-state flows with free-surfaces of complex shape.

Numerical methods for computing free-surface flows for Newtonian fluids are well established (Reddy and Tanner, 1978) and considerable work has also been done with non-Newtonian fluids. As the fluid is moved further from Newtonian behaviour these models begin to break down, either by the solutions becoming inaccurate or by numerical instability. We are therefore exploring progressively more complex viscoelastic fluid models, the principal of which is the convected Maxwell fluid (Bird et al, 1977).

The flows mentioned above, and some results to be given later, were obtained using a Finite Element algorithm. The Finite Element Method has the advantage for free-surface flows that it is easily adapted to boundary shapes which are complex and change at each iteration of the calculation, and indeed the FEM has proved a very successful technique for general fluid dynamics.

With non-Newtonian jets, and particularly with the Maxwell fluid, we have found the main difficulty in the calculation to be associated with the singularity at the lip.

There is a singularity here in the boundary conditions, which change from no-slip to zero stress, and this leads to very high rates of change of velocity and stress in the fluid nearby. Successful representation of the flow near the singularity requires that the nearby elements be small, and geometrical constraints then require a relatively find grid throughout the region. The resulting large number of nodes makes the problem a large one in terms of computing time and space.

The use of fine grids, necessary for the calculation, produces a great deal of information about the flow which is not necessary, as the main aim of the calculation is to predict the actual shape of the jet and the pressure loss in the nozzle. A boundary method would therefore be attractive. In this type of method, the problem is transformed from one over a region to a corresponding problem on the boundary. As the boundary has one fewer dimensions than the region it encloses, the number of nodes required for equivalent accuracy should be greatly reduced.

We will show that free-surface problems can be successfully transformed to boundary problems, and then formed in terms of discrete variables, but we will first describe the non-Newtonian fluid, and the FEM algorithm we have been using, and give some of our results, so as to provide a comparison for the Boundary Element Method.

2   FREE-SURFACE VISCOUS FLOWS

The non-Newtonian fluid model we have investigated is the Maxwell fluid, a model with a single exponential relaxation process. The stresses in a Maxwell fluid have been calculated by integrating over the history of each fluid particle (Caswell et al, 1978). We have preferred to express the constitutive equation in the form of a global differential equation (Bird et al, 1977)

$$\tau_{ij} = \eta(v_{i,j} + v_{j,i})$$

$$- \lambda(v_k\tau_{ij,k} - v_{i,k}\tau_{kj} - v_{j,k}\tau_{ki}) \qquad (1)$$

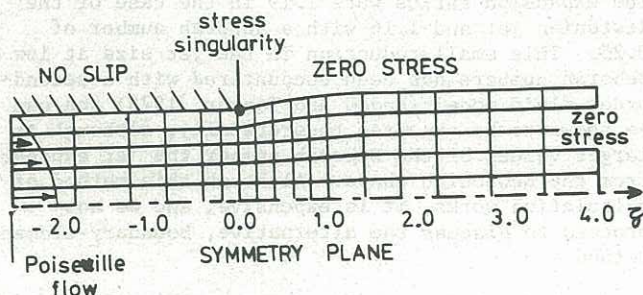where here and elsewhere we use Einstein's summation

Figure 1   Typical grid for jet problem

convention. The τ_{ij} are the deviatoric stresses, so that the total stress is

$$\sigma_{ij} = -p + \tau_{ij} \qquad (2)$$

and the v_i are the velocity components. The Newtonian viscosity is η and λ is a parameter proportional to the relaxation time.

