

Prandtl-Meyer Flow of a Finite Knudsen Number Gas

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SUMMARY The continuum description of a gas expanding around a sharp corner to a vacuum becomes inappropriate when the distance from the corner is small. An existing criterion for the breakdown of continuum flow is used to predict the nature of the process in the classical Prandtl-Meyer expansion flow. These predictions are checked by a numerical analysis of the flow using the direct simulation Monte Carlo method. This verifies that the affected region is largely confined to the gas that is initially within about twenty mean free paths from the corner. It also shows that there is an additional breakdown region near the start of the expansion and extending upstream of it.

1 INTRODUCTION

The Prandtl-Meyer expansion of a gas around a sharp corner to a vacuum is one of the classical flows of continuum gas dynamics. Although the flow is two-dimensional, the properties are a function of angle only, so that it is described by an ordinary differential equation and an exact solution is readily obtained (e.g. Liepmann and Roshko, 1957). A sonic flow at zero flow angle expands to some limiting flow angle at which the pressure falls to zero. This angle is 90° for a monatomic gas with a specific heat ratio of $5/3$ and increases to 180° as the specific heat falls to $5/4$.

The mean free path of a real gas introduces a typical dimension to the flow and the simple angular dependence of the flow no longer applies. The continuum flow model requires that the mean free path should be negligibly small in comparison with the scale length of the macroscopic flow variations. The classical theory for the Prandtl-Meyer expansion may therefore be expected to fail at progressively larger distances from the corner as the gas density decreases with the increasing flow angle and Mach number. The breakdown is initially marked by the temperature remaining above the continuum value. Moreover, the temperature components cease to be in equilibrium.

The nature of the continuum flow breakdown process in a Prandtl-Meyer expansion has implications for a number of flows of engineering significance. These include jet plumes from rocket motors and control jets in space and supersonic free jets for molecular beam generation. While these flows are axially symmetric, the departures from two-dimensional flow are negligible when the distance from the corner is small in comparison with the radius.

The direct simulation Monte Carlo method has been applied to one-dimensional steady and unsteady expansion flows (Bird, 1970). The study led to an empirical criterion for the breakdown of continuum flow and this was justified by subsequent experiments (Cattolica et al, 1974). The criterion is applied here to the Prandtl-Meyer expansion. The predictions from this are then tested by Monte Carlo calculations for the two-dimensional flow.

2 NOTATION

a	speed of sound
M	Mach number
P	continuum breakdown criterion
q	flow speed
r	radial coordinate
s	distance along streamline
T	temperature
v_s	mean molecular speed
γ	specific heat ratio
θ	angular coordinate
λ	molecular mean free path
μ	coefficient of viscosity
ν	molecular collision frequency
ρ	density
ω	temperature exponent of μ
*	denotes sonic value
c	denotes continuum value

3 THEORETICAL PREDICTIONS

The empirical criterion (Bird, 1970) for the breakdown of continuum flow in a steady expansion flow is that

$$P \equiv \frac{q}{\rho v} \left| \frac{dp}{ds} \right| \approx 0.05. \quad (1)$$

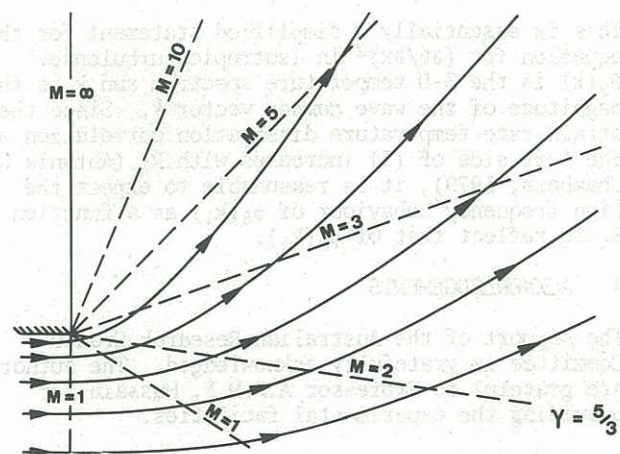


Figure 1. Streamlines and lines of constant Mach number in a Prandtl-Meyer expansion.

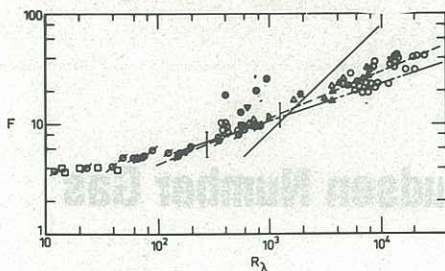


Figure 6 Variation of flatness factor of \bar{u} with R_λ . Symbols are as in Figure 5.

