7th Australasian Hydraulics and
Fluid Mechanics Conference,
Brisbane, 18-22 August, 1980

Natural Convection in Inclined Cavities Filled with a Compressible Fluid with Variable Properties: A Numerical Study

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1. INTRODUCTION

Natural convection in rectangular cavities with a horizontal temperature gradient has been extensively studied numerically with the result that this problem has become a "comparison problem" for the testing of various numerical techniques (Jones, 1979). However the comparison study and most of the previous work (e.g. Mallinson and de Vahl Davis, 1977 and Roux et. al., 1978) has been based on a set of equations which have been simplified by the introduction of the Boussinesq Approximation (Gray and Giorgini, 1976). Only a small number of workers have considered all properties to be variable. Rubel and Landis (1970) used a linearized approach while Polezhaev (1967) solved the complete equations, including the continuity equation, for an air filled cavity, for one value of temperature difference and one aspect ratio only.

Leonardi and Reizes (1979, 1980) developed a method for solving the complete equations without the need for the solution of the continuity equation, so that full advantage could be taken of the method of false transients (Mallinson and de Vahl Davis, 1973) for speeding up convergence and presented results for air, for aspect ratios one and two in both vertical and inclined cavities. They concluded that whilst the flow field was significantly affected by the introduction of variable properties, in particular the variable density, the rate of heat transfer across the cavity differed by a maximum of ten percent from values obtained using the Boussinesq Approximation, in the Rayleigh number range $10^4 \leq Ra \leq 10^6$. At low Rayleigh numbers the differences were substantial and could not be neglected.

The method developed by us (Leonardi and Reizes, 1980) has been extended to include water as the fluid filling the cavity. A systematic study of the effects of fluid property variations on the flow and temperature fields is presented for water filled cavities inclined at 60° to the vertical.

2. MATHEMATICAL FORMULATION

2.1 Governing Equations

The two dimensional cavity considered is shown in Fig. 1. Choosing Y_o', the size of the box in the y direction, as the scale factor for length, the shape of the box is determined by the aspect ratio $L=X_o'/Y_o'$. The use of p_r', an arbitrary reference pressure; ρ_r' the density at pressure p_r' and temperature T_c' (the cold boundary absolute temperature); the coefficient of viscosity and conductivity μ_c' and k_c' at T_c'; $Y_o'^2/\alpha_c'$ (where α_c' is the thermal diffusivity at the temperature T_c') and α_c'/Y_o' as scaling factors for pressure, density, coefficients of viscosity and thermal conductivity,

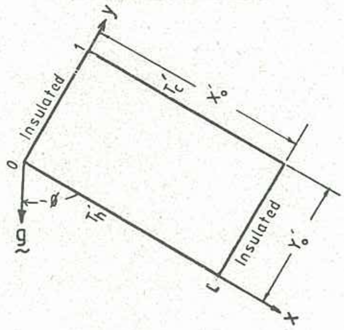

Figure 1. Definition sketch of the cavity

time and velocity and the introduction of the non-dimensional temperature $\theta=(T'-T_c')/(T_h'-T_c')$ (where T' is the local absolute temperature of the fluid and T_h' is the absolute temperature of the hot boundary) permits the equations representing the conservation of momentum (written as a vorticity transport equation) and the energy equation to be written (Leonardi and Reizes, 1980),

$$\frac{\partial \rho u \zeta}{\partial x} + \frac{\partial \rho v \zeta}{\partial y} = \frac{Ra Pr}{\eta \varepsilon}\left(\frac{\partial \rho}{\partial x}\sin\phi - \frac{\partial \rho}{\partial y}\cos\phi\right) + Pr\left[\frac{\partial^2 \mu\zeta}{\partial x^2} + \frac{\partial^2 \mu\zeta}{\partial y^2}\right] + \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)\frac{\partial \rho}{\partial y} - \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)\frac{\partial \rho}{\partial x} + 2Pr\left[\frac{\partial u}{\partial y}\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial x \partial y}\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) - \frac{\partial v}{\partial x}\frac{\partial^2 \mu}{\partial y^2}\right] + 2Pr\left[\frac{\partial \mu}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial \mu}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right)\right] \quad (1)$$

