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Finite Element Analysis of Transonic Flow in Nozzles

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SUMMARY The Finite Element Method (F.E.M.) in conjunction with the Galerkin approach has been used to calculate the inviscid subsonic and transonic flow of a perfect gas in axisymmetric nozzles of arbitrary shape. The present formulation uses the full equations of motion and the irrotationality condition written in terms of the density and velocity components. To avoid solving the resulting non-linear system of equations, the density is approximated or updated before each iteration. Thus the result is a system of linear equations in terms of the velocity components. Starting from the 1-D solution, the method requires few iterations to converge for low subsonic flow. But as the flow gets closer to the choked condition, more iterations are needed for a satisfactory answer. The present method gives solutions up to the choking flow. This solution is compared with experiment.

1 INTRODUCTION

The flow of a fluid in the transonic regime is described mathematically by highly non-linear partial differential equations of mixed elliptic-hyperbolic type. No analytic solution has been obtained and the prospect of obtaining one with our present knowledge appears remote, thus numerical techniques offer the only avenue for solving the equations and obtaining useful information. During the past few years, many numerical techniques have been developed to solve the direct transonic flow problems, such as Finite Difference Method (F.D.M.) [1], Method of Integral Relations (M.I.R.) [2] and Method of Lines (M.O.L.) [3]. But most of them are either time consuming, or encounter difficulties when treating problems with complicated boundary conditions, geometric shapes or singularities.

The F.E.M., which was developed in the field of structural mechanics [4], has since found many applications in the field of fluid mechanics due to ease of computation and flexibility in choice of arbitrary mesh arrangements to suit the boundary. There is freedom of choice of element shape, size and order of approximation to the unknown dependent variables.

There have been many attempts to apply F.E.M. to potential flow problems [5,6,7]. Most of these so far, have employed the velocity potential or the stream function as dependent variables in the full equations of motion. Results have been obtained for the subcritical flow case, but none have been successful for flows containing a supersonic region. While [8] tries to simplify the differential equations in each element by a local linearization process and subsequently iterates to a converged solution, the method works only if the maximum local Mach number is near sonic at 1.07 or less. In [9], groups of variables $(\rho u, \rho u^2, \rho uv)$ are used as unknowns. For the high subsonic flow case, this formulation is not successful if the Galerkin criterion is employed. Nevertheless, in the present paper, the F.E.M. is used in conjunction with the Galerkin criterion to calculate the subsonic and transonic flow of a perfect gas in axisymmetric nozzles of arbitrary shape. The equations of motion are written in terms of primitive variables: velocities and density, and are used together with the irrotationality condition. The density is first approximated (by the incompressible flow solution or 1-D theory for compressible flow) and later updated using the latest information on velocities. This approximation leads to a system of linear algebraic equations in terms of the velocity components, which can be solved using standard methods.

The method iterates rapidly from the 1-D solution in low subsonic flow but as choking is approached, more interations are required. The homentropic field case containing a transonic zone at the nozzle throat has been computed.

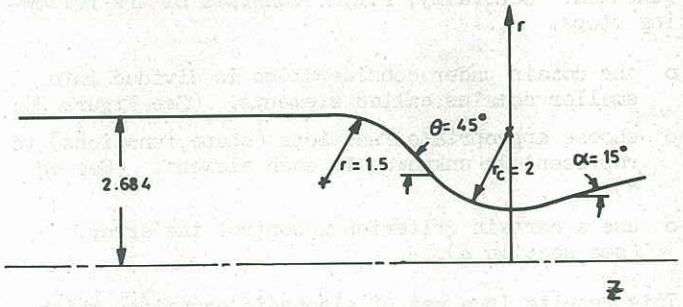

Figure 1 Nozzle Geometry

2 BASIC EQUATION AND ASSUMPTIONS

Let (z, r) be the cartesian co-ordinates of a point in the flow field, V is the velocity vector with components (w, u) in the z and r directions respectively and density ρ, pressure p. The governing equations in the steady state condition are then:

continuity: $$\nabla\cdot\rho\vec{V}r^{\ell} = \frac{\partial}{\partial r}(\rho u r^{\ell}) + \frac{\partial}{\partial z}(\rho w r^{\ell}) = 0 \qquad (1)$$

momentum (Euler):
$$\rho(\vec{V}\cdot\nabla)\vec{V} = -\nabla p$$

$$\frac{1}{\rho}\cdot\frac{\partial p}{\partial z} + w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial r} = 0 \qquad (2)$$

$$\frac{1}{\rho}\cdot\frac{\partial p}{\partial r} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = 0 \qquad (3)$$

entropy: $$p = k\cdot\rho^{\gamma} \qquad (4)$$

where $\ell = 0$ plane flow, $\ell = 1$ axisymmetric flow, γ is the specific heat ratio, k is a constant.

Assuming that there are no shock waves and that

boundary layer effects can be neglected, we have the irrotationality condition:

$$\nabla \times \vec{V} = \underset{\sim}{k}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right] = 0 \qquad (5)$$

With pressure and density normalised on the stagnation values, lengths normalised on the throat half height and velocities normalised on the maximum velocity:

$$V_m = \sqrt{\frac{2\gamma}{\gamma-1} \cdot \frac{p_o}{\rho_o}} \ ,$$

combining eqs. (2), (3), (4) and (5) we have the following relation, where ρ, u, w are now normalised values:

$$\rho = \left(1-V^2\right)^{\frac{1}{\gamma-1}} \qquad (6)$$

where $V^2 = u^2 + w^2$.

At a finite distance upstream from the nozzle entrance, the flow is uniform, i.e. $u = 0$, w = constant.

3 FINITE ELEMENT METHOD (F.E.M.)

The present F.E.M. makes use of the Galerkin criterion to control the error. The criterion is a means of obtaining an approximate solution to the differential equation by requiring the error between the approximate solution of the current iteration and true solution to be orthogonal to the function used in the approximation, i.e.

$$\int_V N_j \cdot R \cdot dV = 0 \ , \quad \text{for each node } j \text{ in the domain}$$

where R is the error and N_j is the approximate function. Generally, F.E.M. consists of the following steps.

- o the domain under consideration is divided into smaller domains called elements. (See Figure 2).
- o choose appropriate functions (shape functions) to represent the unknowns in each element. (See eq. 8).
- o use a certain criterion to control the error. (see section 4).

This results in a set of algebraic equations which can be solved numerically.

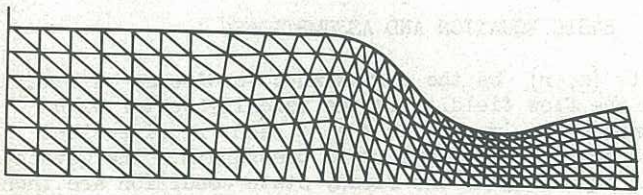

Figure 2 Generation F.E.M.

4 NUMERICAL TECHNIQUES (GALERKIN APPROACH)

If the Galerkin method is applied to eqs. (1) to (4), the result is a system of non-linear equations in which difficulty has been experienced in obtaining a converged solution.

To avoid solving the non-linear equations, it is suggested that one of the dependent variables, ρ, u or w be prescribed beforehand, thus rendering all equations linear and permitting iteration to a converged solution. By application of the Galerkin approximation to eqs. (1), (5) and (6), it will become clear below that the obvious choice of the three is the density ρ.

First, chain differentiate eq. (1), to obtain:

$$\rho r^{\ell}\frac{\partial u}{\partial r} + \rho u \ell r^{\ell-1} + u r^{\ell}\frac{\partial \rho}{\partial r} + \rho r^{\ell}\frac{\partial w}{\partial z} + w r^{\ell}\frac{\partial \rho}{\partial z} = 0 \qquad (7)$$

This equation together with eq. (5) is used to solve for the unknowns u and w.