The equation of motion is

$$\rho v_j v_{i,j} = -p_{,i} + \tau_{ij,j}. \qquad (3)$$

The deviatoric stresses can be divided into Newtonian and non-Newtonian parts:

$$\tau_{ij} = \tau_{ij}^{(newt)} + \tau_{ij}^{(non)} \qquad (4)$$

where

$$\tau_{ij}^{(newt)} = \eta(v_{i,j} + v_{j,i}). \qquad (5)$$

The equation of motion can then be recast:

$$\rho \frac{Dv_i}{Dt} + p_{,i} - \tau_{ij,j}^{(newt)} = \tau_{ij,j}^{(non)}. \qquad (6)$$

Using the Galerkin method and Green's theorem, we can rewrite (6) as

$$\sum_e \left\{ \int_{v_e} \left[\rho \frac{Dv_i}{Dt} + p_{,i} \right] u_{im} dV + \int_{v_e} \tau_{ij}^{(newt)} \frac{\partial u_{im}}{\partial x_j} dV \right.$$

$$\left. - \int_{s_e} \tau_{ij}^{(newt)} n_j u_{im} dS \right\}$$

$$= \sum_e \left\{ \int_{s_e} \tau_{ij}^{(non)} n_j u_{im} dS - \int_{v_e} \tau_{ij}^{(non)} \frac{\partial u_{im}}{\partial x_j} dV \right\}. \qquad (7)$$

The term

$$\int_{s_e} \tau_{ij}^{(newt)} n_j u_{im} dS$$

vanishes except on the boundary of the region, although the corresponding term for the non-Newtonian part of the deviatoric stress does not vanish, due to the lack of continuity in the stress fields used. Equation (7) can therefore be written

$$\sum_e \left\{ \int_{v_e} \left[\rho \frac{Dv_i}{Dt} + p_{,i} \right] u_{im} dV + \int_{v_e} \tau_{ij}^{(newt)} \frac{\partial u_{im}}{\partial x_j} dV \right.$$

$$\left. - \int_{s_R} t_i u_{im} dS \right\}$$

$$= \sum_e \left\{ \int_{s_e} \tau_{ij}^{(non)} n_j u_{im} dS - \int_{v_e} \tau_{ij}^{(non)} \frac{\partial u_{im}}{\partial x_j} dV \right\} \qquad (8)$$

where the t_i are tractions acting on the surface s_R of the region. The terms on the left-hand side of (8) are a standard Newtonian finite element algorithm.

The solution procedure used is to alternately calculate the velocity-pressure field and the deviatoric stresses. A Newtonian velocity field is used to begin the process and the parameter λ is incremented until the desired value is reached. It can be shown that the velocity fields for small

values of λ are very similar to the Newtonian values for plane flow.

The deviatoric stresses are calculated with a FEM using the same element grid and many subroutines in common with the velocity-pressure calculation. The boundary conditions applied are values of the deviatoric stresses on the upstream boundary. These can be easily calculated if the corresponding velocity condition is Poiseuille flow.

On the free-surface a zero-stress boundary condition is applied in the velocity-pressure calculation, and the surface is them moved to obtain zero normal velocity.

3 RESULTS FROM THE FINITE ELEMENT METHOD

The algorithm described above was used to simulate plane jet flow for a Newtonian fluid and a Maxwell fluid with Deborah number 0.25. The element grid used was that shown in Figure 1 which has 594 degrees of freedom. The boundary conditions imposed were fully developed Poiseuille flow upstream, no slip on the solid boundary, no stress on the free-surface and the downstream boundary and symmetry conditions on the central plane. The appropriate deviatoric stresses for Poiseuille flow were imposed upstream for the stress calculation.
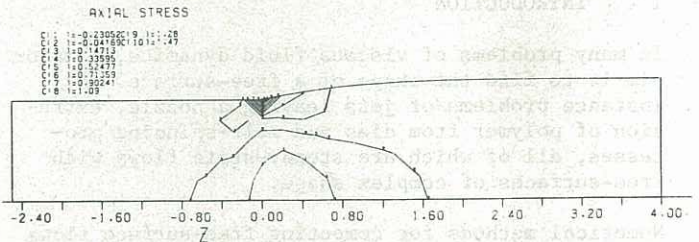

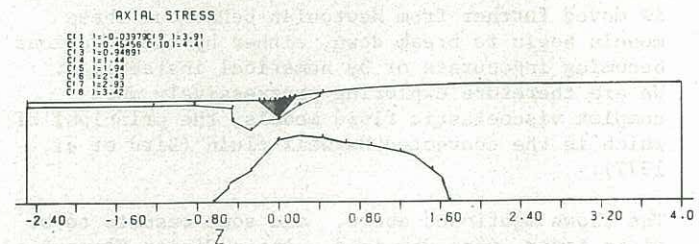Figure 2   Deviatoric tensile stress for Newtonian fluid.