and

$$\frac{\partial \rho u \theta}{\partial x} + \frac{\partial \rho v \theta}{\partial y} = k\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + \frac{\partial k}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial \theta}{\partial y} + \frac{PN}{\varepsilon}\eta\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) + Gn\left(\frac{\eta_r^2}{Ra^2 Pr \varepsilon}\right)^{1/3}\Phi \quad (2)$$

where parameters without the prime are non-dimensional, u and v are the velocity components in the x and y directions respectively, ζ is the vorticity ($\zeta=\nabla \times V$), Ra is the Rayleigh number ($\beta_c' g \rho_r'^2 Y_o'^3 (T_h'-T_c') \tilde{C}_{pc}' / \mu_c' k_c'$), β_c' is the coefficient of volumetric expansion at temperature T_c' and pressure p_r', g' is the gravitational acceleration, C_p' the specific heat capacity at constant pressure, Pr is the Prandtl number ($\mu_c'/\rho_r'\alpha_c'$), $\eta=\beta' T'$, ε is the non-dimensional temperature difference ($(T_h'-T_c')/T_c'$), $Pn=p_r'/(\rho_r' C_{pc}' T_c')$, $Gn=(\mu_c' g'/\rho_r')^{2/3}/C_{pc}' T_c'$ and Φ is the dissipation function.

Since the density is pressure dependent the value of the pressure is required. The equation for the pressure is obtained by taking the divergence of the Navier-Stokes equations to yield (Leonardi and Reizes, 1980),

$$\left\{\frac{Pn}{Gn}\left(\frac{RaPr^{\frac{1}{2}}}{\varepsilon\eta_r}\right)^{2/3}\right\}\nabla^2 p = \left\{\frac{Ra}{\varepsilon\eta_r}\right\}\hat{\underset{\sim}{g}}\cdot\nabla\rho - Pr^{-1}\nabla\cdot(\rho\underset{\sim}{V}\cdot\nabla\underset{\sim}{V})$$
$$+ (\nabla\times\underset{\sim}{\zeta})\cdot\nabla(\lambda+\mu) + \nabla^2\lambda(\nabla\cdot\underset{\sim}{V}) + \nabla(\lambda+\mu)\cdot\nabla(\nabla\cdot\underset{\sim}{V})$$
$$+ (\lambda+2\mu)\nabla\cdot(\nabla^2\underset{\sim}{V}) + \nabla^2\underset{\sim}{V}\cdot\nabla(\lambda+2\mu) + \nabla\cdot(\underset{\sim}{V}\cdot\nabla\nabla\mu)$$
$$+ \nabla\cdot\left((\nabla\underset{\sim}{V})\cdot(\nabla\mu)\right) - \underset{\sim}{V}\cdot\nabla(\nabla^2\mu) \qquad (3)$$

where λ is the non-dimensional second coefficient of viscosity (λ'/μ_c')

and $$p = p_o + \Delta p \qquad (4)$$

where p_o is the pressure at ($x=0, y=0$). (3) has been written in vector form since the expanded equation in two dimensions has thirty six terms!

The necessity for solving the continuity equation is avoided by defining a compressible stream function ξ, such that,

$$u = (1/\rho)(\partial\xi/\partial y) \quad \text{and} \quad v = -(1/\rho)(\partial\xi/\partial x) \qquad (5)$$

so that in the steady state the equation of continuity is automatically satisfied. It may be shown (Leonardi and Reizes, 1980) that the vorticity and compressible stream function are related by,

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} = -\rho\zeta + \frac{1}{\rho}\left[\frac{\partial\xi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\xi}{\partial y}\frac{\partial\rho}{\partial y}\right] \qquad (6)$$

In deriving (2) it has been assumed that the variation of specific heat capacity at constant pressure, C_p, is small in comparision with other property variations (Mayhew and Rogers, 1973) and that little error is therefore introduced for moderate temperature and pressure changes. This however is not true for the other properties.

