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Linearized Low Froude Number Flows Through Obstacles

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1 INTRODUCTION

The zero Froude number approximation to low Froude number flows simply assumes that the free surface remains straight at all times. Analytical solutions can generally be found in cases where the flow field contains a certain type of wall boundaries. For the presence of more complicated wall boundaries, the numerical solutions are comparatively simple. The zero Froude number approximation seem reasonable since the free surface elevations for such flows are usually very small. However, such approximation is over simplified when considering that the kinetic energy trapped locally by the presence of a floating or a submerged obstacle. Hence the added mass of the obstacles derived in this way may not be a good estimation to the real situations.

The presence of an obstacle in an otherwise uniform flow, generates two types of free surface waves : One propagates down stream of the obstacle. It is of order $F^{-2} \exp(-F^{-2})$ where F is the Froude number (Lamb 1932). As it represents the transfer of energy down stream to infinity and so the obstacle experiences the wave pattern drag. The second type of free surface wave is caused by the blockage of the obstacle. These waves occur near to the obstacle and travel with it. They disappear rapidly with distance away from the obstacle. The wave height and velocities related to this type of waves are of order F^2. They contribute to the added mass of the obstacle but exert no force on it.

In this paper, the solution of low Froude number flows assume the disappearance of the first type of waves and only retains the structure of the second type. This allows the free surface condition to be expressed as an integral equation. The solution for zero Froude number is introduced into this equation and a power series in F^2 which satisfies both the free surface and the rigid wall boundary conditions can be generated.

2 NOTATION

a	:	semi axis of ellipse; coincides with calm free surface
b	:	semi axis of ellipse; normal to calm free surface
D	:	reference length of obstacle; for ellipse: $D = a + b$
F	:	Froude number; $F = U (g D)^{-\frac{1}{2}}$
g	:	acceleration of gravity
h	:	free surface elevation
M	:	added mass of object
n	:	direction normal to the line of integration
s	:	direction tangential to the line of integration
U	:	reference velocity; velocity of free stream or velocity of an object
u	:	non dimensional horizontal velocity component on free surface
V	:	integral operater; equations (6), (11) and (19)
v	:	non dimensional vertical velocity component on free surface
x	:	horizontal coordinate
y	:	vertical coordinate
z	:	complex variable $z = x + i\,y$
λ	:	a parameter : $\lambda = (a - b)(a + b)^{-1}$
σ	:	complex variable: equation (9)
ϕ	:	velocity potential
ζ	:	complex variable; equation (8)

3 FORMULATIONS

The following analysis assumes that the Froude number is small and steady state is achieved in the flows. All variables are non-dimensionalized using the reference velocity U which is generally referred to as the velocity of the free stream and the reference length D which associate with the geometry of the obstacle.

The presence of an obstacle in an otherwise uniform flow field, the horizontal and vertical velocities components along the free surface may be expressed as :

$$u = 1 + u_0 + F^2 u_1 \quad . \; . \; . \; (1)$$

$$v = F^2 v_1 \quad . \; . \; . \; (2)$$

where F is the Froude number.

The component u_0 is the zero Froude number solution which assumes that the free surface is a straight stream line and upon superposition with the free stream velocity, satisfies the rigid boundary condition on the obstacle. The velocity components in the second and higher order of Froude number are denoted by u_1 and v_1. It will become evident that u and v are series in even power of F.

Let h be the elevation of the free surface. For a steady flow, the linearized free surface condition is :

$$h = - F^2(u_0 + F^2 u_1)$$

$$= F^2(h_0 + F^2 h_1) \quad \ldots (3)$$

The kinematic condition of the free surface is given by :

$$(1 + u_0 + F^2 u_1) \frac{\partial h}{\partial x} = F^2 v_1 \quad \ldots (4)$$

Now equations (3) and (4) are combined to eliminate h and yields

$$v_1 = - (1 + u_0 + F^2 u_1) \left(\frac{\partial u_0}{\partial x} + F^2 \frac{\partial u_1}{\partial x} \right) \quad \ldots (5)$$

