7th Australasian Hydraulics and
Fluid Mechanics Conference,
Brisbane, 18-22 August, 1980

An Improved Solution for the Infiltration-Advance Problem in Irrigation Hydraulics

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SUMMARY A solution of the infiltration-advance problem in irrigation hydraulics is found for the physically-based "linear soil" infiltration function, which is known to have the correct behaviour for both short and long times. The Philip and Farrell advance function which uses the Philip two-parameter infiltration equation is shown to be valid at short times, but to fail to predict the correct long-time behaviour.

1 THE INFILTRATION-ADVANCE PROBLEM

The simultaneous infiltration and advance of water across the soil surface is of importance in surface irrigation from a border. Figure 1 schematically describes the problem in which a flow of q cm^2 sec^{-1} of water is introduced at the border at time zero. The average water depth on the surface is denoted by c cm, and x(t) is the distance the stream has advanced across the surface at time t. The cumulative infiltration at the point s at time t is denoted by y(t,s) and t_s represents the time at which x = s.

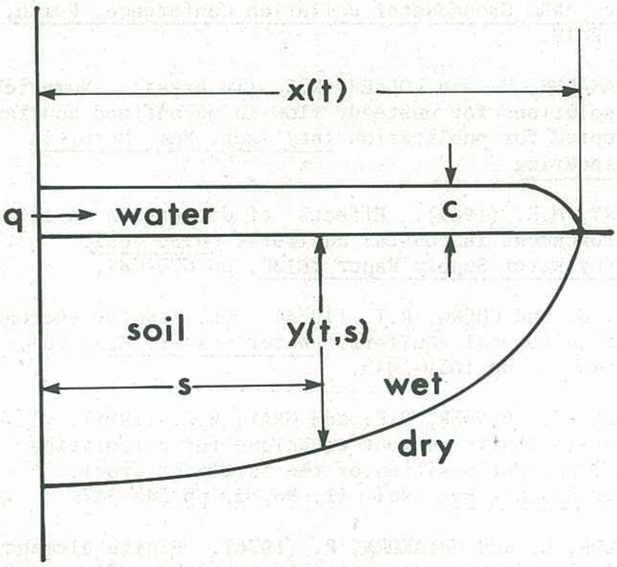

Figure 1
Schematic view of infiltration-advance

If we assume that the infiltration proceeds one-dimensionally (vertically) at each point s and depends only on the elapsed time $t-t_s$, then $y(t-t_s) = y(t,s)$. Subject to certain reasonable conditions listed by Philip and Farrell (1964), it may be shown from volume balance considerations that for each assumed functional relationship $y(t-t_s)$, x(t) satisfies the integral equation of Lewis and Milne (1938)

$$qt = cx + \int_0^t y(t-t_s)x'(t_s)dt_s \tag{1}$$

where x' denotes the rate of advance.

With the dimensionless variables $\tau = K_1^2t/S^2$, $Y = K_1y/S^2$, $C = K_1c/S^2$, $X = K_1x/q$, where S is the sorptivity of the soil, and K_1 the saturated hydraulic conductivity, the Lewis-Milne equation becomes

$$\tau = CX + \int_0^\tau Y(\tau-\tau_s)X'(\tau_s)d\tau_s$$

where X' denotes $dX/d\tau$. (2)

2 THE GENERAL SOLUTION

Philip and Farrell (1964) used the fact that the Lewis-Milne equation is of convolution type to show that the Laplace transform of the general solution is given by

$$L\{X\} = 1/(p^2C + p^3L\{Y\}), \tag{3}$$

where L{Y} denotes the Laplace transform of Y; namely,

$$L\{Y\} \equiv \int_0^\infty \exp(-p\tau)Y(\tau)d\tau \tag{4}$$

This general solution is useful only when the Laplace transform of Y is known, and when X may be readily evaluated from its transform.

