7th Australasian Hydraulics and
Fluid Mechanics Conference,
Brisbane, 18-22 August, 1980

A Spectral Model of MHD Finite Amplitude Overstability

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SUMMARY The application of a magnetic field to a layer of ionized fluid heated from below has the two-fold effect of inhibiting the onset of dynamic convection and instigating the occurrence of finite amplitude overstability. The transient nonlinear flow equations, which have been obtained by an averaging over the convection cells, are solved using the spectral method employing the trigonometric functions. A description of the method is given and graphs are presented depicting the variations of the dependent variables. The results indicate a rapid growth of the overstable oscillations with increasing Rayleigh number, when close to the marginal value.

1 INTRODUCTION

In magnetohydrodynamics it is found that the magnetic field imparts a certain rigidity to the fluid and yet also imparts certain properties of elasticity which enable it to transmit disturbances by new waves of propagation.

This paper deals with the effects of the application of a magnetic field to a layer of ionized fluid thermally stratified, that is one in which there is a temperature gradient imposed across the fluid. The imposition of a magnetic field will cause a suppression of cellular convection and perhaps the introduction of finite amplitude overstability. An ionized fluid in the form of a layer of thickness d and of infinite horizontal extent is heated from below and a magnetic field is imposed. It is known experimentally that such a physical phenomena produces a tesselation of cells and it is on this physical basis that the imposition of "averaging" these periodic cells over the horizontal x - y plane has been taken. There is a large amount of reasonably straightforward vector calculus involved, details of which may be found in Van der Borght [1974], Mann [1974], with further general fundamentals elucidated in Roberts [1966]. The resultant equations are simply quoted in this mathematical model, which assumes the Boussinesq approximation.

The study is confined to one of free boundary conditions in which the velocity and temperature perturbations vanish at the top and bottom boundaries but the mean temperature at each boundary is constant.

Local Analysis, or the Hurwitz Criterion (Mann [1974]), give the marginal stability regions for the various modes of instability. This analysis illustrated that the imposition of the magnetic field suppresses cellular convection so that it occurs at a higher Rayleigh number but introduces finite-amplitude overstability at a Rayleigh number which is lower than that of the cellular convection but higher than that at which cellular convection would occur without the imposition of the magnetic field.

The spectral method is used to solve the nonlinear system of partial differential equations enabling an accurate solution to be made of the approximating equations.

2 NOTATION

- x,y,z : Cartesian coordinate system
- t : independent variable, time.
- $\underset{\sim}{u}$: velocity of the fluid $\equiv (u,v,w)$.
- ρ : density.
- $\underset{\sim}{H}$: magnetic field intensity.
- μ^* : permeability of the medium to the magnetic field.
- η : resistivity of the medium to current transmission.
- μ : kinematic viscosity.
- ϕ : force potential.
- ΔT : temperature difference across the layer.
- T : temperature (absolute).
- α : coefficient of thermal expansion.
- c_V : specific heat at constant volume.
- κ : thermometric diffusion coefficient.
- K : thermal diffusion coefficient $= \frac{\kappa}{\rho\, c_V}$.
- p : pressure in the fluid.

$$D \equiv \frac{d}{dz}$$

- d : thickness of convection layer.
- k : horizontal wavenumber, giving the horizontal extent of the convection cells.
- a : non-dimensional value of k.
- R : Rayleigh number $= \frac{g\rho\alpha\,\Delta T\, d^3}{\kappa\mu}$
- Q : Chandrasekhar number $= \frac{\mu^* H_0^2\, d^2}{4\pi\eta\mu}$
- σ : Prandtl number $= \frac{\mu}{\rho\kappa}$
- τ : magnetic Prandtl number $= \frac{\eta}{\kappa}$
- H : the non-dimensional perturbation from the constant impressed magnetic field intensity.

F : temperature fluctuations about the average in an x-y plane.

g : gravitational acceleration, assumed constant.

3 MATHEMATICAL MODEL

The basic equations for conservation of mass, momentum and energy together with a combination of Maxwell's equations and the generalised Ohm's law may be written as follows where μ^* is constant.