Figure 3   Deviatoric tensile stress for Maxwell fluid, Deb = $\lambda\dot{\gamma}$ = 0.25

In Figures 2 and 3 the deviatoric stress parallel to the flow for Newtonian and Maxwell cases are shown. The concentration of the changes in tensile stress in the region near the slip-no slip junction for the Maxwell fluid is clearly shown. The centreline pressure, Figure 4, is seen in the Maxwell fluid to be parallel to but slightly displaced from the Newtonian pressure curve.

The expansion ratios were 1.19 in the case of the Newtonian jet and 1.16 with a Deborah number of 0.25. This small reduction in the jet size at low Deborah numbers has been encountered with a second-order fluid model (Reddy and Tanner, 1978) and can be shown to be expected theoretically, although at larger values of the Deborah number the jet expands from the Newtonian shape. Although this method of calculation works, it is expensive, and we now proceed to discuss the alternative, boundary-element method.
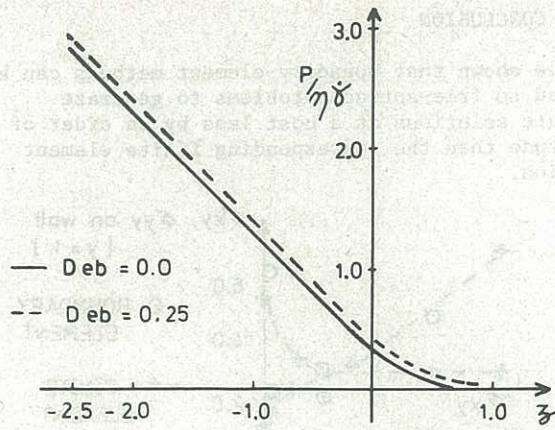
Figure 4 Centreline pressure $p/\eta\dot{\gamma}$

Deb = 0.0
Deb = 0.25

4 RECIPROCAL THEOREM FOR INCOMPRESSIBLE VISCOUS FLOWS

Because of the interest in finding the boundary configuration at various times, and the comparative lack of interest in the internal state of the body in the extrusion problems discussed above, we have begun to explore the possibility of using the so-called boundary-element procedure. Essentially, the technique is derived from reciprocal theorems familiar in linear elasticity, combined with a knowledge of the relevant Green's function. The effective adaptation of these methods for computer use in elasticity seems to be due to Rizzo (1967) and Cruse (Cruse and Rizzo, 1968). Here we begin by discussing viscous incompressible flows; the problem follows closely the discussion for linear elasticity (Brebbia, 1978).

The equations of motion given above in Equation (3), and the conservation of mass equation takes the form $\partial v_i/\partial x_i = 0$. Suppose σ_{ij} are the stresses in the fluid, ρ is the fluid density, f_i and a_i the body forces (per unit mass) and acceleration components respectively, and v_i are the velocity components; x_j denotes the position in the fluid.

We consider an arbitrary set of fields v_i^*, p^*, σ_{ij}^*. We multiply the equation of motion and the mass conservation equation by v_i^*, and p^* respectively and integrate over the body; this is the familiar Galerkin procedure already used in the finite element approach, but we also add on a surface term over the part S_1 of the surface (Figure 5) where the velocities are given. Thus we have

$$\int_\Omega \left[\frac{\partial \sigma_{ij}}{\partial x_j} + \rho(f_i - a_i) \right] v_i^* dV + \int_\Omega p^* \frac{\partial v_i}{\partial x_i} dV$$

$$+ \int_{S_1} (v_i - \hat{v}_i) t_i^* dS = 0 \qquad (9)$$

where t_k^* is the traction vector $\sigma_{kj}^* n_j$ formed from the starred stress tensor and the outward unit normal vector $\underline{n}$, and $\hat{v}_i$ are the given boundary conditions. Any solution (σ_{ij}, p, v_i) that satisfies the equations of motion, the mass conservation equation and the boundary condition $v_i = \hat{v}_i$ on S_1 will make (9) vanish, and hence (9) is true for arbitrary $(\sigma_{ij}^*, p^*, v_i^*)$.