Historically, infiltration functions such as those of Kostiakov (1937) and Horton (1940) have usually been chosen to contain empirical parameters of no particular significance, and to have a very simple functional form so that they may be easily fitted to experimental infiltration curves. A more complicated form for Y is not a disadvantage if the constants are identifiable soil parameters such as sorptivity and saturated hydraulic conductivity that may be independently measured in the field (Talsma (1969), Smiles and Knight (1976)).

Philip and Farrell gave solutions corresponding to the infiltration functions of Horton and of Kostiakov, and also the two-parameter equation of Philip (1954)

$$y = St^{\frac{1}{2}} + At \tag{5}$$

This equation is valid at short time, since it is a truncated form of the series

$$y = \sum_{n=1}^{\infty} a_n t^{n/2} \qquad (6)$$

Philip and Farrell derived the corresponding advance function

$$x/q = \frac{1}{A}\left\{1 - \frac{1}{\alpha-\beta}\left[\alpha\exp(\beta^2 t)\operatorname{erfc}(\beta t^{\frac{1}{2}}) - \beta\exp(\alpha^2 t)\operatorname{erfc}(\alpha t^{\frac{1}{2}})\right]\right\} \qquad (7)$$

where

$$\alpha,\beta = \frac{1}{4c}\left[\pi^{\frac{1}{2}}S \pm (\pi S^2 - 16Ac)^{\frac{1}{2}}\right].$$

When $\alpha = \beta$, the advance function becomes

$$x/q = \frac{1}{A}\left\{1-2\alpha(t/\pi)^{\frac{1}{2}} + (2\alpha^2 t-1)\exp(\alpha^2 t)\operatorname{erfc}(\alpha t^{\frac{1}{2}})\right\}. \qquad (8)$$

Philip and Farrell discussed the short- and long-time behaviour of this expression. However, as pointed out by Parlange (1973), since (5) is valid only for short times, so also is (7). Parlange proposed that the infiltration function of Talsma and Parlange (1972) be used, but was able to derive only a short-time series for the advance function. Philip and Farrell suggested that (5) could be extended to long times by using the approximation

$$y = y_o + K_1 t \qquad (9)$$

Collis-George (1974) derived the solution for this case to be

$$x/q = \frac{1}{K_1}\left\{1 - \exp[-K_1 t/(y_o+c)]\right\}. \qquad (10)$$

This has the correct asymptote for large time, but will be inaccurate for small times.

He found that the advance functions for some materials were best described by (7), whereas others best fitted (10). As a compromise, Collis-George and Freebairn (1979) suggested that the value $A = K_1$ be used in (7), in order to give x the correct value of K_1/q at long times.

3 THE INFILTRATION FUNCTION

None of the above infiltration functions satisfy the criteria of having the correct behaviour for both short and long times, and of possessing a simple Laplace transform.

Philip (1966, 1969) assumed a "linear" soil with constant diffusivity $D = \pi S^2/4$, and hydraulic conductivity K which is a linear function of moisture content θ; viz $K(\theta) = K_1(\theta-\theta_o)/(\theta_1-\theta_o)+K_o$. Assuming surface ponding, and an initially dry soil such that K_o may be neglected in comparison with K_1, he obtained the cumulative infiltration function in terms of the reduced variables as

$$Y(\tau) = \tfrac{1}{2}\tau+\tfrac{1}{2}\tau^{\frac{1}{2}}\exp(-\tau/\pi)+(\pi/4+\tfrac{1}{2}\tau)\operatorname{erf}[(\tau/\pi)^{\frac{1}{2}}]. \qquad (11)$$

As this function comes from the solution of a linear partial differential equation, which may be solved using Laplace transforms, we expect it to have a relatively simple Laplace transform, which is

$$L\{Y\} = [1+ (1+\pi p)^{\frac{1}{2}}]/2p. \qquad (12)$$

This gives the series for Y as

$$Y(\tau) = \tfrac{1}{2}\tau + \tau^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\tau^n}{(4n^2-1)n!\pi^n} \qquad (13)$$

At long times,

$$Y(\tau) \sim \pi/4+\tau - \exp(-\tau/\pi)[1/\pi\tau^{3/2}+\ldots].$$

Therefore, (12) behaves like (6) at small times with $A/K_1 = 1/2$, and behaves like (9) at long times with $y_o = \pi S^2/4K_1$. Although this infiltration function does not have a simple functional form, it does satisfy the criteria of Collis-George and Freebairn (1979), mentioned in section 2.