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underset{\sim}{u}) = 0 \tag{1}$$

$$\rho\, c_v \frac{\partial T}{\partial t} + \rho\, c_v\, \underset{\sim}{u}.\nabla T + p\nabla.\underset{\sim}{u} - K\nabla^2 T = 0 \tag{2}$$

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{u}.\nabla \underset{\sim}{u} + \nabla p + \rho \nabla \phi + \frac{\mu^*}{4\pi} \underset{\sim}{H} \times (\nabla \times H) - \mu[\nabla^2 \underset{\sim}{u} + \frac{1}{3}\nabla(\nabla.\underset{\sim}{u})] = 0 \tag{3}$$

$$\frac{\partial \underset{\sim}{H}}{\partial t} - \text{curl}\,(\underset{\sim}{u} \times \underset{\sim}{H}) + \eta\, \text{curl curl}\, \underset{\sim}{H} = 0 \tag{4}$$

The variational principle of Prigogine and Glansdorff [1964, 1965] has been applied together with an "averaging" over the cells by Van der Borght [1974]. With the assumption that the convection cells are of either square or rectangular planform, and with the Boussinesq approximation, the system, after non-dimensionalization reduces to

$$\frac{\partial T}{\partial t} = D^2 T - D(WF) \tag{5}$$

$$\frac{\partial F}{\partial t} = (D^2 - a^2)F - WDT \tag{6}$$

$$\frac{1}{\sigma}(D^2 - a^2)\frac{\partial W}{\partial t} = (D^2 - a^2)^2 W - Ra^2 F + \tau Q D(D^2 - a^2)H \tag{7}$$

$$\frac{\partial H}{\partial t} = \tau(D^2 - a^2)H + DW \tag{8}$$

A detailed analysis of this is given in Mann [1974] and symbols are defined in the notation section.

The non-dimensionalized layer of fluid is of depth unity and the non-dimensionalization has introduced the flow "number", Rayleigh, Chandrasekhar, Prandtl and magnetic Prandtl.

3.1 Boundary Conditions

It is assumed:

(a) that there is no overshooting of the convected fluid in the layer

$$\therefore \quad W = 0 \quad \text{at } z = 0,1 \tag{9}$$

(b) that the two surfaces are kept at constant temperature, that is, the boundaries are conducting boundaries in which the boundaries conduct the fluctuations out of the flow faster than the layer itself does

$$\therefore \quad F = 0 \quad \text{at } z = 0,1 \tag{10}$$

(c) that the layer of fluid is insulated against magnetic field perturbations from the adjoining medium and so have the condition

$$DH = 0 \quad \text{on} \quad z = 0,1 \tag{11}$$

A typical current-free adjoining medium is a vacuum and the corresponding boundary is a free boundary. Across such a boundary the tangential stresses are continuous. Since the above conditions show that the magnetic part of the tangential stress is continuous across a free boundary, and moreover there are no viscous stresses outside the fluid, the tangential viscous stresses must vanish on free surfaces, resulting in

$$D^2 W = 0 \quad \text{on} \quad z = 0,1 \tag{12}$$

The steady-state non-dimensional temperatures of the free surfaces are

$$T = 0 \quad \text{on} \quad z = 0$$
$$T = -1 \quad \text{on} \quad z = 1 \tag{13}$$

Van der Borght [1974] reduced equations (5)-(8) to a three equation system independent of H and then solved using a one-mode approximation. It can be shown (MANN [1974]) that the three equation system of Van der Borght is equivalent to the use of equations (5)-(8) together with an explicit boundary condition on H of DH = 0 at z = 0,1

3.2 Parameter Values

For point of easy comparison with the one-mode model of Van Der Borght [1974] the major part of the analysis will be undertaken with the following values of the parameters

$$a = \pi$$
$$\sigma = 1$$
$$\tau = 0.1$$
$$Q = 10^5 \tag{14}$$

and varying values of R, starting from the linear value.