It is now possible to relate u_1 and v_1 by the potential theory which is commonly used in thin aerofoil theory (Cheng & Rott 1954), that is :

$$u_1 = V (v_1) \quad \ldots (6)$$

The integral operator V depends on the geometry of the obstacle. Thus equation (6) contains the free surface condition and also the rigid wall boundary condition. The key to the solution of (6) lies in the construction of the operator V. For once it is known v_1 in (5) and (6) are replaced by the following :

$$u_1 = - V \left[\frac{\partial u_0}{\partial x} (1 + u_0) \right] - F^2 V \left[\frac{\partial u_1}{\partial x} (1 + u_0) + u_1 \frac{\partial u_0}{\partial x} \right] - F^4 V \left[u_1 \frac{\partial u_1}{\partial x} \right] \quad \ldots (7)$$

The value of u_1 can now be solved as a series in F^2 : that is, the first term of u_1 is derived from the known function u_0 and $\frac{\partial u_0}{\partial x}$. Further terms in higher order of F^2 are obtained by repeat substitutions of the current solution of u_1 into equation (7).

4 APPLICATION TO FLOATING ELLIPSES

As an illustration to the method of solution using the above analysis, the obstacle is an ellipse floating on the free surface. The diameter along this axis is $1 + \lambda$ and the vertical diameter is $1 - \lambda$. This parameter λ varies from -1 to +1 according to the eccentricity of the ellipse.

4.1 Evaluation of u_1 and h_1

Consider the conformal transformations as shown in figure 1 : the ellipse in the z-plane is transformed into a circle in the ζ- plane, by the relation :

$$z \ (= x + i\, y) = \frac{1}{2} \left(\zeta + \frac{\lambda}{\zeta} \right) \quad \ldots (8)$$

This is followed by the transformation of the free surface into the interval (-1,1) along the real axis of the σ- plane :

$$\sigma = \frac{2\,\zeta}{\zeta^2 + 1} \quad \ldots (9)$$

By considering the ζ- plane, the zero Froude number solution is found to be

$$u_0 = \frac{\lambda - 1}{\zeta^2 - \lambda} \quad \ldots (10)$$

The operater V may be constructed from the σ-plane: Along the real axis, the vertical velocity component $v_1 \frac{dz}{d\sigma}$ vanishes outside the free surface interval (-1,1) since this is the boundary of the ellipse. The singular integral formula related to finite

$$u_1 = - \frac{1}{\pi} \frac{d\sigma}{dz} \int_{-1}^{1} \frac{v_1 (\sigma') \frac{dz}{d\sigma'}}{\sigma - \sigma'} \, d\,\sigma' \quad \ldots (11)$$

where the Cauchy Principal value of the integral has to be considered whenever σ lies within this interval (-1,1).

It is evident that V can now be identified.

The first term of u_1 is calculated from the substitution of u_0 into equation (7). From the value of u_0 and u_1, the components of free surface elevations given in (3) are obtained. The typical values of h_0 and h_1, corresponding to $\lambda = 0$ which is a floating circle, are plotted in figure 2. Note that these elevations are symmetrical and only the symmetrical half are shown. This symmetry also implies that no force is exerted on the ellipse so long as the velocity of the free stream is constant. However as energy is trapped around the ellipse and this energy increases with respect to the increase of free stream velocity, the effect of this type of free surface waves is important in the studies of quasi-static flows.

4.2 The Added Mass

The added mass of the ellipse is calculated from the energy associated with its motion in a still fluid medium. The velocity potential ϕ correspond to the flow given in equations (1) and (2).

$$\phi = x + \phi_0 + F^2 \phi_1 \quad \ldots (12)$$

where ϕ_0 is the zero Froude number solution and ϕ_1 is associated with the velocities component u_1 and v_1.

The non dimensional added mass M for an obstacle such as the ellipse is given by

$$M = - \oint \phi_r \frac{\partial \phi_r}{\partial n} \, d\,s \quad \ldots (13)$$

where the contour of integration consists of the free surface and all rigid wall boundaries and the derivative $\frac{\partial \phi_r}{\partial n}$ is in the direction normal to this contour. The velocity potential ϕ_r is given by