4 THE ADVANCE FUNCTION

For the "linear" soil of section 3, the Laplace transform of the advance function is given by

$$L\{X\} = 2p^{-1}[2pC+1+(1+\pi p)^{\frac{1}{2}}]^{-1}. \qquad (14)$$

When $C \neq \pi/4$, this may be inverted to give

$$X(\tau) = 1 - \frac{2\tau^{\frac{1}{2}}}{(4C-\pi)}\exp(-\tau/\pi) + \left[\frac{2}{(4C-\pi)} - \tfrac{1}{2} - \frac{\pi^2}{2(4C-\pi)^2}\right]\operatorname{erfc}\left[\left(\frac{\tau}{\pi}\right)^{\frac{1}{2}}\right] - \frac{4C(2C-\pi)}{(4C-\pi)^2}\exp\left[\frac{(\pi-4C)\tau}{4C^2}\right]\operatorname{erfc}\left[\frac{(\pi-2C)}{2C}\left(\frac{\tau}{\pi}\right)^{\frac{1}{2}}\right]. \qquad (15)$$

When $C = \pi/4$,

$$X(\tau) = 1-\tau^{\frac{1}{2}}(2/\pi + 4\tau/\pi^2)\exp(-\tau/\pi) + (4\tau^2/\pi^2 + 4\tau/\pi - 1)\operatorname{erfc}[(\tau/\pi)^{\frac{1}{2}}] \qquad (16)$$

At small times, when $C \neq 0$,

$$X(\tau) = \frac{\tau}{C} - \frac{2\tau^{3/2}}{3C^2} - \frac{(2C-\pi)\tau^2}{8C^3} - \frac{(2C^2-4C\pi+\pi^2)\tau^{5/2}}{15\pi C^4} \qquad (17)$$

and when $C = 0$,

$$X(\tau) = \frac{4\tau^{\frac{1}{2}}}{\pi} - \frac{2\tau}{\pi} + \frac{4\tau^{3/2}}{3\pi^2} - \frac{2\tau^{5/2}}{15\pi^3} \qquad (18)$$