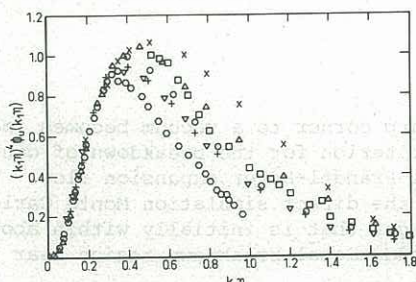


Figure 7 Fourth moment of Kolmogorov normalised spectra; ordinate $\times 10^2$. Circular jet ($d = 18$ cm): \times , $x/d = 50$; Plane jet: $+$, 60; ∇ , 80; \odot , 100; Δ , 160; \square , circular jet, Champagne (1978); \diamond , cylinder wake, Champagne (1978)

sity with R_λ as $k_1 \eta$ increases. Champagne's (1978) circular jet and cylinder wake data are shown for comparison. For locally isotropic turbulence, the turbulent vorticity budget (e.g. Wyngaard & Tennekes, 1970) reduces to

$$S = -116 \int_0^\infty (nk_1)^4 \phi_u(nk_1) d(nk_1) \quad (4)$$

Values of the left and right sides of (4) have been calculated for the plane jet and circular jet ($d = 18$ cm). In all cases the magnitude of left side is smaller by about 60-100% than that of right side. It should be noted that no corrections (for the effect of turbulence intensity on Taylor's hypothesis) have been made to the spectra in Fig. 7. This is because of the inadequacy in the assumptions made in obtaining the correction for spectra (Antonia et al, 1980). It is of interest to note that the analogous expression to (4) when considering temperature fluctuations is

$$(\partial u / \partial x) (\partial \theta / \partial x)^2 \sim \int_0^\infty k^4 F_\theta(k) dk \quad (5)$$

This is essentially a simplified statement for the equation for $(\partial \theta / \partial x)^2$ in isotropic turbulence. $F_\theta(k)$ is the 3-D temperature spectrum and k is the magnitude of the wave number vector k . Since the strain rate-temperature dissipation correlation on the left side of (5) increases with R_λ (Antonia & Chambers, 1979), it is reasonable to expect the high frequency behaviour of $\phi_\theta(k_1)$ as a function of R_λ to reflect that of $\phi_u(k_1)$.

4 ACKNOWLEDGEMENTS

The support of the Australian Research Grants Committee is gratefully acknowledged. The authors are grateful to Professor A.K.M.F. Hussain for providing the experimental facilities.

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Typical streamlines and lines of constant Mach number in a Prandtl-Meyer expansion are illustrated in Fig. 1. Note that the lines of constant Mach number are the Mach lines of the flow, so that

$$ds = Mrd\theta. \quad (2)$$

The angular coordinate θ is measured from the sonic Mach line and is related to the Mach number by the standard result,

$$\theta = \left(\frac{\gamma+1}{\gamma-1} \right)^{1/2} \tan^{-1} \left[\left(\frac{\gamma-1}{\gamma+1} \right)^{1/2} (M^2-1)^{1/2} \right],$$

so that,

$$\frac{d\theta}{dM} = \frac{(\gamma+1)/2}{1+[(\gamma+1)/2]M^2} \frac{M}{(M^2-1)^{1/2}}, \quad (3)$$

while the standard one-dimensional result for the variation of density along a streamline is

$$\frac{1}{\rho} \frac{d\rho}{ds} = - \frac{M}{1+[(\gamma+1)/2]M^2} \frac{dM}{ds}. \quad (4)$$

The collision frequency ν is related to the mean free path by

$$\nu = v_s/\lambda = [8/(\gamma\pi)]^{1/2} a/\lambda$$

and, noting that $q = aM$, (2) to (4) may be substituted into (1) to give,

$$P = \left(\frac{\gamma\pi}{2} \right)^{1/2} \frac{(M^2-1)^{1/2}}{\gamma+1} \frac{\lambda}{r}. \quad (5)$$