2.2 Fluid Properties

All but one of the relationships presented in this section were obtained from Bretsznajder (1971).

The Tamman-Hesse equation for the viscosity can be written in non-dimensional form as,

$$\mu = e^{D[(\varepsilon\theta+1)^{-3} - 1]} \qquad (7)$$

where $D = 6.18 \times 10^7 T_c'^{-3}$

and the equation for the coefficient of thermal conductivity as,

$$k = 1 + \varepsilon\theta T_c'/10^3 \qquad (8)$$

Since the pressure has to be evaluated, not all approximate equations of state for water were suitable. The Tumlirz equation (Eckart, 1958) was the most easily handled and can be written as,

$$\rho = \frac{1}{\rho_r'}\left[6.98 \times 10^{-4} + \frac{K_t'}{pp_r' + p_t'}\right]^{-1} \qquad (9)$$

where $K_t' = 1.80308 + 1.13991 \times 10^{-2}t - 7.54871 \times 10^{-5}t^2$

$$p_t' = 5.96804 \times 10^3 + 38.5035t - 0.379968t^2$$

and $$t = T_c'(1 + \varepsilon\theta) - 273.1$$

From (3) it is possible to calculate the value of Δp and since the actual pressure has to be known in (9), p_o has to be evaluated. Now if it is assumed that the initial mass in the cavity is $X_o'Y_o'\rho_r'$ and that the mass of the cavity remains constant it follows that,

$$\frac{1}{L}\int_A \rho dA = 1, \qquad (10)$$

where A is the non-dimensional area of the cavity. The substitution of (9) and (4) into (10) together with some algebraic manipulation leads to,

$$p_o = \frac{L - p_r'\int_A \frac{1}{K_t'}\left(\frac{1}{\rho_r'} - 6.98 \times 10^{-4}\rho\right)\left(\Delta p + \frac{p_t'}{p_r'}\right)dA}{p_r'\int_A \frac{1}{K_t'}\left(\frac{1}{\rho_r'} - 6.98 \times 10^{-4}\rho\right)dA} \qquad (11)$$

Thus from (7, 8, 9 and 11) it is possible to obtain the values of all the properties, if the temperature field is known. A similar set of relations can be written for air (Leonardi and Reizes, 1980).

2.3 Boundary Conditions

At $y=0$ and $y=1$, $\theta=1$ and $\theta=0$ respectively (Fig. 1). The other boundaries are adiabatic that is, $(\partial\theta/\partial n)_B=0$ where n is the coordinate normal to the boundary and the subscript B specifies evaluation at the boundary. The compressible stream function $\xi=0$, at the boundaries and the non-slip condition $(\partial\xi/\partial n)_B=0$, can be substituted into (6) to yield the vorticity boundary condition $\zeta_B=-1/\rho(\partial^2\xi/\partial n^2)_B$. The normal pressure gradients on the boundaries are obtained by a numerical evaluation of the two dimensional form of the Navier-Stokes equations.

The numerical technique used, is fully described in Leonardi and Reizes (1980).

3. RESULTS AND DISCUSSION

Since results for air filled cavities have been presented earlier (Leonardi and Reizes, 1979, 1980) results for water filled cavities only will be discussed. Although variations in the reference temperature, T_c' (also the cold boundary temperature), have an effect on the solutions for water filled cavities, the general conclusions are not altered, so that only T_c'=288K is used for the results presented here. Solutions have been obtained for cavities inclined at $\phi=-60$ (see Fig. 1), $10^3\leqslant Ra\leqslant 10^6$, $0.001\leqslant\varepsilon\leqslant 0.3$, $Gn=4.138\times10^{-10}$, $Pn=8.298\times10^{-5}$ and $Pr=8.1$.