A linear analysis similar to that in Chandrasekhar [1961], to determine the region of flow can produce, for these conditions, dynamic convection at $R = 1.97\ 10^6$ and overstable oscillations at $R = 1.09508\ 10^5$ Full details in Mann [1974]

4 SPECTRAL METHOD

The Fourier-trigonometric approximations for the dependent variables are chosen to be

$$W = \sum_{i=1}^{n} W_{2i-1}(t)\sin(2i-1)\pi z$$

$$F = \sum_{j=1}^{n} F_{2j-1}(t)\sin(2j-1)\pi z$$

$$T = -z + \sum_{\ell=1}^{n} C_{2\ell}(t)\sin 2\pi\ell z$$

$$H = \sum_{k=1}^{n} H_{2k-1}(t)\cos(2k-1)\pi z \tag{15}$$

It can be readily verified that these satisfy the boundary conditions. The form of T is chosen so as to closely approximate the expected profile of the temperature distribution in the unsteady state e.g. the variations occur asymmetrically about the steady-state profile.

Substitution of (15) into the transient nonlinear system of partial differential equations (5)-(8) results in the following, where ' denotes the derivative with respect to time, t

(5) becomes

$$\sum C'_\ell \sin 2\pi\ell z = -4\pi^2 \sum C_\ell \ell^2 \sin 2\pi\ell z$$

$$- \sum_i \sum_j W_{2i-1}(2i-1)\pi \cos(2i-1)\pi z\, F_{2j-1}\sin(2j-1)\pi z$$

$$- \sum_i \sum_j W_{2i-1}\sin(2i-1)\pi z\, F_{2j-1}(2j-1)\pi \cos(2j-1)\pi z \qquad (16)$$

To extract the z-dependence, a Galerkin approximation using the orthogonality of the circular functions, is employed. Multiplication by sin $2\pi mz$ and integration over z for $z \in [0,1]$ reduces (16) to a system of ordinary differential equations.

Similar approximations are carried out on equations (6)-(8).

So, for various $M = 1,\ldots.n$ where n is the order of the highest harmonic, the resultant system of ordinary differential equations describing the problem may, with some minor modifications, be summarized as follows

$$C'_M = -4\pi^2 M^2 C_M - 2\pi \sum_I \sum_J W_{2I-1}F_{2J-1}[(2I-1)AI(I,J,M) + (2J-1)AJ(I,J,M)] \qquad (20)$$

$$F'_{2M-1} = -F_{2M-1}[\pi^2(2M-1)^2 + a^2] + W_{2M-1} - 4\pi \sum_I \sum_J W_{2I-1}\, C_J\, J\, AK(I,J,M) \qquad (21)$$

$$H'_{2M-1} = -\tau H_{2M-1}[\pi^2(2M-1)^2 + a^2] + \pi(2M-1)W_{2M-1} \qquad (22)$$

$$W'_{2M-1} = -\sigma\left\{W_{2M-1}[(2M-1)^2\pi^2 + a^2] - \frac{Ra^2F_{2M-1}}{(2M-1)^2\pi^2+a^2} + \tau Q(2M-1)\pi H_{2M-1}\right\} \qquad (23)$$

where

$$AI(I,J,M) = \int_0^1 \cos(2I-1)\pi z \sin(2J-1)\pi z \sin 2\pi Mz\, dz \qquad (24)$$

$$AJ(I,J,M) = \int_0^1 \sin(2I-1)\pi z \cos(2J-1)\pi z \sin 2\pi Mz\, dz \qquad (25)$$

$$AK(I,J,M) = \int_0^1 \sin(2I-1)\pi z \cos 2\pi Jz \sin(2M-1)\pi z\, dz \qquad (26)$$

In the evaluation of the integrals for AI, AJ and AK as in (24)-(26), due to the orthogonality of the trigonometric functions only certain combinations of the I, J, M will render these integrals non-zero. The integrals can be evaluated analytically and expressed as constants for various combinations of I, J, M.