A mean free path may be defined for all gases by assuming the universal application of the hard sphere relation

$$\lambda = (16/5) (\mu/\rho) (2\pi RT)^{-1/2}.$$

The effect of molecular model may be introduced through the temperature exponent ω of the coefficient of viscosity. The extreme models of hard sphere and Maxwell molecules correspond to $\omega = 1/2$ and $\omega = 1$, respectively, while most real gases have values of ω around 0.75. Density and temperature may be related to the sonic values by the standard one-dimensional relationships involving the Mach number. This means that the value of P for a given gas at a particular Mach number may be expressed as a function of r/λ^* , where λ^* is the mean free path in the initial sonic flow. That is,

$$P = \left(\frac{\gamma\pi}{2} \right)^{1/2} \frac{(M^2-1)^{1/2}}{\gamma+1} \left\{ \frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2} \right\}^{1/(\gamma-1)-\omega+1/2} \frac{\lambda^*}{r}. \quad (6)$$

The Mach number is related to the angular coordinate θ by the standard result

$$M^2 = \frac{\gamma+1}{\gamma-1} \tan^2 \left[\left(\frac{\gamma-1}{\gamma+1} \right)^{1/2} \theta \right] + 1. \quad (7)$$

Note that both the parameter P and the ratio λ^*/r may be regarded as Knudsen numbers of the flow and that continuum theory applies as these tend to zero.

Curves of constant value of the breakdown criterion P are plotted with typical streamlines in Fig. 2. This is for $\gamma = 5/3$ and $\omega = 0.75$, but the general features are insensitive to these parameters.

The constant P curves cut sharply across the streamlines near to the corner, but progressively become almost coincident with the streamlines. The critical $P = 0.05$ curve approaches the streamline

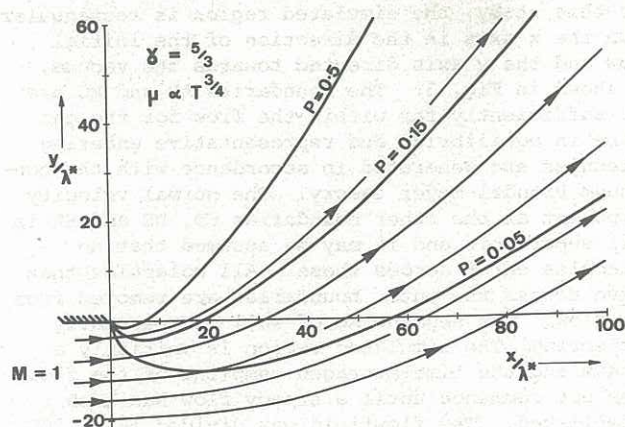


Figure 2. Contours of constant value of the breakdown criterion P .

which enters the expansion at 16 mean free paths from the corner. This means that the gas flowing along streamlines that are further out in the flow may be expected to remain in thermal equilibrium indefinitely.

While the average exit Mach number of a rocket nozzle is supersonic, the sonic line in the boundary layer intersects the lip of the nozzle (Bird, 1974). The Prandtl-Meyer solution therefore applies in the locally two-dimensional region near the lip. However, the effects due to the axial symmetry and finite extent of the plume cause the curves of constant P to cut back across the streamlines to the centreline. The flow outside twenty mean free paths from the corner may not, therefore, remain in equilibrium for a sufficient distance to produce a continuum backflow. It should be noted that, for $\gamma = 1.4$, a streamline which starts at $20\lambda^*$ from the corner will be $3.79 \times 10^4 \lambda^*$ when it reaches $\theta = 180^\circ$ and no less than $1.74 \times 10^{16} \lambda^*$ when it reaches $\theta = 210^\circ$.

For small jets at least, it appears that most of the plume backflow originates in the highly non-equilibrium region within some tens of mean free paths from the nozzle lip.

4 NUMERICAL RESULTS.

A numerical analysis of the region adjacent to the centre of the expansion was made with the direct simulation Monte Carlo method (Bird, 1976). This is a technique for the computer modelling of a real gas by some thousands of simulated molecules. The velocity components and position coordinates of these molecules are stored in the computer and are modified with time as the molecules are concurrently followed through representative intermolecular collisions and boundary interactions in simulated physical space.

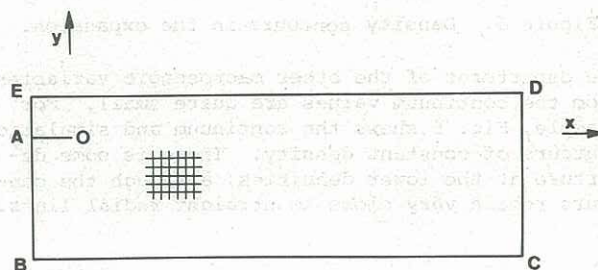


Figure 3. Geometry of the simulated region of the flow.