Now consider the following expression denoted by $I(*, 0)$ where

$$I(*,0) = \int_\Omega \frac{\partial \sigma_{ij}^*}{\partial x_j} v_i dV - \int_S v_n t_n^* dS - \int_\Omega p^* \frac{\partial v_i}{\partial x_i} dV. \qquad (10)$$
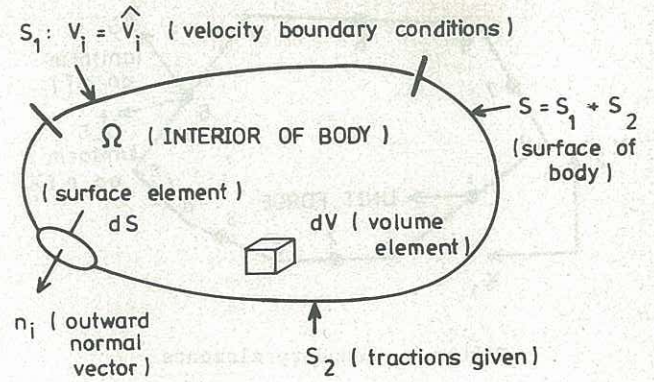

Figure 5 Boundary condition definitions.

By using Green's theorem we can show that for a Newtonian fluid there is a reciprocal theorem

$$I(0,*) = I(*,0). \qquad (11)$$

This reciprocal theorem can be used to replace some of the terms in (9), finding, when both v_i^* and v_i are incompressible fields,

$$0 = \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j} v_i dV - \int_S v_i t_i^* dS + \int_S v_i^* t_i^* dS$$

$$+ \int_\Omega \rho(f_i - a_i) v_i^* dV + \int_S (v_i - \hat{v}_i) t_i^* dS. \qquad (12)$$

We can combine the second and last terms to read $-\int_S v_i t_i^* dS$, understanding that $v_i = \hat{v}_i$ on S_1.

We shall also suppose in this paper that f_i and a_i are known; for the latter this implies an iteration scheme that is not expected to converge for large Reynolds numbers (Gartling et al, 1977). For the examples given here, both f_i and ρ are assumed zero, so we have creeping flow under no body forces. We will also restrict ourselves to plane flows.

5 THE BOUNDARY ELEMENT METHOD

We now assume that the (*)-fields are produced by a unit force in the ℓ-direction. The plane flow solutions for the $t_k^{*(\ell)}$ and $v_k^{*(\ell)}$ are well-known (Brebbia, 1978)

$$v_k^{*(\ell)} = \frac{1}{4\pi\eta} \left[(-\ln r)\delta_{\ell k} + \frac{\partial r}{\partial x_\ell} \frac{\partial r}{\partial x_k} \right], \qquad (13)$$

$$t_k^{*(\ell)} = -\frac{1}{\pi r} \left[\frac{\partial r}{\partial n} \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial x_\ell} \right]. \qquad (14)$$

Consider the body of fluid in Figure 6. Suppose the boundary is discretized into linear "elements" ab, bc, cd, etc. Suppose at the centre of ab we have node 1. Suppose we assume a (*)-field which consists of a unit force in the x_1-direction applied at node 1. Then we have (Brebbia, 1978), from (14)

$$0 = -\tfrac{1}{2}v_1 - \int_S v_k t_k^* dS + \int_S v_k^* t_k dS \qquad (15)$$

where v_k^* and t_k^* are known. If we now assume that the (0)-field is uniform over each segment, so that we may speak of $v_1(m)$, $v_2(m)$, $t_1(m)$ and $t_2(m)$ as the uniform components on the segment containing the mth node, then (15) becomes a linear equation
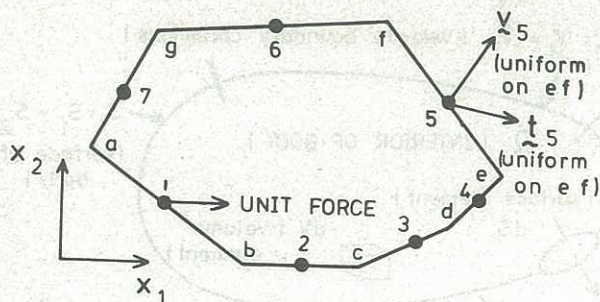
Figure 6 Boundary elements