As usual in F.E.M., the unknown is locally approximated by a set of trial (shape) functions, i.e.

$$\begin{bmatrix} u \\ w \end{bmatrix} = \sum N_i(r,z) \cdot \begin{bmatrix} u_i \\ w_i \end{bmatrix} \qquad (8)$$

where i is the i^{th} node, N_i is an arbitrarily chosen shape function appropriate to the i^{th} node and u_i, w_i are the undetermined parameters at node i.

From eq. (7), because the first derivatives of ρ are involved, it is required to interpolate for these derivatives. From eqs. (6) and (4), it can be seen that

$$\rho = 0\left[V^{\frac{2}{\gamma-1}}\right] ,$$

i.e. order 5 if $\gamma = 1.4$. However, because in a small element, the variation in nodal values is not large, a low order of approximation can be used without introducing large errors, i.e.

$$\rho = \sum_k N_k(r,z) \cdot \rho_k \qquad (9)$$

In this paper, we will use the same order of approximation for ρ as for u and w.

Substitution of (8) and (9) into eqs. (5) and (7), gives the following residuals:

$$R_1 = \sum_i u_i \left[\frac{\partial N_i}{\partial r} r^{\ell} \sum_k N_k \rho_k + \ell r^{\ell-1} N_i \sum_k N_k \rho_k + r^{\ell} N_i \sum_k \frac{\partial N_k}{\partial r} \cdot \rho_k\right]$$

$$+ \sum_i w_i \left[\frac{\partial N_i}{\partial z} r^{\ell} \sum_k N_k \rho_k + r^{\ell} N_i \sum_k \frac{\partial N_k}{\partial z} \cdot \rho_k\right]$$

$$R_2 = \sum_i u_i \frac{\partial N_i}{\partial z} - \sum_i w_i \frac{\partial N_i}{\partial r}$$

With the assumption that the ρ_i's are known before an iteration, apply the Galerkin method to the above residuals. We have:

$$\left.\begin{aligned} \iint_A N_j \cdot R_1 \cdot dr \cdot dz = 0 \\ \iint_A N_j \cdot R_2 \cdot dr \cdot dz = 0 \end{aligned}\right\} \ j = 1, n \qquad (10)$$

where n is the number of nodes in the domain. Application of Green's theorem to these equations and after some manipulation, yields

$$\left.\begin{aligned}
&\sum_{i=1}^{n} a_{ji} u_i + \sum_{i=1}^{n} b_{ji} w_i = 0 \quad j = 1, n \\
&\sum c_{ji} u_i + \sum d_{ii} w_i = 0 \quad j = 1, n
\end{aligned}\right\}$$

where

$$\left.\begin{aligned}
a_{ji} &= \sum_k \rho_k \int r^{\ell} N_i N_j N_k dz - \sum \rho_k \int r^{\ell} N_i N_k \frac{\partial N_j}{\partial r} dr dz \\
b_{ji} &= \sum_k \rho_k \int r^{\ell} N_i N_j N_k dr - \sum \rho_k \int r^{\ell} N_i N_k \frac{\partial N_j}{\partial z} dr dz \\
c_{ji} &= \int N_i N_j dr - \int N_i \frac{\partial N_j}{\partial z} dr dz \\
d_{ji} &= -\int N_i N_j dz + \int N_i \frac{\partial N_j}{\partial r} dr dz
\end{aligned}\right\} \quad (11)$$

Calculations of c_{ji} and d_{ji} can be carried out once and for all using a dummy element [11] to save some computation time. a_{ji} and b_{ji} can be best calculated by numerical integration. This results in a linear algebraic system of equations which can be solved using the Gaussian elimination technique. Solving this system subjected to prescribed boundary conditions, we find the velocities (u, w) at nodal points. Then we update the density field by using relation (6). The above process is repeated until the solution converges. The convergence criterion presently used is that for which the relative change in local density between two consecutive iterations should not be greater than a given small number ε for all nodes in the flowfield, i.e.

$$\frac{\left|\rho^{(n)} - \rho^{(n-1)}\right|}{\left|\rho^{(n)}\right|} < \varepsilon$$

This criterion is found to be more stringent than the use of local velocity V.