Whereas for air filled cavities it has been shown (Leonardi and Reizes, 1980) that the solutions obtained using variable properties did not differ significantly from the solutions generated using the Boussinesq Approximation (B.A.) for $\varepsilon<0.1$, for water filled cavities significant differences occur at $\varepsilon=0.01$ as may be seen in Fig. 2a. For the case given in Fig. 2 the B.A. solution yielded a minimum value of the compressible stream function of -23.54 and the value obtained with variable properties was -25.01; a difference of 6% when the temperature difference between the hot and cold boundaries is less than 3K. As may be seen in Fig. 2b, there is very little difference between the two solutions for the temperature fields. Of course, no discernible differences can be seen between the solution for any of the parameters at low values of ε ($\varepsilon\leqslant 0.001$) with variable properties and the B.A. solution for the same Rayleigh number. However, even at these low values of ε, if the solutions are compared point by point, slight differences (usually in the fifth significant figure) occur; the most obvious being

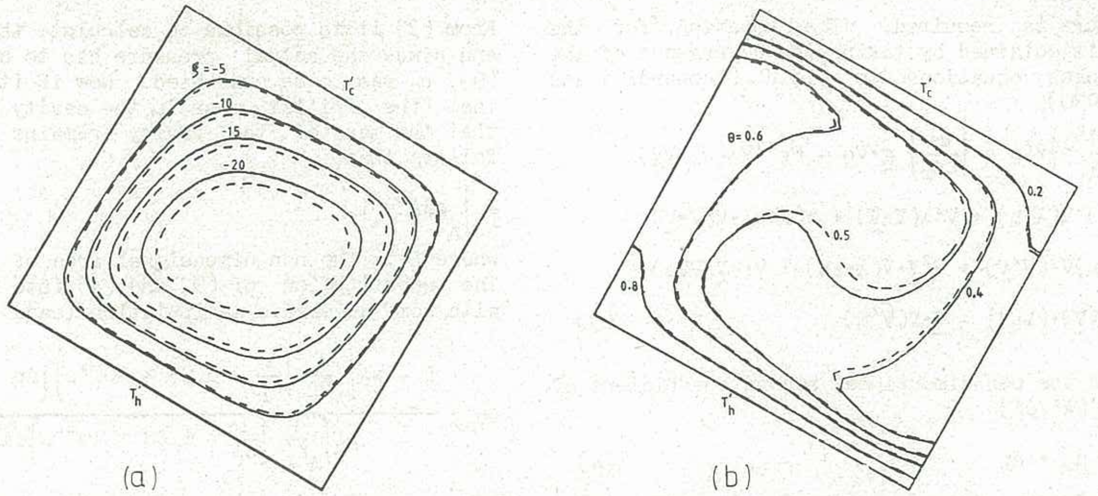

Figure 2. Streamlines (a) and Isotherms (b) for Rayleigh No.=10^5, Aspect Ratio=1,Angle of Inclination=-60°, Ref. Temp.=15°C, Ref. Pres.=1Bar, Boussinesq Approx. (dashed lines) and ε=0.01 (full lines)