$$\phi_r = \phi - x = \phi_0 + F^2 \phi_1 \quad \ldots (14)$$

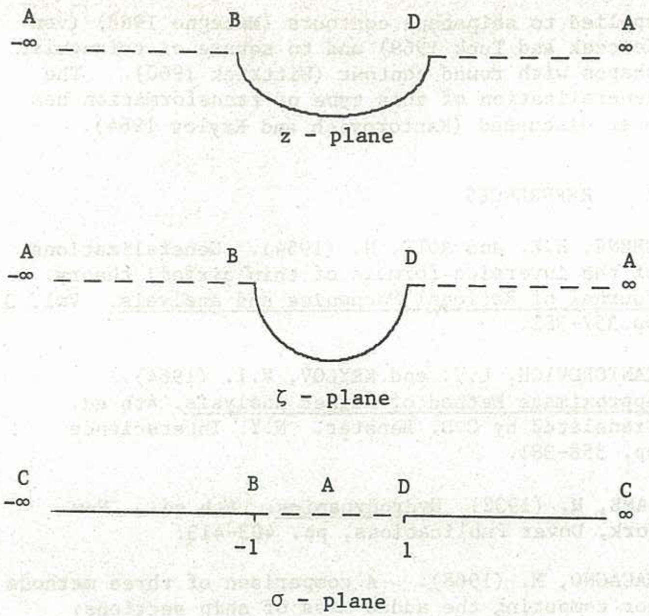

Figure 1 Conformal transformations of an ellipse in the z - plane

The substitution of equation (14) into equation (13) gives :

$$M = -\oint \phi_0 \frac{\partial \phi_0}{\partial n}\, ds - F^2 \oint \left[\phi_0 \frac{\partial \phi_1}{\partial n} + \phi_1 \frac{\partial \phi_0}{\partial n} \right] ds - F^4 \oint \phi_1 \frac{\partial \phi_1}{\partial n} ds \quad . . . (15)$$

For a floating ellipse considered here, the first term of (15) is equal to half of the added mass

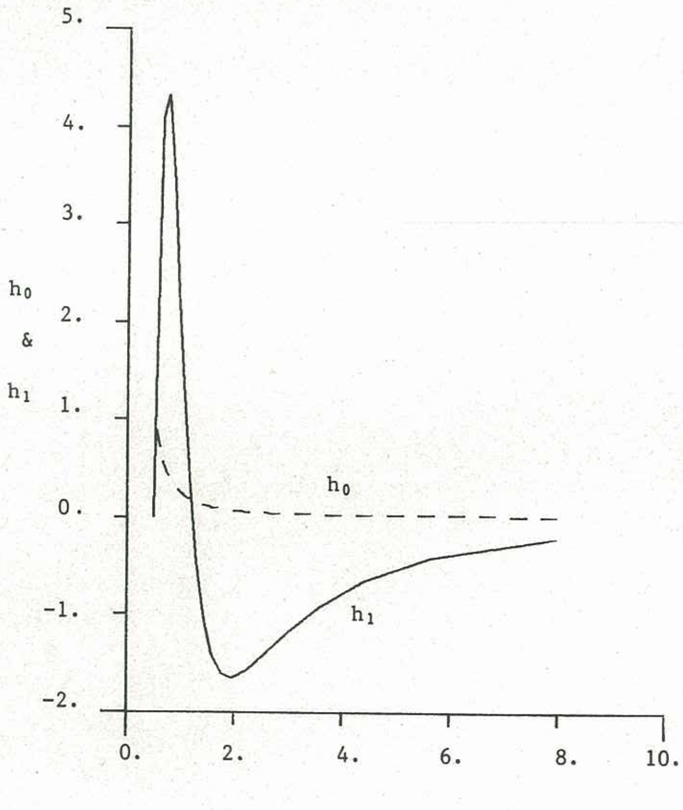

Figure 2 Components of free surface elevations h_0 and h_1 in the neighbourhood of a floating circle centered at 0.;0)