5 NUMERICAL RESULTS AND DISCUSSION

The choice of the shape function N in the F.E.M. must be such as to ensure continuity of the derivatives across the element's boundaries. The maximum order of derivatives which must be continuous is one order less than the maximum order of derivatives occurring in the finite element equations (10), (see [10]). With the formulation in terms of velocity components, the highest derivative is the first. Applying Green's theorem, the derivative terms are removed. Thus the lowest order of shaping function which can be chosen is linear, giving a linear variation of velocity within each element. Obviously, quadratic or higher order representations can be used to improve accuracy but would involve more complicated programming and longer execution time. In all results presented here, linear approximation in triangular elements is used.

Calculation can be done for subsonic flow up to the maximum (choked) flow case. As mentioned, the density field is required to be prescribed before each iteration. We can either start from the incompressible flow solution or the 1-D compressible flow solution. We chose the latter in the cases presented here.

The method iterates rapidly for low subsonic flow. It takes 3 iterations to converge to an error $\varepsilon = O(10^{-5})$ for low subsonic flow (curve 1, Fig. 3). 8 iterations are needed for a medium flow (curve 2, Fig. 3), more iterations are needed as the flow gets closer to the choked condition, where 43 iterations are required to converge (curve 3, Fig. 3). But

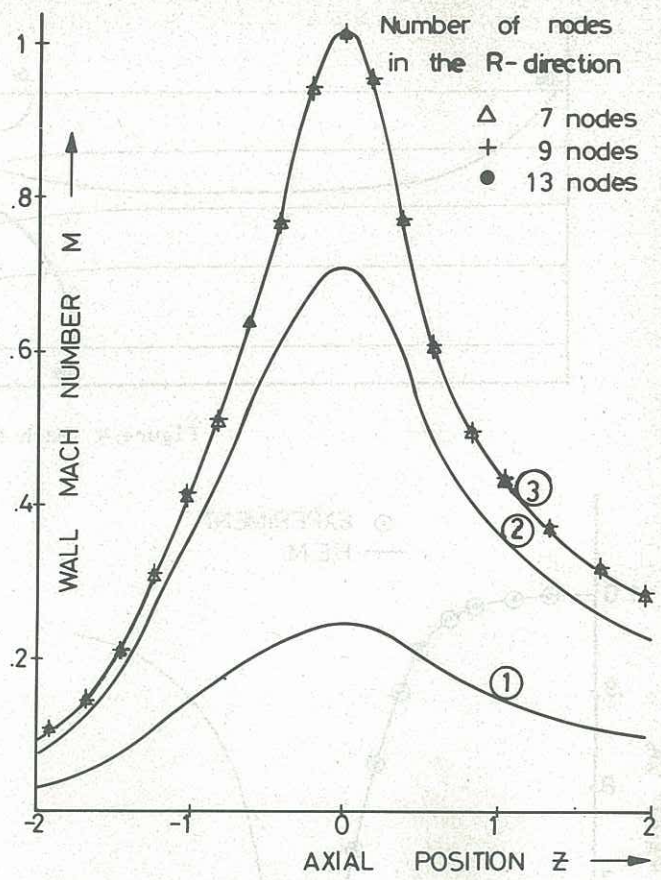

Figure 3 Nozzle Wall Mach Number for Various Mass Flow Rates

note also that in this case the search for the maximum possible mass flow rate requires iteration at the same time as density. The figure also shows the effect of increasing the number of elements. It can be seen that there is little difference if either 7-nodes, 9-nodes or 13-nodes are used in the r-direction.

Figure 4 shows the velocity field and stream function contour for the case of maximum possible flow. Figure 5 shows the wall pressure distribution for the same case. This compares very well with the experimental data from [2].