For this study, the simulated region is rectangular with the x axis in the direction of the initial flow and the y axis directed towards the vacuum, as shown in Fig. 3. The boundaries AB and BC are set sufficiently far within the flow for the gas to be in equilibrium and representative entering molecules are generated in accordance with the continuum Prandtl-Meyer theory. The normal velocity component at the other boundaries CD, DE and EA is well supersonic and it may be assumed that no molecules enter across these. All molecules that leave across the outer boundaries are removed from the flow. The segment AO of wall is specularly reflecting. The simulated region is initially a vacuum and the time-averaged sampling of the flow does not commence until a steady flow has been established. The flowfield was divided into 1200 uniform cells and approximately 12,000 simulated molecules were present.

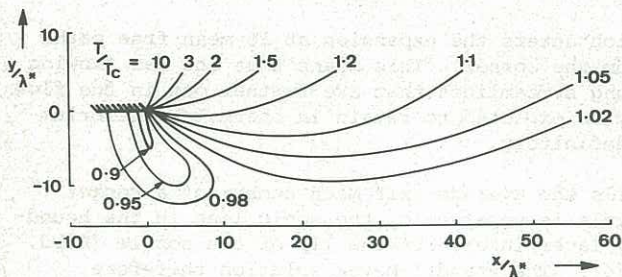


Figure 4. Temperature contours in the expansion.

Figure 4 shows contours of constant value of the ratio of the actual temperature T to that based on the continuum theory T_c . As expected, breakdown with the temperature in excess of the continuum value occurs in the region where the breakdown parameter P is in excess of 0.05. The ratio becomes very large in the near free molecule region with P greater than unity. The large gradients normal to the streamline near the corner produce a second breakdown region in which the temperature falls below the continuum value. Note that this region extends upstream of the corner.

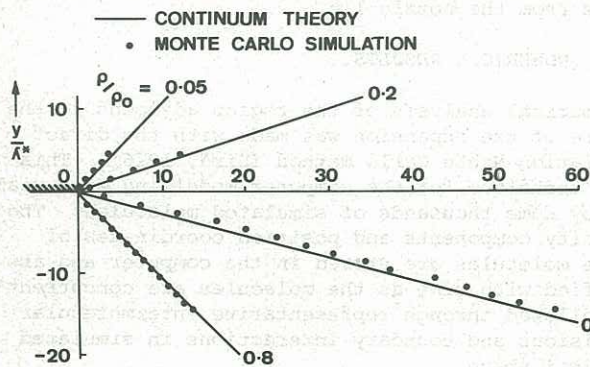


Figure 5. Density contours in the expansion.

The departures of the other macroscopic variables from the continuum values are quite small. For example, Fig. 5 shows the continuum and simulation contours of constant density. There is some departure at the lower densities, although the contours remain very close to straight radial lines.

However, the overall major qualitative departure from the continuum theory is that there is a small but finite gas density in the region beyond a flow angle of 90° where the continuum theory predicts a complete vacuum. The density contours in this region are shown in Fig. 6.

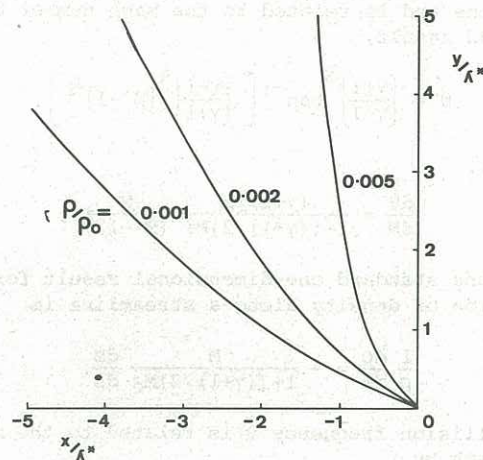


Figure 6. Density contours in the "cavitation" region.

5 CONCLUSIONS.

Non-continuum effects in a Prandtl-Meyer expansion are largely confined to the gas which enters the expansion within twenty mean free paths from the corner. These effects lead to a significant gas density within the "cavitation region" where the continuum theory predicts a vacuum. In practical free jet and jet plume flows, the consequences of axial symmetry and finite flow extent will cause the breakdown to spread to all parts of the flow. However, the overall flow geometry is such that, with the exception of very large rocket motors, backflow will largely arise in the essentially two-dimensional breakdown region very close to the corner.

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