Because these variables, I, J, M, are intrinsically limited to positive integers the selection is even more limited, with the result as follows

$$\begin{aligned}
AI(I,J,M) &= 0.25 \text{ if } M = I + J - 1 \quad \text{or} \quad M = J - I\\
&= -0.25 \text{ if } M = I - J\\
AJ(I,J,M) &= 0.25 \text{ if } M = I + J - 1 \quad \text{or} \quad M = I - J\\
&= -0.25 \text{ if } M = J - I\\
AK(I,J,M) &= 0.25 \text{ if } M = I + J \quad \text{or} \quad M = I - J\\
&= -0.25 \text{ if } M = J - I + 1
\end{aligned} \qquad (27)$$

A Runge-Kutta-Gill numerical integration of this system was carried out. The choice of M Fourier terms for each dependent variable resulted in there being a system of 4 M nonlinear ordinary differential equations to integrate. From early runs of the program it was found that only the first two terms of the Fourier series for W and H were significant while the higher harmonies of F and T retained their importance. Hence the program was modified, fixing W and H each at a 2-term Fourier expansion and allowing for an ever-increasing number of terms for F and T. Typically for the value of $R = 2.9218 \times 10^5$, the increasing of the number of Fourier terms for the dependencies of F and T required the increment size (δt) to be reduced to preserve numerical stability, ranging in value from 0.0002 for a one-term expansion to 0.00005 for a fourteen-term expansion.

5 RESULTS

The fluid oscillates to and fro rather than in a circular motion with the transmission of heat through the fluid being due to diffusion from this oscillatory path: a much slower transmission rate than for dynamic convection.

Figure 1 illustrates that when W was at its maximum in amplitude, whether it be positive or negative, the temperature distribution assumed a stable stratification across the bulk of the fluid whereas at the boundaries the unstable gradient was still dominant. The effect of the formation of this stable gradient is to draw potential from the flow in order to slow it down. This contrasts with the effect of the inverse gradient, which is to supply potential to the flow to enable growth of the disturbances.

The solutions of F (see Figure 4) display sharp peaks near the boundaries. If we consider equation (6), we realise that for steady finite amplitude of z oscillating in time, then the peaks of the curve of F exist where D^2F is large and the requirement that DT_0 is large is satisfied near the boundaries (see Figure 1). Physically, the heat flux increases when convection occurs and this requires that the heat input must increase which implies that the gradient at the outside of the boundary must increase to input more heat flux. To satisfy continuity considerations, the temperaturn gradient on the inside of the boundary must increase and hence form a thermal boundary layer. Hence the D^2F increases in this layer and this is depicted on the graphs. An increased value of the Rayleigh number (R) corresponds to an increase in the temperature difference across the flow where the other terms in the Rayleigh number equivalence are taken to be constant.

Figure 5 provides a comparison between the solutions for the one-mode approximation (Van der Borght [1974]) and the spectral method utilized here. It can be seen that the nonlinearity is dominant at values of the Rayleigh number very close to the linear value. The one-mode approximation does not indicate the rapid growth of the maximum velocity with increasing Rayleigh number. This rapidity of rate of growth is most dominant for R values near the linear, a slower increase being prevalent for higher values of R.

The results indicate that at a point of the flow in an x-y plane at a specific depth, the vertical velocity is oscillating in time (t). At a specific x-y value, there is oscillatory motion both in the time and depth dependencies.

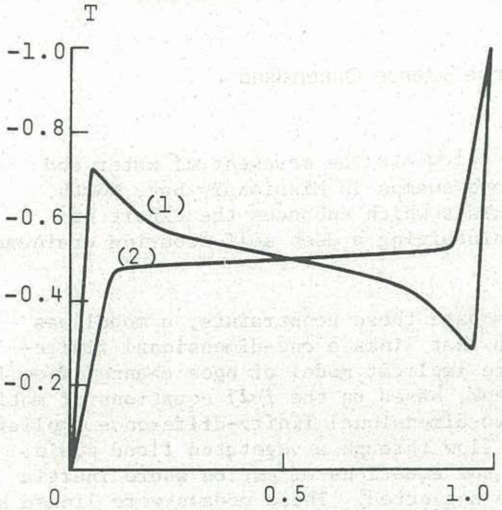

Figure 1 Variation of the temperature (T) with vertical height (Z). R = 2.9218 10^5 Curves (1) When W(MAX) is positive or negative, (2) When $|W|$ minimum