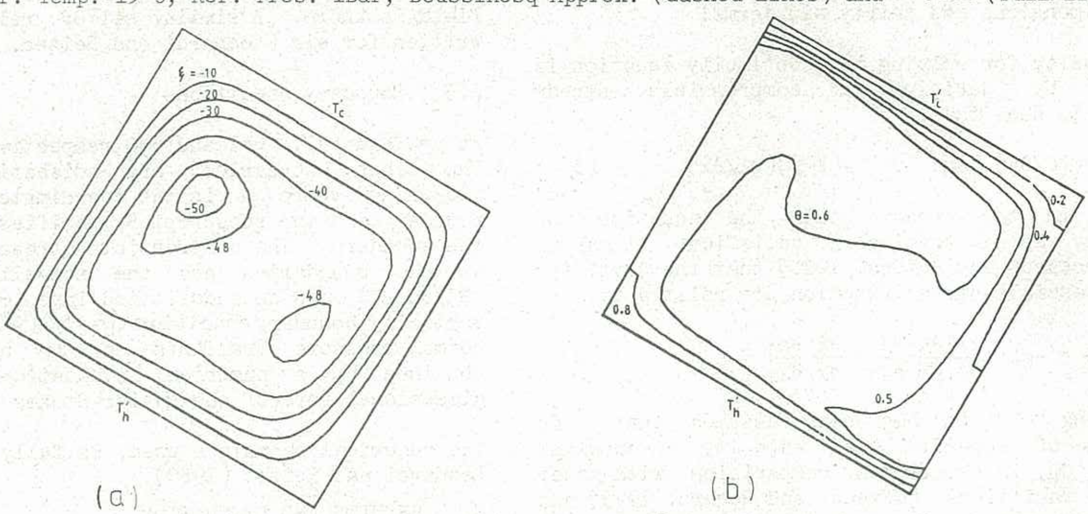

Figure 3. Streamlines (a) and Isotherms (b) for Rayleigh No.=10^5, Aspect Ratio=1,Angle of Inclination=-60°, Ref. Temp.=15°C, Ref. Pres.=1Bar, ε=0.3

that the B.A. solution of the flow field is necessarily symmetric about a diagonal and the variable properties solution is not.

As the temperature difference between the hot and cold boundaries is increased the flow and temperature fields change beyond recognition, as may be seen in Fig. 3. The single cell structure in Fig. 2 has now become a two cell structure with the two cells occupying a much larger proportion of the cavity so that the flow is essentially of the "boundary layer" type. The two cells are neither symmetrically located nor of equal strength as occurs in B.A. solutions at higher Rayleigh numbers. The minimum value of the compressible stream function has now become -50.95; more than twice the B.A. solution! Since the coefficient of viscosity on the hot boundary is approximately one fifth of the value at the cold boundary, the "boundary layer" on the hot boundary is thinner than the one on the cold boundary.

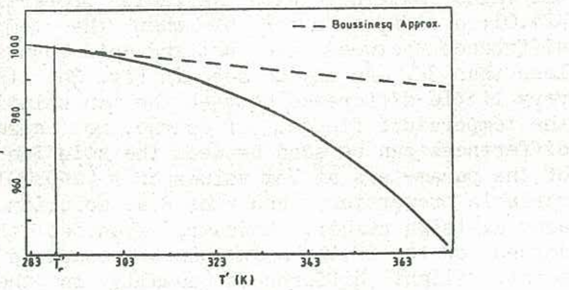

Figure 4. Comparison of density change with temperature and the Boussinesq Approx.

In a previous publication (Leonardi and Reizes, 1980) we explained the reduction of convection between a B.A. solution and solutions with variable properties for an air filled cavity at high temperature differences between the boundaries (ε>0.1), by showing that the B.A. solution over-estimated the density changes in the cavity. In water the reverse is true! The B.A. approach leads to a serious under-estimate of the density change, as may be seen in Fig. 4. For example, in air at ε=2 the over-estimate in the change in density between the cold and hot boundaries is 48% if the B.A. is used whereas in water at ε=0.3, the under-estimate of the change in density is 370% (Fig. 4). This leads to a considerably increased convection in the cavity with the result that the changes in the flow and temperature fields in a water filled cavity as ε is increased, are considerably greater than those in an air filled cavity. Similar results are obtained at aspect ratios other than one and for Rayleigh numbers other than 10^5 as may be seen in Fig. 5.

The reduction in convection together with an increase in the coefficient of heat transfer at the hot boundary led, in the case of an air filled cavity, to Nusselt numbers which were some 10% higher (in the worst case of an inclined cavity) than those predicted by the B.A. approach (Leonardi and Reizes, 1980). As may be seen in Figs. 6 and 7, for water filled cavities at a given Rayleigh number, the Nusselt number increases significantly as the non-dimensional temperature difference ε,