At large times, for $C<\pi/2$,

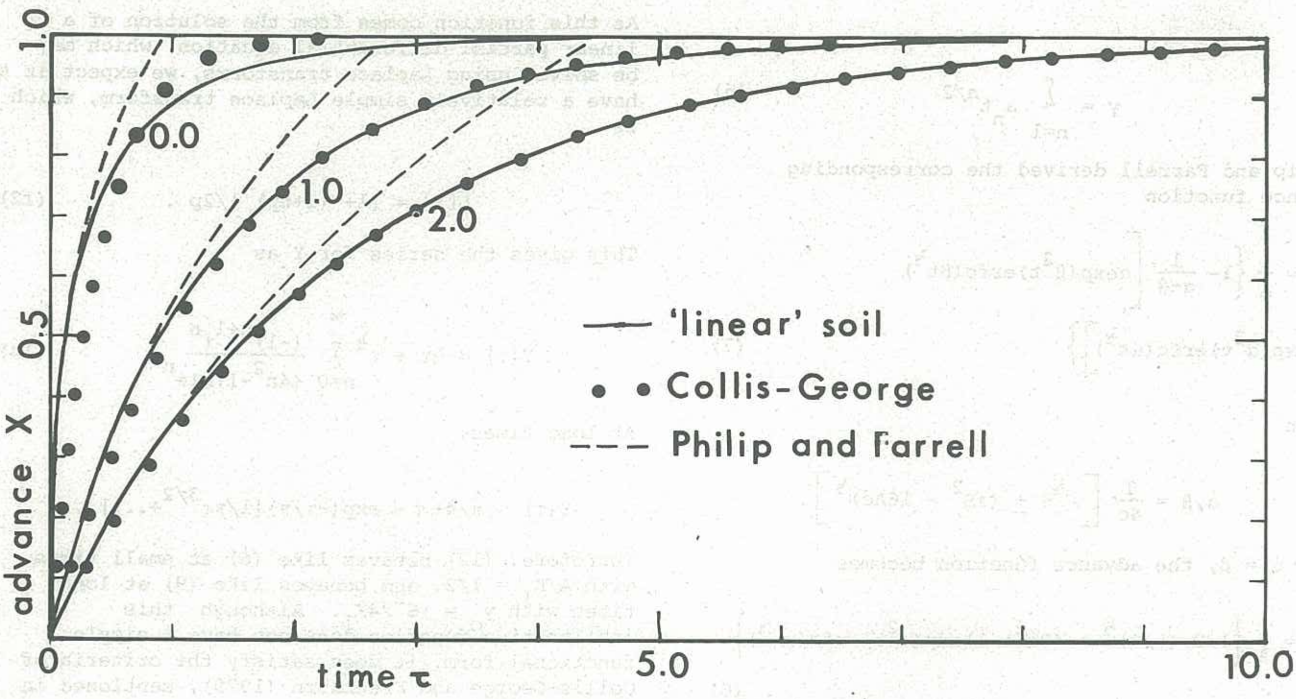

Figure 2 Advance functions. Numbers on curves are values of C

$$X(\tau) \sim 1-\tau^{-3/2}\exp(-\tau/\pi)\left[\pi^3/(\pi-2C)^2-..\right], \qquad (19)$$

for $C = \pi/2$,

$$X(\tau) \sim 1-\tau^{-\frac{1}{2}}\exp(-\tau/\pi)\left[2-..\right], \qquad (20)$$

and for $C>\pi/2$,

$$X(\tau) \sim 1 - \frac{8C(2C-\pi)}{(4C-\pi)^2}\exp\left[\frac{-(4C-\pi)\tau}{4C^2}\right] + \ldots \qquad (21)$$

The Philip and Farrell advance function corresponding to the truncated infiltration function $Y(\tau) = \tau^{\frac{1}{2}}+\frac{1}{2}\tau$, may be considered a short-time approximation to (15). The first three terms agree with those of (17), but the long-time behaviour is quite different from that of (19), (20) and (21). The fact that the Philip infiltration function $y(t) = St^{\frac{1}{2}}+At$ gives the correct asymptote for the infiltration-advance function only if $A = K_1$ has caused many workers to be confused about the meaning of the coefficient A and the term At, which Philip and Farrell say "consists, essentially, of that [contribution to infiltration] arising from gravity". The use of the "linear soil" infiltration function removes this difficulty.

In figure 2, the infiltration advance X is graphed against reduced time τ for various values of the parameter C. The small-time approximation of Philip and Farrell given by eq (7) is shown for the same values of C.

Also shown in Figure 2 for the same values of C is the advance function (10) corresponding to the long-time infiltration equation $Y(\tau) = \pi/4+\tau$, which in dimensionless variables is

$$X(\tau) = 1-\exp\left[-\tau/(\pi/4+C)\right] \qquad (22)$$

5 CONCLUSIONS

The advance function for the "linear" soil gives a test of the usefulness of the Philip and Farrell advance function corresponding to the Philip two-parameter infiltration equation, and of the Collis-George advance function corresponding to the long-time infiltration equation. As expected, the former is accurate at small times but fails to predict the correct long-time asymptote. The latter is extremely accurate for C about 1.0 or larger, and reasonably useful for all C. For a real soil, it is suggested that, with the correct choice of the physical soil parameters, the "linear" soil advance function provides a useful approximation to the true function, and is reasonably accurate at all times.

6 ACKNOWLEDGEMENTS

Part of this work was done while the author held a CSIRO postdoctoral studentship in the Soil Science Department, University of Wisconsin, Madison, Wisconsin.

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