6 CONCLUSION

The F.E.M. using the Galerkin approach has been adapted to predict the transonic flow in a nozzle of arbitrary shape having radius of curvature of 2. Satisfactory agreement with experiment has been obtained.

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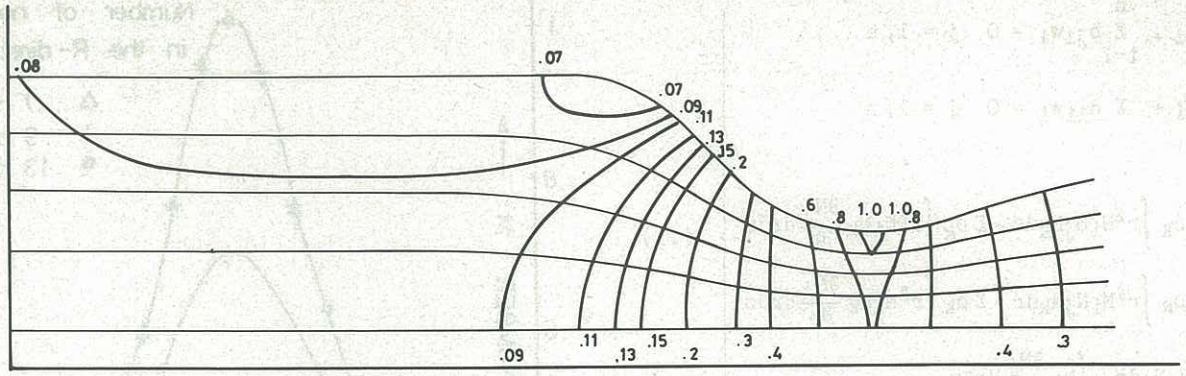

Figure 4 Mach Number Contours

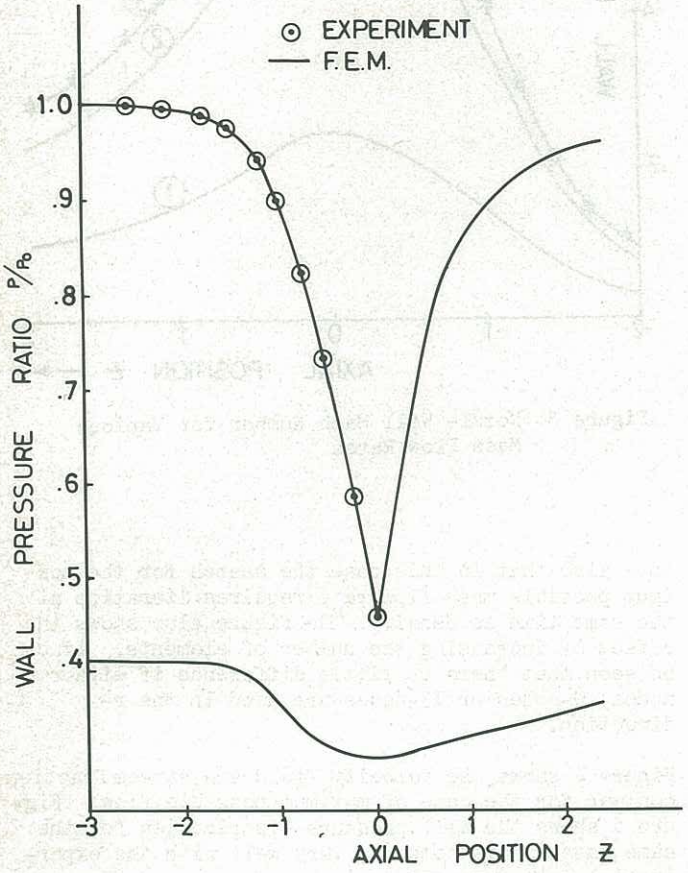

Figure 5 Wall Static to Stagnation Pressure Ratio (Choking Flow Case)

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