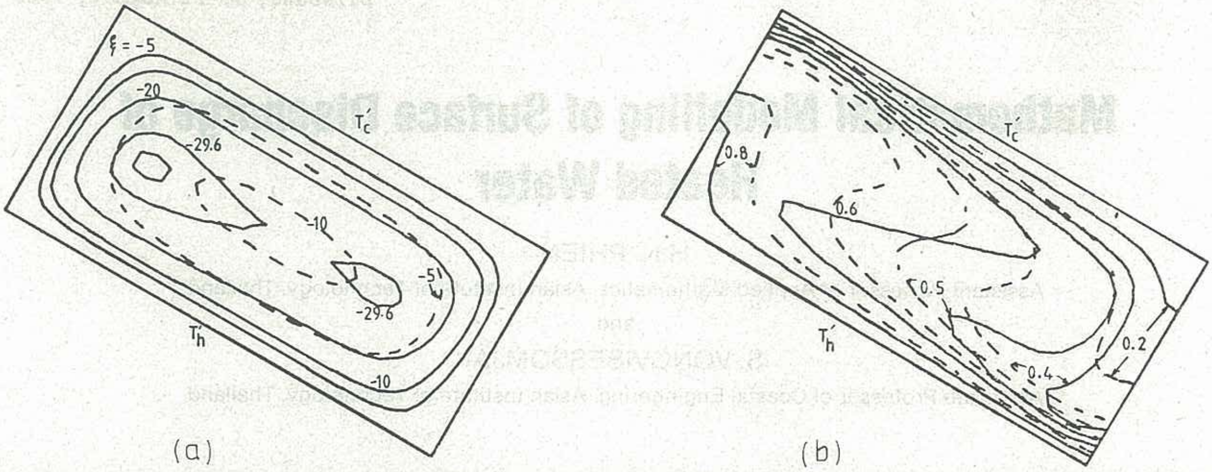

Figure 5. Streamlines (a) and Isotherms (b) for Rayleigh No.=10^4, Aspect Ratio=2, Angle of Inclination= 60°, Ref. Temp.=15°C, Ref. Pres.=1Bar, Boussinesq Approx. (dashed lines) and ε=0.3 (full lines)

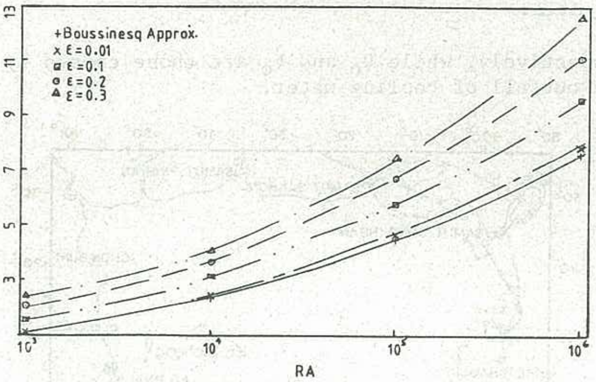

Figure 6. Relation between Nu and Ra, L=1.

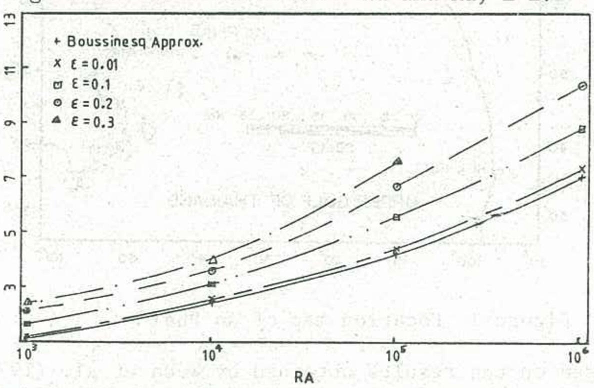

Figure 7. Relation between Nu and Ra, L=2.

increases. This is due to the large increase in the buoyancy force, considerable reduction in the coefficient of viscosity and a slight, but significant, increase in the coefficient of heat transfer, at the hot wall as ε is increased. These effects lead to much larger Nusselt numbers when variable properties are used than those obtained when the B.A. is invoked. It follows that the Boussinesq Approximation cannot be used to obtain reasonable results if the temperature difference between the hot and cold boundaries exceeds 3K (i.e. ε>0.01) in a water filled cavity.

4. CONCLUSION

Numerically calculated flow and temperature fields and heat transfer data for water filled inclined cavities have been presented. It has been shown that the flow and thermal fields are substantially affected by the introduction of variable properties and that the simpler Boussinesq Approximation cannot be used to obtain reasonably accurate heat transfer data for temperature differences between the hot and cold wall greater than 3K.

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