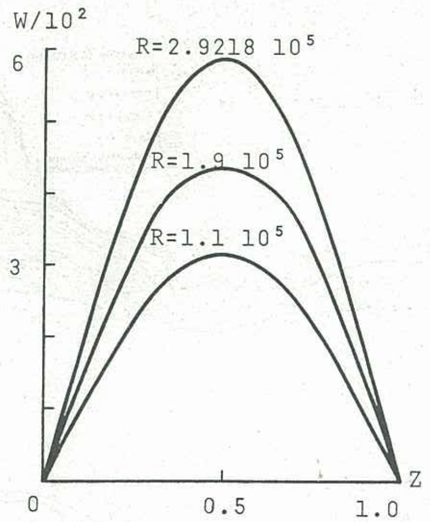

Figure 2 The variation of the vertical velocity (W) with the vertical height (Z), for selected values of the Rayleigh number (R).

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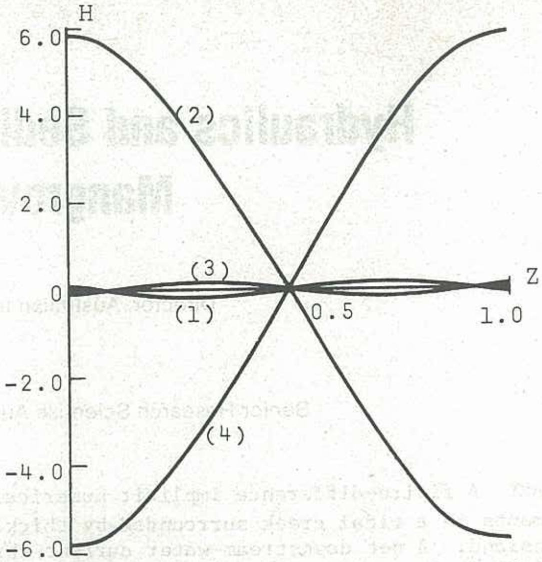

Figure 3 Variation of the magnetic perturbations (H) with the vertical height (Z). Curves (1) When W(MAX) is positive (2) When $|W|$ minimum after W(MAX) is positive (3) When W(MAX) is negative (4) When $|W|$ minimum after W(MAX) is negative

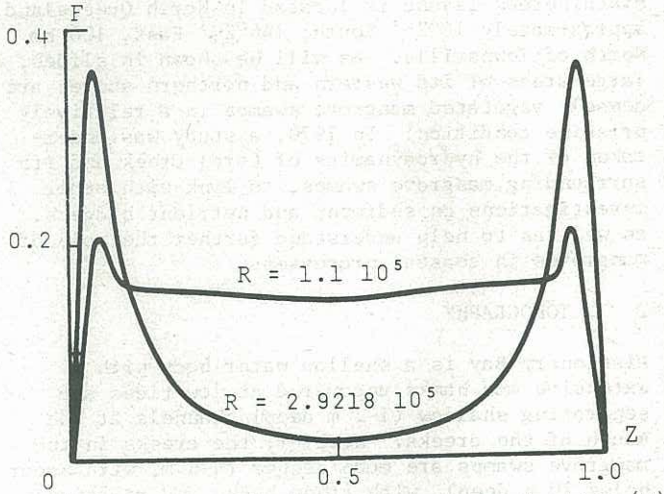

Figure 4 Variation of the temperature fluctuation (F) with the vertical height (Z) for various values of Rayleigh number R when W(MAX) is positive

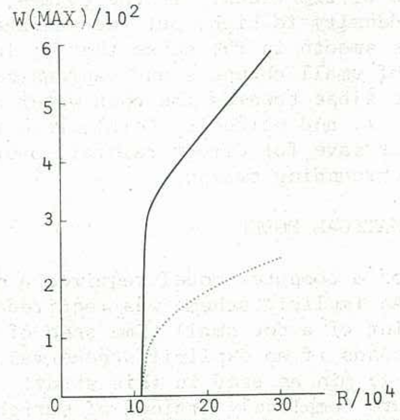

Figure 5 Variation of W(MAX) of overstable oscillations with the Rayleigh number (R). Line of dots represents the one-mode curve; The spectral curve is represented by the continuous line.

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