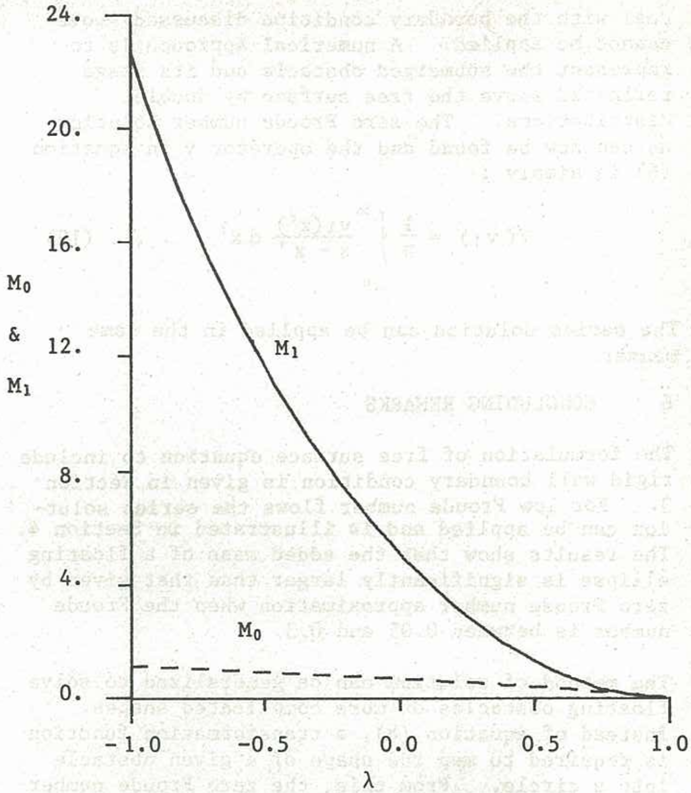

Figure 3 Components of added mass M_0 and M_1 of ellipses plotted against λ

for a submerged ellipse moving in the direction of one of its axes. Denote this as M_0 and :

$$M_0 = -\oint \phi_0 \frac{\partial \phi_0}{\partial n}\, d\,s = \frac{\pi}{8}(1 - \lambda)^2 \quad . . . (16)$$

Let the contribution to the added mass by the calculated first term of u_1 be M_1 so that (15) is approximately

$$M = M_0 + F^2 M_1 \quad . . . (17)$$

The values of M_0 and M_1 for λ varying between -1 and 1 are plotted in figure 3. The contribution of M_1 is progressively large as λ approaches -1 where the ellipse is degenerated toward a vertical flat plate. For example, if $F = 0.3$, the ratio of $F^2 M_1(\lambda = -1)$ to $M_0(\lambda = -1)$ is approximately 1.25.

On the other hand, where λ is near to 1.0, the ellipse is practically a horizontal straight line section on the free surface. In this region the value of M_1 approaches a simple function of λ given by

$$M_1(\lambda \doteq 1) \doteq \frac{1 - \lambda}{\lambda} \quad . . . (18)$$

By comparison with the value of M_0 given in equation (16), it is evident that the ratio M_1/M_0 approaches an infinity large value as λ approaches 1. That is, the added mass in this region is dominated by the higher order term.

5 A NOTE ON APPLICATION TO FLOWS WITH SUBMERGED OBSTACLES

The analysis given in Section 3 can be applied to submerged obstacles. In contrast to floating obstacle as discussed in Section 4, the rigid wall boundary does not come in contact with the free surface. Thus the region bounded by the free surface and the obstacle is not simply connected. The technique using conformal transformation to deal with the boundary condition discussed above cannot be applied. A numerical approach is to represent the submerged obstacle and its image reflected above the free surface by doublet distributions. The zero Froude number solution u_0 can now be found and the operator V in equation (6) is simply :

$$V(v_1) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v_1(x')}{x - x'} \, dx' \quad . . . (19)$$

The series solution can be applied in the same manner.

6 CONCLUDING REMARKS

The formulation of free surface equation to include rigid wall boundary condition is given in Section 3. For low Froude number flows the series solution can be applied and is illustrated in Section 4. The results show that the added mass of a floating ellipse is significantly larger than that given by zero Froude number approximation when the Froude number is between 0.05 and 0.3.

The method of solution can be generalized to solve floating obstacles of more complicated shapes. Instead of equation (8), a transformation function is required to map the shape of a given obstacle into a circle. From this, the zero Froude number solution u_0 can be derived and so allows the series solution to be applied.

There are various methods for transforming certain types of shape into a circle. For example those applied to shipshape contours (Macagno 1968) (von Kerczek and Tuck 1969) and to square or triangular shapes with round contour (Wittrick 1960). The generalization of this type of transformation has been discussed (Kantorovich and Krylov 1964).

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