$$0 = \tfrac{1}{2}v_i - \sum_{m=1}^{N} \left[\int_{s_m} t_1^* dS \int_{s_m} t_2^* dS \right] \begin{bmatrix} v_1^{(m)} \\ v_2^{(m)} \end{bmatrix}$$

$$+ \sum_{m=1}^{N} \left[\int_{s_m} v_1^* dS \int_{s_m} v_2^* dS \right] \begin{bmatrix} t_1^{(m)} \\ t_2^{(m)} \end{bmatrix} \qquad (16)$$

connecting 2N components of velocity and 2N components of traction. Similarly one can apply a unit force in the x_2-direction at node 1, generating a second equation, and so on for all nodes, finally generating 2N equations. There are 2N v-components and 2N t-components, but only a total of 2N unknowns. Thus one can solve a set of linear equations for all boundary unknowns. To obtain values in the interior, one places a point force where needed as the (*)-field, and uses (14) again, thus producing the needed value of v; a slightly different field will produce the tractions and stresses. Thus results at all points of the body can be found.

6   AN EXAMPLE

We exhibit solutions for a creeping Newtonian film extrusion. This problem has previously been solved using the finite element method (FEM) by Reddy and Tanner (1978) and we shall compare the solutions. The problem is shown in Figure 1. The object is to find the free-surface shape by iteration. Results for the velocities and stresses found by FEM and boundary element methods are compared in Figure 7. The agreement between the two methods of calculation is seen to be excellent in that the stress singularity near the exit lip is represented well; this is the most difficult feature to capture numerically. The time difference between the FEM and the present program is striking: using 24 elements (343 unknowns) the FEM program took 80 secs. per iteration; the boundary method (18 nodes or elements and 36 unknowns) took 4.4 secs. on the updated version of the same CDC Cyber 72 machine. Thus, allowing for increased machine speed, the boundary method is roughly ten times quicker.

7   CONCLUSION

We have shown that boundary-element methods can be applied to free-surface problems to generate accurate solutions at a cost less by an order of magnitude than the corresponding finite element solution.
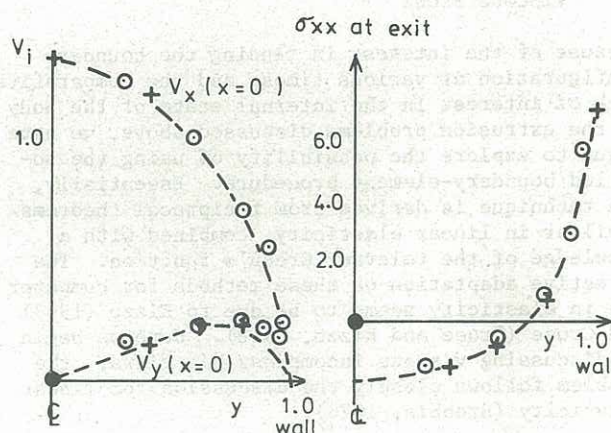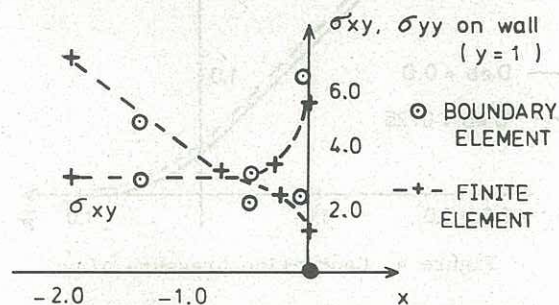

Figure 7 Boundary Element Solutions

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